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Some solvable compatible Lie extensions of a nilpotent compatible Lie algebra

Gaybullaev Rustamjon Kakhramonovich
National University of Uzbekistan named after M.Ulugbek, Tashkent, Uzbekistan
r.gaybullaev@nuu.uz
Solijanova Gulkhayo Oybek kizi
National University of Uzbekistan named after M.Ulugbek, Tashkent, Uzbekistan
gulhayo.solijonova@mail.ru

RESUME

Throughout this paper we construct some solvable compatible Lie algebras by including maximal nilpotent ideal $\mathcal{N}$ which its component Lie algebras are classical filiform Lie algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$.

Key words: compatible Lie algebras, derivations, nilpotency, solvability.

Introduction

Compatible algebraic structures arise when a vector space carries two product operations whose arbitrary linear combinations remain valid products of the same type. In recent years, such structures especially compatible Lie algebras have attracted considerable attention in mathematics and mathematical physics.

One of the central problems in the field is the classification of these algebras. A major step in this direction has been the classification of nilpotent compatible Lie algebras. This approach constructs nilpotent algebras as central extensions of smaller ones and has proved effective for obtaining complete classifications in low dimensions.

After understanding the nilpotent case, a natural continuation is the study of solvable compatible Lie algebras. In the classical theory of Lie algebras, solvable algebras are typically built by first determining their maximal nilpotent ideal the nilradical and then forming solvable extensions through non-nilpotent derivations. This strategy, often attributed to Mubarakzyanov, forms the basis for many classification results; see, for example, [1,2,6,9].

The aim of this paper is to adapt this construction method to compatible Lie algebras. However, several fundamental difficulties arise. For instance, in a compatible Lie algebra, the sum of two nilpotent ideals is not necessarily nilpotent, so the usual notion of a nilradical does not behave well.

We analyze the compatibility conditions for semidirect products and employ maximal tori of diagonal derivations to construct solvable extensions. Within this framework, we develop solvable compatible Lie algebras whose nilpotent ideal is a nilpotent compatible Lie algebra whose Lie components are classical filiform Lie algebras, such as $\mathcal{L}_n$, and $\mathcal{Q}_n$. In particular, we treat the compatible pairs $(\mathcal{L}_n, \mathcal{Q}_n)$ obtaining explicit one-dimensional solvable extensions by means of non-nilpotent derivations of $\mathcal{L}_n$ and proving the nonexistence of higher-dimensional ones.

Throughout the paper, all vector spaces and algebras are considered over the complex numbers. We assume that the reader is familiar with the basic concepts of Lie theory.

Preliminaries

In this section we define compatible Lie algebras and the basics about them. We let $\mathbb{K}$ be an arbitrary field of characteristic different from 2.

Proposition 1. [4] Let $\mathfrak{g}_1 = (\mathfrak{g}, [-,-]_1)$ and $\mathfrak{g}_2 = (\mathfrak{g}, [-,-]_2)$ be two Lie algebras over the same vector space $\mathfrak{g}$. Then the following conditions are equivalent:

(*i*) $(\mathfrak{g}, [-,-])$ is a Lie algebra, where $[x, y] = [x, y]_1 + [x, y]_2$ for all $x, y \in \mathfrak{g}$;
(*ii*) $(\mathfrak{g}, [-,-]_{\lambda_1,\lambda_2})$ is a Lie algebra for all $\lambda_1,\ \lambda_2 \in \mathbb{K}$, where

$$[x, y] = \lambda_1[x, y]_1 + \lambda_2[x, y]_2;$$

(*iii*) The following identity (named the mixed Jacobi identity) holds for all $x, y, z \in \mathfrak{g}$

$$[[x, y]_1, z]_2 + [[y, z]_1, x]_2 + [[z, x]_1, y]_2 + [[x, y]_2, z]_1 + [[y, z]_2, x]_1 + [[z, x]_2, y]_1 = 0.$$

Let denote the second cocycle identity by

$$Z(x, y, z) = [x, \varphi(y, z)] - [\varphi(x, y), z] + [\varphi(x, z), y] + \varphi(x, [y, z]) - \varphi([x, y], z) + \varphi([x, z], y).$$

Thus, we have the definition of compatible Lie algebras.

Definition 1. [4] A compatible Lie algebra is a triple $(\mathfrak{g}, [-,-]_1, [-,-]_2)$, where $\mathfrak{g}_1 = (\mathfrak{g}, [-,-]_1)$ and $\mathfrak{g}_2 = (\mathfrak{g}, [-,-]_2)$ are Lie algebras satisfying any of the three equivalent conditions in above Proposition.

