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Criterion of completeness for two-sided ideals of compact operators

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RESUME

In this paper established a sufficient condition for a two-sided ideal of compact operators to form a Banach ideal. We prove that any proper, commutatively complete ideal $X \subseteq \mathcal{K}(\mathcal{H})$, which equipped with a monotone norm $\|\cdot\|_X$, is a Banach ideal.

Key words: non-increasing rearrangement, compact operator, Calkin's correspondence, Banach ideals of compact operators.

Preliminaries and main result. Let l^∞ (respectively, c_0) be the Banach space of bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^\infty$ of complex numbers equipped with the uniform norm $\|\{\xi_n\}\|_\infty = \sup\limits_{n\in\mathbb{N}} |\xi_n|$, where $\mathbb{N}$ is the set of natural numbers.

In l^∞ we consider the natural partial order

$$\{\xi_n\} \leq \{\eta_n\} \iff \xi_n \leq \eta_n \quad \text{for all} \quad n \in \mathbb{N}.$$

If $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$, then the *non-increasing rearrangement* $\xi^* : (0,\infty) \to (0,\infty)$ of ξ is defined by

$$\xi^*(t) = \inf\{\lambda : \mu\{|\xi| > \lambda\} \leq t\}, \quad t > 0,$$

(see, for example, [2, Ch. 2, Definition 1.5]). As such, the non-increasing rearrangement of a sequence $\{\xi_n\}_{n=1}^\infty \in l^\infty$ can be identified with the sequence $\xi^* = \{\xi_n^*\}_{n=1}^\infty$, where

$$\xi_n^* = \inf\left\{\sup_{n\notin F} |\xi_n| : F \subset \mathbb{N},\ |F| < n\right\}.$$

If $\{\xi_n\} \in c_0$, then $\xi_n^* \downarrow 0$; in this case there exists a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $|\xi_{\pi(n)}| = \xi_n^*$, $n \in \mathbb{N}$.

Hardy-Littlewood-Polya partial order in the space l^∞ is defined as follows:

$$\xi = \{\xi_n\} \prec\prec \eta = \{\eta_n\} \iff \sum_{n=1}^m \xi_n^* \leq \sum_{n=1}^m \eta_n^* \quad \text{for all} \quad m \in \mathbb{N}.$$

A non-zero linear subspace $E \subset l^\infty$ with a Banach norm $\|\cdot\|_E$ is called a *symmetric* (*fully symmetric*) sequence space if

$$\eta \in E,\ \xi \in l^\infty,\ \xi^* \leq \eta^* \ (\text{resp., } \xi^* \prec\prec \eta^*) \implies \xi \in E \ \text{ and } \ \|\xi\|_E \leq \|\eta\|_E.$$

Every fully symmetric sequence space is a symmetric sequence space. The converse is not true in general. At the same time, any separable symmetric sequence space is a fully symmetric space.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space, then

$$\|\xi\|_E = \|\,|\xi|\,\|_E = \|\xi^*\|_E \quad \text{for all} \ \ \xi \in E.$$

Immediate examples of fully symmetric sequence spaces are $(l^\infty, \|\cdot\|_\infty)$, $(c_0, \|\cdot\|_\infty)$ and the Banach spaces

$$l^p = \left\{\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty :\ \|\xi\|_p = \left(\sum_{n=1}^\infty |\xi_n|^p\right)^{1/p} < \infty\right\},\ 1 \leq p < \infty.$$

For any symmetric sequence space $(E, \|\cdot\|_E)$ the following continuous embeddings hold [2, Ch. 2, § 6, Theorem 6.6]: $(l^1, \|\cdot\|_1) \subset (E, \|\cdot\|_E) \subset (l^\infty, \|\cdot\|_\infty)$. Besides, $\|\xi\|_E \leq \|\xi\|_1$ for all $\xi \in l^1$ and $\|\xi\|_\infty \leq \|\xi\|_E$ for all $\xi \in E$.

If there is $\xi \in E \setminus c_0$, then $\xi^* \geq \alpha \mathbf{1}$ for some $\alpha > 0$, where $\mathbf{1} = \{1, 1, ...\}$. Consequently, $\mathbf{1} \in E$ and $E = l^\infty$. Therefore, either $E \subset c_0$ or $E = l^\infty$.

Let now $(\mathcal{H}, (\cdot, \cdot))$ be an infinite-dimensional separabel Hilbert space over $\mathbb{C}$, and let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the $C^\star$–algebra of all bounded linear operators in $\mathcal{H}$. Denote by $\mathcal{K}(\mathcal{H})$ ($\mathcal{F}(\mathcal{H})$) the two-sided ideal of compact (respectively, finite rank) linear operators in $\mathcal{B}(\mathcal{H})$.

