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Capacity of the α-Brjuno-Rüssmann set

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RESUME

In this work, we study generalized Brjuno-Rüssmann condition and prove new capacity estimate for the sets of parameters of this condition. In particular, we show that the complement of parameters of α-Brjuno-Ruüssmann condition has zero logarithmic capacity.

Key words: α-Brjuno-Russmann condition, Brjuno condition, C_σ-capacity, h-Hausdorff measure.

1. Introduction

In dynamical systems and Hamiltonian mechanics, main difficulty in providing the persistence of invariant tori comes from small divisors. The classic way to control them is the Diophantine condition on the frequency vectors. However, A. Brjuno (see [1]) and later H. Rüssmann (see [11]) introduced a weaker arithmetical condition that still allows KAM-type results.

The α-Bruno-Russmann condition was introduced by A. Bounemoura and J. Féjoz (see [7]) and important in the study quasi-periodic Hamiltonian systems, particularly relevant when analyzing persistence of invariant tori under perturbation.

Consider a vector $\omega \in \mathbb{R}^n$ and define the function $\Psi_\omega : [1, +\infty) \to [\Psi_\omega(1), +\infty]$ measures the size of the so-called small denominators

$$\Psi_\omega(Q) = \max\{|k\omega|^{-1} : k \in \mathbb{Z}^n, 0 < |k| < Q\}.$$

where $k\omega = k_1\omega_1 + k_2\omega_2 + ... + k_n\omega_n$ and $|k| = k_1 + k_2 + \cdots + k_n$.

Now assume that ω is non-resonance that is $|k\omega| \neq 0$ for any non zero $k \in \mathbb{Z}^n$. The function Ψ_ω is non-decreasing, piecewise constant, and has a countable number of discontinuities (see [7]). Then, α-Brjuno-Rüssmann condition defined as follows:

Definition 1.1. (see [7]) A vector ω is said to satisfy α-Brjuno-Rüssmann condition if

$$\int_1^{+\infty} \frac{\ln \Psi_\omega(Q)}{Q^{1+\frac{1}{\alpha}}} dQ < \infty,\ \alpha \geq 1. \tag{1}$$

This condition prevents Ψ_ω from growing too fast at large Q. The set of such vectors is denote by BR_α. This set becomes smaller as α increases and when $\alpha = 1$ the condition reduces to the classic Brjuno-Rüssmann condition.

In this area significant results obtained by H. Rüssmann, C. Chavaudret, S. Marmi, S. Fischler, A. Bounemoura and J. Féjoz (see [11], [7-9]) and others.

Now, let ν be an irrational number. Denote $v_0 = [\nu]$ the whole part of ν and define

$$\nu_0 = \nu - v_0,\ \nu_{k+1} = \frac{1}{\nu_k} - \left[\frac{1}{\nu_k}\right],\ v_{k+1} = \left[\frac{1}{\nu_k}\right],\ k \geq 0$$

Since ν is irrational, it can be written as the continued fraction

$$\nu = v_0 + \cfrac{1}{v_1 + \cfrac{1}{v_2 + ... \cfrac{1}{v_n + ...}}} = [v_0; v_1, v_2, ..., v_n, ...].$$

The finite part of this continued fraction becomes a rational number

$$[v_0; v_1, v_2, ..., v_n] = \frac{P_n}{Q_n}. \tag{2}$$

Fix $\beta, \gamma \in \mathbb{R}^+$ and define the set

$$\mathcal{A}(\beta,\gamma) = \left\{ \nu \in \mathbb{R} : \sum_{n=1}^{\infty} \frac{\ln^{\beta} Q_{n+1}}{Q_n^{\gamma}} < +\infty \right\}. \tag{3}$$

This set contracts as β increases or γ decreases. Conversely, as β decreases or γ increases, the set $\mathcal{A}(\beta,\gamma)$ expands. When $\beta = 1$ and $\gamma = 1$, the condition reduces to the classic Brjuno condition in $\mathbb{C}$ (see [1]).

In earlier work, A. Sadullaev and K. Rakhimov (see [2]) studied capacity dimension of the Brjuno set in $\mathbb{C}$. Definition of C_σ-capacity provided in Section 2. Our main results generalize their work by considering the condition (3). The relationship between the set $\mathcal{A}(\beta,\gamma)$ and the set BR_α established in Section 2.

