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STRONG LAW OF LARGE NUMBERS FOR DEPENDENT RANDOM VARIABLES WITH VALUES IN RADEMACHER TYPE p BANACH SPACES

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RESUME

We consider functionals of the sequences of independent random variables with values in Rademacher type p Banach spaces. Under some additional conditions we prove a strong law of large numbers for the sequences of such functionals.

Key words: Banach space, a functional of the sequence random variables, strong law of large numbers.

Strong law of large numbers for Banach space valued independent and weakly dependent random variables were studied by many authors, see for instance [1]-[15] and references therein. We are interested in strong laws of large numbersfor dependent random variables with values in Rademacher type p Banach spaces.

We say that the sequence $\{X_n,\ n \geq 1\}$ satisfies a strong law of large numbers if (assuming $EX_i = 0,\ i \in N$) as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0 \text{ a.s.}$$

Our goal is to establish strong laws of large numbers under some dependence conditions.

Definition 1. A separable Banach space B(with a norm $\|\cdot\|$) is called Rademacher type p $(1 \leq p \leq 2)$ Banach space if for any finite collection of B-valued independent random variables $X_1, X_2, ..., X_n$ with $EX_i = 0$, $E\|X_i\|^p < \infty,\ \ i = 1, 2, ..., n$ there exists a constant $C(B, p)$ depending on B and p only such that the following inequality

$$E\left\|\sum_{j=1}^{n} X_j\right\|^p \leq C(B,p)\sum_{j=1}^{n} E\|X_j\|^p,$$

holds.

For the sequence $\{X_n,\ n \geq 1\}$ of independent random variables with values in Rademacher type p Banach space the following result is known.

Theorem 1([9]). Let B be a Rademacher type p $(1 < p \leq 2)$ Banach space and assume that $\{X_n,\ n \geq 1\}$ is a sequence of independent random variables such that the following hold

$$EX_k = 0, E\|X_k\|^p < \infty,\ \ k = 1, 2, ...,$$

$$\sum_{k=1}^{\infty} \frac{E\|X_k\|^p}{k^p} < \infty.$$

Then $\{X_n,\ n \geq 1\}$ satisfies the strong law of large numbers i.e. as $n \to \infty$,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0, \text{ a.s.}$$

Theorem 1 was extended to the mixing random variables with values in Rademacher type p Banach spaces.

Definition 2. For the sequence of B-valued random variables $\{X_n,\ n\geq 1\}$, the mixing coefficients are defined as following:

$$\psi(k)=\sup\left\{\left|\frac{P(AB)-P(A)P(B)}{P(A)P(B)}\right|:\ A\in F_1^n,\ B\in F_{n+k}^{\infty}, n\in N\right\}$$

where F_a^b is a $\sigma-$field generated by the random variables $X_a, X_{a+1}, ..., X_b$.

We say that $\{X_n,\ n\geq 1\}$ is ψ-mixing if $\lim\limits_{n\to\infty}\psi(n)=0$.

The following theorem was proved in [12].

Theorem 2. Let B be a separable Rademacher type p $(1<p\leq 2)$ Banach space. Assume that $\{X_n,\ n\geq 1\}$ is ψ-mixing sequence of random variables with values in B satisfying the following conditions:

$$X_k=0,\quad E\left\|X_k\right\|^p<\infty,\ \ k=1,2,...\ ,$$

$$\psi=\prod_{k=1}^{\infty}(1+\psi(k))<\infty,$$

$$\sum_{k=1}^{\infty}\frac{E\left\|X_k\right\|^p}{k^p}<\infty.$$

Then as $n\to\infty$,

$$\frac{1}{n}\sum_{k=1}^{n} X_k \to 0, \text{ a.s.}$$

Our goal is to prove strong laws of large numbers for dependent random variables with values in Rademacher type p Banach spaces.

Main result.

Consider a two-sided, sequence $\{Y_n, n\in Z\}$ of independent random variables with values in a separable measurable space T. We say that $\{X_n, n\in Z\}$ is a functional of $\{Y_n, n\in Z\}$ if there exists a measurable function $f: T^Z\to B$ such that

$$X_n=f\left((Y_{n+i})_{i\in N}\right) \tag{1}$$

We say that f is a p-approximating functional (or near epoch dependent) if there exist sequences $\{\alpha_n(m),\ n\geq 1,\ m\geq 1\}$ with $\alpha_n(m)\to 0$ for any fixed n as $m\to\infty$ and for every m a function $f_m: T^{2m+1}\to B$ such that

$$E\left\|X_n-f_m(Y_{n-m},...Y_{n+m})\right\|^p\leq\alpha_n(m) \text{ for all }\ m\in N,\ \ n\in Z. \tag{2}$$

Examples of such functionals can be found in [16].

As an example, consider a sequence $\{Y_n, n\in Z\}$ of independent random variables with values in a separable Rademacher type p Banach spaces.

Set $X_n=\sum_{i\in Z} a_i Y_{n+i},\ n\in Z$ assuming that the series converges almost surely.