Denote $\mathcal{B}_h(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : x = x^*\}$, $\mathcal{B}_+(\mathcal{H}) = \{x \in \mathcal{B}_h(\mathcal{H}) : x \geq 0\}$, and let $\tau : \mathcal{B}_+(\mathcal{H}) \to [0, \infty]$ be the *canonical trace* on $\mathcal{B}(\mathcal{H})$, that is,

$$\tau(x) = \sum_{j \in J} (x\varphi_j, \varphi_j), \quad x \in \mathcal{B}_+(\mathcal{H}),$$

where $\{\varphi_j\}_{j \in J}$ is an orthonormal basis in $\mathcal{H}$ (see, for example, [6, Ch. 7, E. 7.5]).

Let $\mathcal{P}(\mathcal{H}) = \{e \in \mathcal{B}(\mathcal{H}) : e = e^2 = e^*\}$ be the lattice of projectors in $\mathcal{B}(\mathcal{H})$. If $\mathbf{1}$ is the identity of $\mathcal{B}(\mathcal{H})$ and $e \in \mathcal{P}(\mathcal{H})$, we will write $e^\perp = \mathbf{1} - e$.

Let $x \in \mathcal{B}(\mathcal{H})$, and let $\{e_\lambda(|x|)\}_{\lambda \geq 0}$ be the spectral family of projections for the absolute value $|x| = (x^*x)^{1/2}$ of x, that is, $e_\lambda(|x|) = \{|x| \leq \lambda\}$. If $t > 0$, then the *t-th generalized singular number* of x, or the *non-increasing rearrangement* of x, is defined as

$$\mu_t(x) = \inf\{\lambda > 0 : \ \tau(e_\lambda(|x|)^\perp) \leq t\}$$

(see [3]).

A non-zero linear subspace $X \subset \mathcal{B}(\mathcal{H})$ with a Banach norm $\|\cdot\|_X$ is called *symmetric* (*fully symmetric*) if the conditions

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \mu_t(y) \leq \mu_t(x) \quad \text{for all} \quad t > 0$$

(respectively,

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \int_0^s \mu_t(y) dt \leq \int_0^s \mu_t(x) dt \quad \text{for all} \quad s > 0 \quad (\text{writing } y \prec\prec x))$$

imply that $y \in X$ and $\|y\|_X \leq \|x\|_X$.

The spaces $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ as well as the classical Banach two-sided ideals

$$\mathcal{C}^p = \{x \in \mathcal{K}(\mathcal{H}) : \ \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, \ 1 \leq p < \infty,$$

are examples of fully symmetric spaces.

It should be noted that for every symmetric space $(X, \|\cdot\|_X) \subset \mathcal{B}(\mathcal{H})$ and all $x \in X$, $a, b \in \mathcal{B}(\mathcal{H})$,

$$\|x\|_X = \| \, |x| \, \|_X = \|x^*\|_X, \quad axb \in X, \quad \text{and} \quad \|axb\|_X \leq \|a\|_\infty \|b\|_\infty \|x\|_X.$$

Remark 1. *If $X \subset \mathcal{B}(\mathcal{H})$ is a symmetric space and there exists a projection $e \in \mathcal{P}(\mathcal{H}) \cap X$ such that $\tau(e) = \infty$, that is, $\dim e(\mathcal{H}) = \infty$, then $\mu_t(e) = \mu_t(\mathbf{1}) = 1$ for every $t \in (0, \infty)$. Consequently, $\mathbf{1} \in X$ and $X = \mathcal{B}(\mathcal{H})$. If $X \neq \mathcal{B}(\mathcal{H})$ and $x \in X$, then $e_\lambda(|x|)^\perp = \{|x| > \lambda\}$ is a finite-dimensional projection, that is, $\dim e_\lambda(|x|)^\perp(\mathcal{H}) < \infty$ for all $\lambda > 0$. This means that $x \in \mathcal{K}(\mathcal{H})$, hence $X \subset \mathcal{K}(\mathcal{H})$. Therefore, either $X = \mathcal{B}(\mathcal{H})$ or $X \subset \mathcal{K}(\mathcal{H})$.*

Throughout this paper, we assume that for a symmetric space $(X, \|\cdot\|_X)$ the embedding $X \subset \mathcal{K}(\mathcal{H})$ holds, and we call the pair $(X, \|\cdot\|_X)$ a *Banach ideal of compact operators* (cf. [4, Ch. III]).