Theorem 1.2. *Let $\beta > 0$ and $\gamma > 0$. The complement $\mathbb{C}^n \setminus \mathcal{A}(\beta,\gamma)$ has zero C_σ-capacity with respect to the logarithmic kernel*

$$k_1(z,\xi) = |\ln|z-\xi||^{\sigma},$$

where $\sigma > \max\left\{\frac{2\beta}{\gamma}, \beta\right\}$. In particular, it has zero h-Hausdorff measure with respect to the function $h(t) = |\ln t|^{-\delta}$ for any $\delta > \max\left\{\frac{2\beta}{\gamma}, \beta\right\}$.

Special case of Theorem 1.2 when $\beta = 1$ and $\gamma = 1 + \frac{1}{\alpha}$ for $\alpha \geq 1$ is given by the following corollary.

Corollary 1.3. *The complement of the α-Brjuno-Rüssmann set has zero C_σ-capacity with respect to the kernel*

$$k_\sigma(z,\xi) = |\ln|z-\xi||^{\frac{2\alpha}{\alpha+1}}$$

for any $\alpha \geq 1$. In particular, it has zero h-Hausdorff measure with respect to the function $h_\delta(t) = |\ln t|^{-\delta}$ for any $\delta > \frac{2\alpha}{\alpha+1}$.

2. Preliminary

2.1. Some properties of the continued fraction. Let ν be an irrational number and $\frac{P_n}{Q_n}$ be as in (2). We will use the following properties of the sequence of fraction $\frac{P_n}{Q_n}$ for any $n \geq 1$ (see [6]).

1. Denominator Q_n satisfies

$$Q_n > \frac{1}{2} \cdot \left(\frac{\sqrt{5}+1}{2}\right)^{n-1}.$$

2. We have classic two-sided inequality:

$$\frac{1}{2Q_nQ_{n+1}} < \left|\nu - \frac{P_n}{Q_n}\right| < \frac{1}{Q_nQ_{n+1}}.$$

3. For any $0 < Q < Q_n$, $P \in \mathbb{Z}$, $(P,Q) \neq (P_n,Q_n)$, we have

$$|Q\nu - P| > |Q_n\nu - P_n|.$$

2.2. Connection between BR_α and $\mathcal{A}(\beta,\gamma)$. We now prove the following proposition concerning the complement of the α-Brjuno Rüssmann set and the complement of the $\mathcal{A}(\beta,\gamma)$ set.

Proposition 2.1. *Let $\omega = (1,\nu) \in \mathbb{R}^2$ non-resonant. If*

$$\sum_{n=1}^{\infty} \frac{\ln^{\beta} Q_{n+1}}{Q_n^{\gamma}} = +\infty,$$

for $\beta=1$, $\gamma=1+\frac{1}{\alpha}$ *and* $\alpha\geq 1$, *then*

$$\int_1^{+\infty}\frac{\ln\Psi_\omega(Q)}{Q^{1+\frac{1}{\alpha}}}dQ=+\infty.$$

Proof. Equivalence of α-Brjuno-Rüssmann condition and

$$\sum_{Q=1}^{\infty}\frac{\ln\Psi_\omega(Q)}{Q^{1+\frac{1}{\alpha}}}<+\infty$$

is established in [7]. By Property 2 and 3 of the continued fractions, we have

$$\sum_{Q=1}^{\infty}\frac{\ln\Psi_\omega(Q)}{Q^{1+\frac{1}{\alpha}}}=\sum_{Q=1}^{\infty}\frac{\ln|Q_n\nu-P_n|^{-1}}{Q^{1+\frac{1}{\alpha}}}\geq\sum_{Q\neq Q_n}\frac{\ln Q}{Q^{1+\frac{1}{\alpha}}}+\sum_{n=1,\ Q=Q_n}^{\infty}\frac{\ln Q_{n+1}}{Q_n^{1+\frac{1}{\alpha}}}\geq\sum_{n=1}^{\infty}\frac{\ln Q_{n+1}}{Q_n^{1+\frac{1}{\alpha}}}.$$

□

2.3. C_σ- capacity. Capacity is one of the main concepts in potential theory. The capacity of the Brjuno set is the one of the connection points between potential theory and complex dynamical systems (see [2]).