In this case we can take $f_m(Y_{n-m},...Y_{n+m})$ as following

$$f_m(Y_{n-m},...Y_{n+m})=\sum_{n-m\leq i\leq n+m} a_i Y_{n+i},$$

and (2) means

$$E\left\| X_n - f_m\left(Y_{n-m}, ...Y_{n+m}\right)\right\|^p = E\left\|\sum_{|i|>m} a_i Y_{n+i}\right\|^p \le \alpha_n(m).$$

The following is our main result

Theorem 3. Let $\{Y_n, n\in Z\}$be a sequence of independent random variables with values in Rademacher type p $(1<p\le 2)$ Banach space B satisfying (1), (2) and $EX_k=0, \quad E\left\|X_k\right\|^p<\infty\ , k=1,2,...,$

$$\max_i \alpha_i\left(m\right)\le \frac{C}{m^\gamma}, \quad for\ \ some\ \ C>0, \quad \gamma>1\ \ .$$

Then as $n\to\infty$,

$$\frac{1}{n}\sum_{i=1}^{n} X_i \to 0, \ \ \text{a.s.}$$

Note that we can consider in (1) one sided sequences with corresponding changes in theorem.

Proof of Theorem 3.

Denote $N_0 = N\cup\{0\}\,, \ \ S_{a,b}=\sum_{k=a}^{a+b-1} X_k\ , \quad a,b\in N.$

In the proof we will use the following results.

Theorem 4 ([5]). Let $1<p<\infty$. Assume that $\{X_n, n\in N_0\}$ is a sequence of random variables with values in a separable Banach space B satisfying the following conditions:

$$EX_i=0, \quad E\left\|X_i\right\|^p<\infty, \quad i\in N_0,$$

$$\sum_{n=0}^{\infty}\sup_{k\in N_0} E\left\|\frac{S_{k,2^n}}{2^n}\right\|^p<\infty.$$

Then the sequence $\{X_n, n\in N_0\}$ satisfies strong law of large numbers.

Theorem 5 ([17]). Let $\{X_n, n\in Z\}$ be a sequence of random variables with values in a separable Rademacher type p $(1<p\le 2)$ Banach space B satisfying conditions (1) and (2). Then there exists a constant $C_1\left(B,p\right)$ such that

$$E\left\|\sum_{i=1}^{n} X_i\right\|^p \le C_1\left(B,p\right)\left(\left(2m+1\right)^{p-1}\sum_{i=1}^{n} E\left\|X_i^m\right\|^p + n^{p-1}\sum_{i=1}^{n}\alpha_i\left(m\right)\right),$$

for all $n\ge 4m+2$.

By C we denote a constant which might be different even in one chain of inequalities.

Denote $X_i^m = f_m\left(Y_{-m}, ..., Y_m\right)$, $S_n=\sum_{i=1}^{n} X_i^m + \sum_{i=1}^{n}\left(X_i - X_i^m\right)$.

Using Theorem 5 we have

$$\sum_{n=0}^{\infty}\sup_{k\ge 0}E\left\|\tfrac{1}{2^n}\sum_{i=k}^{k+2^n-1}X_i\right\|^p\le$$
$$\sum_{n=0}^{\infty}\tfrac{1}{2^{np}}\sup_{k\ge 0}E\left\|\sum_{i=k}^{k+2^n-1}X_i\right\|^p\le$$
$$\sum_{n=0}^{\infty}\tfrac{1}{2^{np}}\sup_{k\ge 0}\left(C(B,p)\left(2m+1\right)^{p-1}\sum_{i=k}^{k+2^n-1}E\left\|X_i^m\right\|^p + 2^{n(p-1)}\sum_{i=k}^{k+2^n-1}\alpha_i\left(m\right)\right)$$

Now using conditions of the theorem we get

$$\sum_{n=0}^{\infty}\tfrac{C}{2^{np}}\sup_{k\ge 0}\left(\left(2m+1\right)^{p-1}\sum_{i=k}^{k+2^n-1}E\left\|X_i^m\right\|^p\right)\le\sum_{n=0}^{\infty}\tfrac{C}{2^{np}}\left(\ \left(2m+1\right)^{p-1}2^n\right)\le C\sum_{n=0}^{\infty}\tfrac{(2m+1)^{p-1}}{2^{np-1}} \tag{3}$$

and

$$\sum_{n=0}^{\infty} \frac{C}{2^{np}} \sup_{k\geq 0} \left(2^{n(p-1)} \sum_{i=k}^{k+2^n-1} \alpha_i(m)\right) \leq \sum_{n=0}^{\infty} \frac{C}{2^{np}} \left(2^{n(p-1)} \sum_{i=k}^{k+2^n-1} \max \alpha_i(m)\right) \leq \sum_{n=0}^{\infty} \frac{C}{m^{\gamma}} \qquad (4)$$

We take $m = m(n) = n^{\frac{\beta}{\gamma}},\ \ 1 < \beta < \gamma,\ for\ n \geq 1\ and\ m(0) = 1$. Then series in (3) and (4) converges. Now Theorem 4 implies Theorem 3. The proof is complete.

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REZYUME

Rademaxer p tipli Banach fazolarida qiymat qabul qiluvchi tasodifiy miqdorlar ketma-ketligining funksionallarini ko‘rib chiqamiz. Ba’zi qo‘shimcha shartlar ostida biz bunday funksionallar ketma-ketligi uchun kuchaytirilgan katta sonlar qonunini isbotlaymiz.

Kalit so‘zlar: Banax fazosi, tasodifiy miqdor, funksional tasodifiy miqdor, kuchaytirilgan katta sonlar qonuni.

РЕЗЮМЕ

Рассматриваются функционалы от последовательностей независимых случайных величин со значениями в банаховых пространствах Радемахера типа p. При некоторых дополнительных условиях доказывается усиленный закон больших чисел для последовательностей таких функционалов.

Ключевые слова: Банахово пространство, функционал последовательности случайных величин, усиленный закон больших чисел.