We say that the norm $\|\cdot\|_X$ of the Banach ideal $(X, \|\cdot\|_X)$ is *monotone* if for any two positive compact operators $A \geq 0$ and $B \geq 0$ with $A \leq B$, the inequality $\|A\|_X \leq \|B\|_X$ holds.

We say that $X \subset \mathcal{B}(\mathcal{H})$ is *commutatively complete* if every Cauchy sequence of pairwise commuting self-adjoint elements in X converges in X with respect to the operator norm (i.e., converges uniformly).

We now state the criterion of completeness for two-sided ideals of compact operators.

Theorem 1. Let X be a proper, commutatively complete two-sided ideal of compact operators in $\mathcal{B}(\mathcal{H})$, equipped with a monotone norm $\|\cdot\|_X$. Then $(X, \|\cdot\|_X)$ is a Banach ideal of compact operators.

Proof. To prove that $(X, \|\cdot\|_X)$ is a Banach ideal, we show that every Cauchy sequence in $(X, \|\cdot\|_X)$ converges to a limit within X with respect to the $\|\cdot\|_X$ norm.

Let $\{T_n\}_{n=1}^{\infty}$ be a Cauchy sequence in $(X, \|\cdot\|_X)$. Since $\|\cdot\|_X$ is a monotone norm on the ideal of compact operators X, it follows from [6, Theorem 2.7 (a)] that

$$\|A\|_\infty \leq \|A\|_X \quad \text{for all } A \in X.$$

Thus, $\{T_n\}$ is also a Cauchy sequence with respect to $\|\cdot\|_\infty$. Since the space of compact operators $K(\mathcal{H})$ is complete, there exists a unique compact operator $T \in \mathcal{K}(\mathcal{H})$ such that

$$\lim_{n\to\infty} \|T_n - T\|_\infty = 0.$$

We must show that $T \in X$. Consider the positive compact operator $|T| = (T^*T)^{1/2}$. By the Spectral Theorem (see, for example, [2]), $|T|$ admits a decomposition in terms of its singular values $s_k(T)$ and corresponding orthogonal rank-one projections $P_k \in \mathcal{F}(\mathcal{H})$:

$$|T| = \sum_{k=1}^{\infty} s_k(T) P_k.$$

Define the sequence of finite-rank partial sums A_m by

$$A_m = \sum_{k=1}^{m} s_k(T) P_k.$$

Since X contains all finite-rank operators, we have $A_m \in X$ for all $m \in \mathbb{N}$. The elements A_m are self-adjoint and pairwise commuting. Therefore, the sequence $\{A_m\}_{m=1}^{\infty}$ converges uniformly to $|T|$, i.e.,

$$\| \, |T| - A_m \, \|_\infty = s_{m+1}(T) \to 0.$$

Hence, $\{A_m\}$ is a Cauchy sequence in $\|\cdot\|_\infty$.

By the commutative completeness of X, it follows that $|T| \in X$.

Now, using the polar decomposition $T = U|T|$ (where $U \in \mathcal{B}(\mathcal{H})$ is a partial isometry) and the two-sided ideal property of X, we conclude that $T \in X$.

Next, we show that $\|T_n - T\|_X \to 0$. Let $\varepsilon > 0$. Since $\{T_n\}$ is Cauchy, there exists N such that for all $n, m \geq N$, $\|T_n - T_m\|_X < \varepsilon$.

A key property of monotone norms on compact operator ideals is lower semi-continuity with respect to the uniform operator topology, i.e., if $S_m \to S$ in $\|\cdot\|_\infty$, then

$$\|S\|_X \leq \liminf_{m\to\infty} \|S_m\|_X$$

(see [6, Theorem 2.7(d)]).

Fix $n \geq N$. The sequence $T_n - T_m$ converges uniformly to $T_n - T$ as $m \to \infty$, because

$$\|(T_n - T_m) - (T_n - T)\|_\infty = \|T - T_m\|_\infty \to 0.$$

Applying the lower semi-continuity property to the sequence $S_m = T_n - T_m$ we get

$$\|T_n - T\|_X \leq \liminf_{m\to\infty} \|T_n - T_m\|_X.$$

Since $\|T_n - T_m\|_X < \varepsilon$ for all $m \geq N$, we obtain

$$\|T_n - T\|_X \leq \varepsilon.$$

Since ε was arbitrary, it follows that

$$\lim_{n\to\infty} \|T_n - T\|_X = 0.$$

Therefore, $(X, \|\cdot\|_X)$ is a Banach ideal.