A subalgebra of a compatible Lie algebra $\mathfrak{g}$ is a vector subspace of $\mathfrak{g}$ which is closed for both products. An ideal $\mathfrak{i}$ of a compatible Lie algebra $\mathfrak{g}$ is a vector subspace such that $[\mathfrak{i}, \mathfrak{g}]_1,\ [\mathfrak{i}, \mathfrak{g}]_2 \subseteq \mathfrak{i}$.

Definition 2. [4] Let $\mathfrak{s}$ and $\mathfrak{t}$ be two subspaces of a compatible Lie algebra $\mathfrak{g}$. The commutator $[\mathfrak{s}, \mathfrak{t}]$ of $\mathfrak{s}$ and $\mathfrak{t}$ is

$$[\mathfrak{s}, \mathfrak{t}] = [\mathfrak{s}, \mathfrak{t}]_1 + [\mathfrak{s}, \mathfrak{t}]_2 = span_{\mathbb{K}}\{[s, t]_1, [s, t]_2 \mid s \in \mathfrak{s}, t \in \mathfrak{t}\}$$

Remark that if $\mathfrak{i}$ and $\mathfrak{j}$ are ideals of $\mathfrak{g}$, then their commutator $[\mathfrak{i}, \mathfrak{j}]$ is a subalgebra and $[\mathfrak{g}, \mathfrak{i}]$ is an ideal.

In general, the *descending central sequence* or *lower central series* of a compatible Lie algebra $\mathfrak{g}$ is defined in the same way as for Lie algebras [4]:

$$\mathcal{C}^0(\mathfrak{g}) := \mathfrak{g}, \quad \mathcal{C}^{k+1}(\mathfrak{g}) := [\mathfrak{g}, \mathcal{C}^k(\mathfrak{g})], \quad \text{for all} \quad k \geq 0.$$

Consequently, if $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ for some k, then the compatible Lie algebra is called *nilpotent*. Then the smallest integer k such that $\mathcal{C}^k(\mathfrak{g}) = \{0\}$ is called the nilindex of the compatible Lie algebra $\mathfrak{g}$.

Applying induction, one can prove that $\mathcal{C}^i(\mathfrak{g})$ is an ideal of $\mathfrak{g}$ and $[\mathcal{C}^i(\mathfrak{g}), \mathcal{C}^j(\mathfrak{g})] \subseteq \mathcal{C}^{i+j}(\mathfrak{g})$ for any $i, j \in \mathbb{N}$.

Likewise, the *derived sequence* of $\mathfrak{g}$ can be defined as follows.

Definition 3. Given a compatible Lie algebra $\mathfrak{g}$ we define the derived sequence as

$$\mathcal{D}^0(\mathfrak{g}) := \mathfrak{g}, \quad \mathcal{D}^{k+1}(\mathfrak{g}) := [\mathcal{D}^k(\mathfrak{g}), \mathcal{D}^k(\mathfrak{g})] \ \text{ for all } \ k \geq 0$$

If this sequence is stabilized in zero, then the compatible algebra is said to be *solvable*.

In this paper, we construct solvable compatible Lie extensions of a compatible Lie algebras with two component Lie algebras $\mathcal{L}_n$ and $\mathcal{Q}_{2m}$ (where $n = 2m$) [3].

1. Let $\mathcal{L}_n$ be the n-dimensional Lie algebra defined by non-zero products

$$[e_1, e_i] = e_{i+1},\ 2 \leq i \leq n-1,$$

   where $\{e_1, \dots, e_n\}$ is a basis of $\mathcal{L}_n$.

2. Let $\mathcal{Q}_{2m}$ be the $2m$-dimensional nilpotent Lie algebra defined in the basis $\{e_1, \dots, e_{2m}\}$ by

$$\{e_1, e_i\} = e_{i+1}, \quad 2 \leq i \leq 2m-1, \quad \{e_j, e_{2m+1-j}\} = (-1)^j e_{2m}, \quad 2 \leq j \leq m.$$

A straightforward verification confirms that the algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$ satisfy the hypotheses of Proposition 1. Consequently, $\mathcal{L}_n$ and $\mathcal{Q}_n$ may be realised as the Lie algebra components of a compatible Lie algebra. We shall denote this compatible Lie algebra by $\mathcal{N}$.