Let $K\subset\{|z|<1\}\subset\mathbb{C}$ be a compact set and consider the kernel $k_\sigma(z,\xi)=|\ln|z-\xi||^\sigma$ with $\sigma>0$. Then, the potential $U^\mu(z)$ for a positive Borel probability measure $\mu\in\mathring{M}_K^+$ at a point $z\in\mathbb{C}$ is defined as

$$U^\mu(z)=\int_K k_\sigma(z,\xi)d\mu(\xi).$$

Let

$$I(\mu)=\int_K U^\mu(z)d\mu(z)=\iint_{K\times K}k_\sigma(z,\xi)\,d\mu(z)\,d\mu(\xi)$$

and $W_\sigma(K)=\inf\{I(\mu):\mu\in\mathring{M}_K^+\}$. Then, C_σ-capacity of the set K is defined as

$$C_\sigma(K)=\frac{1}{W_\sigma(K)}=\left(\inf_{\mu\in\mathring{M}_K^+}\int_K U^\mu(z)d\mu(z)\right)^{-1}=\left(\inf_{\mu\in\mathring{M}_K^+}\iint_{K\times K}k_\sigma(z,\xi)\,d\mu(z)\,d\mu(\xi)\right)^{-1}.$$

The outer and inner C_σ-capacities defined as

$$\underline{C}_\sigma=\sup\{C_\sigma(K):K\subset E,\ K-\text{compact}\}$$
$$\overline{C}_\sigma=\inf\{\underline{C}_\sigma(U):E\supset U,\ U-\text{open}\}$$

We note the following properties of the C_σ-capacity (see [4-5]).

1. For any Borel set $E\subset\mathbb{C}$, the outer and inner C_σ-capacities coincides:
$$\overline{C}_\sigma(E)=\underline{C}_\sigma(E)=C_\sigma(E).$$
2. The capacity of a Borel set is zero $C_\sigma(E)=0$, if and only if there exists a finite Borel measure μ supported on E such that its potential satisfies $U^\mu(z)\equiv+\infty$ for all $z\in E$.
3. If $C_\sigma(E)=0$, then the h_δ-Hausdorff measure of E with gauge function $h(t)=\log^{-\delta}\frac{1}{t}$ is zero for any $\delta>\sigma$ (see [3]).
4. For any sequence of compact sets K_j,
$$\overline{C}_\sigma\left(\bigcup_{j=1}^{\infty}K_j\right)\leq\sum_{j=1}^{\infty}C_\sigma(K_j).$$

2.4. h-Hausdorff measure. Let $h:[0,r_0]\to[0,+\infty)$ be a gauge function - continuous, strictly increasing function with $h(0)=0$ and $r_0>0$. For a bounded subset $E\subset\mathbb{R}^n$ and any $0<\varepsilon<r_0$, consider a finite covering of E by open balls $B_j(x_j,r_j)$ with center at x_j and radius r_j such that $r_j<\varepsilon$. Define

$$H^h(E,\varepsilon)=\inf\left\{\sum_{j=1}^m h(r_j):\ \bigcup_{j=1}^m B_j\supset E,\quad r_j<\varepsilon\right\}$$

for all $1\le j\le m$, where m depends on the chosen cover. It is obvious that, $H^h(E,\varepsilon)$ non-decreasing as ε decreases. Hence, the limit

$$H^h(E)=\lim_{\varepsilon\to 0+}H^h(E,\varepsilon)$$

exists and defines the *h-Hausdorff measure* of E. When $h(t)=t^\delta$ for $\delta>0$, it is called the classic δ-dimensional Hausdorff measure of E.

3. Capacity of the complement of the set $\mathcal{A}(\beta,\gamma)$.

Firstly, we need the following lemma.