□

Application. If $x \in \mathcal{K}(\mathcal{H})$, then $|x| = \sum\limits_{n=1}^{m(x)} s_n(x)p_n$ (if $m(x) = \infty$, the series converges uniformly), where $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of singular values of x, that is, the set of eigenvalues of the compact operator $|x|$ in the decreasing order, and p_n is the projection onto the eigenspace corresponding to $s_n(x)$. Consequently, the non-increasing rearrangement $\mu_t(x)$ of $x \in \mathcal{K}(\mathcal{H})$ can be identified with the sequence $\{s_n(x)\}_{n=1}^{\infty}$, $s_n(x) \downarrow 0$ (if $m(x) < \infty$, we set $s_n(x) = 0$ for all $n > m(x)$).

Let $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ be a symmetric space. Fix an orthonormal basis $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathcal{H}$. Let p_n be the one-dimensional projection on the subspace $\mathbb{C} \cdot \varphi_n \subset \mathcal{H}$. It is clear that the set

$$E(X) = \left\{ \xi = \{\xi_n\}_{n=1}^{\infty} \in c_0 : \ x_\xi = \sum_{n=1}^{\infty} \xi_n p_n \in X \right\}$$

(the series converges uniformly), is a symmetric sequence space with respect to the norm $\|\xi\|_{E(X)} = \|x_\xi\|_X$ (see, for example, [1]). Therefore, each symmetric subspace $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ uniquely generates a symmetric sequence space $(E(X), \|\cdot\|_{E(X)}) \subset c_0$. The converse is also true: every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ uniquely generates a symmetric space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) \subset \mathcal{K}(\mathcal{H})$ by the following rule (see, for example, [4, Ch. 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E.$$

In addition,

$$E(\mathcal{C}_E) = E, \ \|\cdot\|_{E(\mathcal{C}_E)} = \|\cdot\|_E, \ \mathcal{C}_{E(\mathcal{C}_E)} = \mathcal{C}_E, \ \|\cdot\|_{\mathcal{C}_{E(\mathcal{C}_E)}} = \|\cdot\|_{\mathcal{C}_E}.$$

The construction described above is known as Calkin's construction.

From the Theorem 1 we obtain the following useful corollaries, which are represents a key component of Calkin's correspondence.

Corollary 1. Let $(E, \|\cdot\|_E)$ be a symmetric sequence space. Then the associated operator ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a Banach ideal of compact operators.

Corollary 2. Let $(E, \|\cdot\|_E) = (l^p, \|\cdot\|_p)$ be a classical symmetric sequence space. Then the associated operator ideal (Schatten's ideal) $(\mathcal{C}_{l^p}, \|\cdot\|_{\mathcal{C}_{l^p}}) = (\mathcal{C}^p, \|\cdot\|_p)$ is a Banach ideal of compact operators (see, for example, [1]).

REFERENCES

1. A. Azizov, *An abstract characterization of trace class ideal* $\mathcal{C}_1$, Acta NUUz (Exact sciences), No. 2/2 (2025), 23-28.

2. C. Bennett, R. Sharpley, *Interpolation of Operators*, Academic Press Inc., 1988.

3. T. Fack, H. Kosaki, *Generalized s-numbers of τ-measurable operators*, Pacific. J. Math., **123** (1986), 269–300.

4. I.C. Gohberg, M.G. Krein, *Introduction to the theory of linear nonselfadjoint operators*, Translations of Mathematical Monographs **18**, Amer. Math. Soc., Providence, RI 02904, 1969.

5. S. Lord, F. Sukochev, D. Zanin, *Singular Traces*, Walter de Gruyter GmbH, Berlin/Boston, 2013.

6. B. Simon, *Trace Ideals and Their Applications*, **120**, Second edition, Mathematical Surveys and Monographs, American Mathematical Society, Providence, RI, 2005.

REZYUME

Ushbu maqolada kompakt operatorlarning ikki tomonlama ideali Banax idealini hosil qilishi uchun yetarli shart o'rnatilgan. Biz monoton norma $\|\cdot\|_X$ bilan berilgan har qanday xos, kommutativ to'la $X \subset \mathcal{K}(\mathcal{H})$ ideal Banax ideali ekanligini isbotlaymiz.

Kalit so'zlar: o'smaydigan o'rin almashtirish, kompakt operator, Kalkin munosabati, kompakt operatorlar Banax ideali.

РЕЗЮМЕ

В этой статье приводится и доказывается достаточное условие для того, чтобы двусторонний идеал компактных операторов образовывал банахов идеал. Мы доказываем, что любой собственный, коммутативно полный идеал ($X \subseteq \mathcal{K}(\mathcal{H})$), оснащенный монотонной нормой $\|\cdot\|_X$, является банаховым идеалом.

Ключевые слова: невозрастающая перестановка, компактный оператор, соответствие Калкина, Банаховы идеалы компактных операторов.