Main part

In this section we construct solvable extensions of a nilpotent compatible Lie algebra with components $\mathcal{L}_\backslash$ and $\mathcal{Q}_\backslash$, where n is an even natural number. First of all we analyze derivations spaces of the algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$.

Proposition 2. Let $d_1 \in End(\mathcal{L}_n)$ and $d_2 \in End(\mathcal{Q}_n)$ be any derivations of the algebras $\mathcal{L}_n$ and $\mathcal{Q}_n$, respectively. Then the derivations admit the following explicit descriptions.

$$\begin{cases} d_1(e_1) = \sum\limits_{t=1}^{n} \alpha_{1,t}e_t, \\ d_1(e_i) = \big((i-2)\alpha_{1,1} + \alpha_{2,2}\big)e_i + \sum\limits_{t=3}^{n-i+2} \alpha_{2,t}\, e_{t+i-2}, \quad 2 \le i \le n; \end{cases}$$

and

$$\begin{cases} d_2(e_1) = \sum\limits_{t=1}^{2m} \beta_{1,t}e_t, \\ d_2(e_2) = (\beta_{1,1} + \beta_{1,2})e_2 + \sum\limits_{t=1}^{m-1} \beta_{2,2t+1}e_{2t+1} + \beta_{2,2m}e_{2m}, \\ d_2(e_{2j}) = ((2j-1)\beta_{1,1} + \beta_{1,2})e_{2j} + \sum\limits_{t=1}^{m-j} \beta_{2,2t+1}e_{2t+2j-1} + \beta_{1,2m-2j+1}e_{2m}, \\ d_2(e_{2j+1}) = (2j\beta_{1,1} + \beta_{1,2})e_{2j+1} + \sum\limits_{t=1}^{m-j-1} \beta_{2,2t+1}e_{2t+2j} + (-\beta_{1,2m-2j+1} + \beta_{2,2m-2j+1})e_{2m}, \\ d_2(e_{2m}) = ((2m-1)\beta_{1,1} + 2\beta_{1,2})e_{2m}, \end{cases}$$

where $1 \le j \le m-1$ and $n = 2m$. In particular, these formulas completely characterize all derivations of $\mathcal{L}_n$ and $\mathcal{Q}_n$.

Proof. Let take denotations as

$$d_1(e_i) = \sum_{t=1}^{n} \alpha_{i,t}e_t, \quad \text{(respectively,} \quad d_2(e_i) = \sum_{t=1}^{n} \beta_{i,t}e_t) \quad 1 \le i \le 2.$$

By straightforward computations, we get the restrictions for the parameters $\alpha_{i,t}$ (respectively,$\beta_{i,t}$) given in the proposition.

Next, we consider a nilpotent compatible Lie algebra with two component are $\mathcal{L}_n$ and $\mathcal{Q}_n$. Below we describe solvable extensions of $\mathcal{N}$ by means of non-nilpotent derivations of $\mathcal{L}_n$, $\mathcal{Q}_n$ and $\mathcal{N}$.

Let consider two-dimensional non-nilpotent solvable extensions of $\mathcal{L}_n$.

Theorem 1. There is no $(n+2)$-dimensional solvable compatible Lie extension of $\mathcal{N}$ by means of non-nilpotent derivations of $\mathcal{L}_n$.

Proof. Let $\mathcal{R}_1$ be a $(n+2)$-dimensional solvable Lie algebras with nilradical $\mathcal{L}_n$ and $\{e_1, \dots, e_n, x, y\}$ is a basis of $\mathcal{R}_1$. By using derivation properties we have the following products:

$$\begin{cases} [e_1, x] = \sum\limits_{t=1}^{n} \alpha_{1,t}e_t, \\ [e_i, x] = ((i-2)\alpha_{1,1} + \alpha_{2,2})e_i + \sum\limits_{t=i+1}^{n} \alpha_{2,t-i+2}e_t, \quad 2 \le i \le n, \\ [e_1, y] = \beta_{1,1}e_1 + \sum\limits_{t=2}^{n} \beta_{1,t}e_t, \\ [e_i, y] = ((i-2)\beta_{1,1} + \beta_{2,2})e_i + \sum\limits_{t=i+1}^{n} \beta_{2,t-i+2}e_t, \quad 2 \le i \le n, \end{cases}$$

where $\begin{vmatrix} \alpha_{1,1} & \alpha_{2,2} \\ \beta_{1,1} & \beta_{2,2} \end{vmatrix} \neq 0.$