Lemma 3.1. *If*

$$\sum_{n=1}^{\infty}\frac{\ln^\beta Q_{n+1}}{Q_n^\gamma}=+\infty$$

where $\beta>0$ and $\gamma\le 2$. Then for any $\varepsilon>0$ we have

$$\sum_{n=1}^{\infty}\frac{\ln^{\beta\frac{2+\varepsilon}{\gamma}} Q_{n+1}}{Q_n^{2+\frac{\varepsilon}{4}}}=+\infty.$$

Proof. Let $\delta>0$ be so small that $(1-\delta)(2+\varepsilon)\ge 2+\dfrac{\varepsilon}{4}$. Then, by using the Hölder inequality, we have

$$\sum_{n=1}^{\infty}\frac{\ln^\beta Q_{n+1}}{Q_n^\gamma}\le\left(\sum_{n=1}^{\infty}\frac{\ln^{\beta\frac{2+\varepsilon}{\gamma}} Q_{n+1}}{Q_n^{(1-\delta)(2+\varepsilon)}}\right)^{\frac{\gamma}{2+\varepsilon}}\left(\sum_{n=1}^{\infty}\frac{1}{Q_n^{\frac{\delta\gamma(2+\varepsilon)}{2+\varepsilon-\gamma}}}\right)^{\frac{2+\varepsilon-\gamma}{2+\varepsilon}}. \tag{4}$$

By the property 1 of the continues fractions, it is obvious that the second series on the right-hand side converges. Since, $(1-\delta)(2+\varepsilon)\ge 2+\dfrac{\varepsilon}{4}$, we have

$$\sum_{n=1}^{\infty}\frac{\ln^{\beta\frac{2+\varepsilon}{\gamma}} Q_{n+1}}{Q_n^{(1-\delta)(2+\varepsilon)}}<\sum_{n=1}^{\infty}\frac{\ln^{\beta\frac{2+\varepsilon}{\gamma}} Q_{n+1}}{Q_n^{2+\frac{\varepsilon}{4}}}.$$

Therefore, by inequality (4), we have

$$\sum_{n=1}^{\infty}\frac{\ln^{\beta\frac{2+\varepsilon}{\gamma}} Q_{n+1}}{Q_n^{2+\frac{\varepsilon}{4}}}=+\infty.$$

Hence, by the property 2 of the continued fractions

$$\sum_{n=1}^{\infty}\frac{\left|\ln^{\beta\frac{2+\varepsilon}{\gamma}}\left|\nu-\frac{P_n}{Q_n}\right|\right|}{Q_n^{2+\frac{\varepsilon}{4}}}\ge\sum_{n=1}^{\infty}\frac{\ln^{\beta\frac{2+\varepsilon}{\gamma}} Q_nQ_{n+1}}{Q_n^{2+\frac{\varepsilon}{4}}}>\sum_{n=1}^{\infty}\frac{\ln^{\beta\frac{2+\varepsilon}{\gamma}} Q_{n+1}}{Q_n^{2+\frac{\varepsilon}{4}}}=+\infty.$$

□

Now, we going to prove our main results.

Proof. (Proof of Theorem 1.2) To prove the main result by the second property of C_σ capacity, it is enough to show that $U^\mu(\nu) = +\infty$ for any compact $K \subset \mathbb{R} \setminus \mathcal{A}(\beta, \gamma)$ where $\beta > 0$, $\gamma > 0$. For this we use the following Borel measure

$$\mu = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} \frac{\delta_{\frac{p}{q}}}{q^{2+\frac{\varepsilon}{4}}}, \ \varepsilon > 0 \tag{5}$$

where $\delta_{\frac{p}{q}}$ is a Dirac measure supported at the point $\frac{p}{q}$. This measure is finite.

If $\gamma \leq 2$, the potential of μ with respect to the kernel $k_1(z, \xi) = |\ln|z - \xi||^{\frac{2\beta}{\gamma}}$ is

$$U_1^\mu(z) = \int k_1(z, \xi) d\mu(\xi) = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} \frac{1}{q^{2+\frac{\varepsilon}{4}}} \left| \ln \left| z - \frac{p}{q} \right| \right|^{\beta \frac{2+\varepsilon}{\gamma}}.$$

We need to show that $U_1^\mu(z) = +\infty$ for any $z \in K$. Indeed, if $z \in K$ is rational number, it is clean. In the case, when z is irrational number ν we will use the following

$$U_1^\mu(\nu) = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} \frac{1}{q^{2+\frac{\varepsilon}{4}}} \left| \ln \left| \nu - \frac{p}{q} \right| \right|^{\beta \frac{2+\varepsilon}{\gamma}} = \sum_{n=1}^{\infty} \frac{\left| \ln \left| \nu - \frac{P_n}{Q_n} \right| \right|^{\beta \frac{2+\varepsilon}{\gamma}}}{Q_n^{2+\frac{\varepsilon}{4}}} +$$
$$\sum_{n=1}^{\infty} \sum_{p \neq P_n, \ p=1}^{Q_n - 1} \frac{1}{Q_n^{2+\frac{\varepsilon}{4}}} \left| \ln \left| \nu - \frac{p}{Q_n} \right| \right|^{\beta \frac{2+\varepsilon}{\gamma}} + \sum_{q=1}^{\infty} \sum_{p \neq P_n, \ p=1}^{q-1} \frac{1}{q^{2+\frac{\varepsilon}{4}}} \left| \ln \left| \nu - \frac{p}{q} \right| \right|^{\beta \frac{2+\varepsilon}{\gamma}}$$
$$\geq \sum_{n=1}^{\infty} \frac{\left| \ln \left| \nu - \frac{P_n}{Q_n} \right| \right|^{\beta \frac{2+\varepsilon}{\gamma}}}{Q_n^{2+\frac{\varepsilon}{4}}}.$$

Therefore, by lemma 3.1 we can conclude that

$$U_1^\mu(\nu) = +\infty.$$

If $\gamma > 2$, we will use the same measure (5) and the kernel

$$k_2(z, \xi) = |\ln|z - \xi||^\beta.$$

Then the potential is given by

$$U_2^\mu(\nu) = \sum_{q=1}^{\infty} \sum_{p=1}^{q-1} \frac{1}{q^{2+\frac{\varepsilon}{4}}} \left| ln \left| \nu - \frac{p}{q} \right| \right|^{\beta} \geq \sum_{n=1}^{\infty} \frac{\left| \ln \left| \nu - \frac{P_n}{Q_n} \right| \right|^{\beta}}{Q_n^{2+\frac{\varepsilon}{4}}}.$$

As in the previous proof, using property 2 of continued fraction, we obtain

$$\sum_{n=1}^{\infty} \frac{\left| \ln \left| \nu - \frac{P_n}{Q_n} \right| \right|^{\beta}}{Q_n^{2+\frac{\varepsilon}{4}}} \geq \sum_{n=1}^{\infty} \frac{\ln^\beta Q_{n+1}}{Q_n^{2+\frac{\varepsilon}{4}}} \geq \sum_{n=1}^{\infty} \frac{\ln^\beta Q_{n+1}}{Q_n^{\gamma}} = +\infty.$$

Therefore, $C_\sigma(K) = 0$ for the set of real numbers that does not satisfy $\mathcal{A}(\beta, \gamma)$ condition for any $\beta > 0$ and $\gamma > 0$, so $C_\sigma(\mathbb{R} \setminus \mathcal{A}(\beta, \gamma)) = 0$. The last assertion follows from property 4 of C_σ-capacity. □

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REZYUME

Ushbu ishda umumlashtirilgan Brjuno-Ryusman sharti o‘rganilib, ushbu shart parametrlari to‘plamlari uchun yangi sig‘im bahosi isbotlanadi. Xususan, α-Brjuno–Ryusman sharti parametrlari to‘plamining to‘ldiruvchisi nol logarifmik sig‘imga ega ekani ko‘rsatiladi.

Kalit so‘zlar: α-Bryuno-Ryusman sharti, Bryuno sharti, C_σ-sig‘im, h-Hausdorff o‘lchovi.

РЕЗЮМЕ

В данной работе исследуется обобщённое условие Брюно-Рюссмана и доказывается новая оценка ёмкости для множеств параметров этого условия. В частности, показано, что дополнение множества параметров, удовлетворяющих условию α-Брюно-Рюссмана, имеет нулевую логарифмическую ёмкость.

Ключевые слова: α-условие Брюно-Рюссмана, условие Брюно, C_σ-ёмкость, h-мера Хаусдорфа.