Then by taking changing of basis we get

$$x' = (\alpha_{1,1}\beta_{2,2} - \alpha_{2,2}\beta_{1,1})^{-1}\big(\beta_{2,2}x - \alpha_{2,2}y\big), \quad y' = (\alpha_{1,1}\beta_{2,2} - \alpha_{2,2}\beta_{1,1})^{-1}\Big(\alpha_{1,1}y - \beta_{1,1}x\Big),$$

we can suppose:

$$\begin{cases} [e_1,x]=e_1+\sum\limits_{t=2}^{n}\alpha_{1,t}e_t, & [e_i,x]=(i-2)e_i+\sum\limits_{t=i+1}^{n}\alpha_{2,t-i+2}e_t,\\ [e_1,y]=\sum\limits_{t=2}^{n}\beta_{1,t}e_t, & [e_i,y]=e_i+\sum\limits_{t=i+1}^{n}\beta_{2,t-i+2}e_t, \end{cases}$$

where $2\le i\le n$. In addition, setting

$$x'=x+\alpha_{2,3}e_1-\sum_{t=2}^{n-1}\alpha_{1,t+1}e_t,\quad y'=y+\beta_{2,3}e_1-\sum_{t=2}^{n-1}\beta_{1,t+1}e_t,$$

one can assume

$$\begin{array}{lll} [e_1,x]=e_1+\alpha_{1,2}e_2, & [e_i,x]=(i-2)e_i+\sum\limits_{t=i+2}^{n}\alpha_{2,t-i+2}e_t, & 2\le i\le n,\\ [e_1,y]=\beta_{1,2}e_2, & [e_i,y]=e_i+\sum\limits_{t=i+2}^{n}\beta_{2,t-i+2}e_t, & 2\le i\le n. \end{array}$$

Set the following basis transformation:

$$e_1'=e_1,\quad e_i'=e_i+\sum_{t=4}^{n-i+2}\mu_te_{t+i-2},\quad 2\le i\le n,\quad \text{with}\quad \mu_t=\frac{1}{2-t}(\alpha_{2,t}+\sum_{p=4}^{t-2}\alpha_{2,p}\mu_{t-p+2}).$$

Thus we obtain $[e_i,x]=(i-2)e_i,\ 2\le i\le n$.

Note that under the above basis transformation the multiplications table of $\mathcal{Q}_n$ does not change.

Applying the Jacobi identity for the obtained products, we obtain

$$\mathcal{R}_1:\begin{cases} [e_1,x]=e_1+\alpha e_2, & [e_1,y]=-\alpha e_2,\\ [e_i,x]=(i-2)e_i, & [e_i,y]=e_i, \qquad 2\le i\le n. \end{cases}$$

For $\varphi\in Hom(\mathcal{R}_1\wedge\mathcal{R}_1,\mathcal{N})$ such that $\{e_i,e_j\}=\varphi(e_i,e_j)$ and $\varphi(e_i,y)=d_2(e_i)$, we verify the 2-cocycle property.

$$\begin{aligned} Z(e_2,e_{n-1},y) &= [e_2,\varphi(e_{n-1},y)]-[\varphi(e_2,e_{n-1}),y]+[\varphi(e_2,y),e_{n-1}]+\\ &+ \varphi(e_2,[e_{n-1},y])-\varphi([e_2,e_{n-1}],y)+\varphi([e_2,y],e_{n-1})=e_n\neq 0. \end{aligned}$$

This contradiction complete the proof of theorem.

We present the description of one-dimensional solvable extensions of $\mathcal{N}$ by means of non-nilpotent derivations of $\mathcal{L}_n$.

Theorem 2. Let $\mathcal{R}$ be a one-dimensional solvable extension of $\mathcal{N}$ by means of non-nilpotent derivations of $\mathcal{L}_n$. Then it admits a basis $\{e_1,\dots,e_n,x\}$ such that its table of multiplications in the basis has the following form:

$$\begin{cases} [e_1,x] &= \alpha_{1,1}e_1+\alpha_{1,2}e_2,\\ [e_i,x] &= ((i-2)\alpha_{1,1}+\alpha_{2,2})e_i+\sum\limits_{t=i+1}^{n}\alpha_{2,t-i+2}e_t,\\ \{e_1,x\} &= \beta_{1,1}e_1-\alpha_{1,2}e_2+\sum\limits_{t=3}^{2m}\beta_{1,t}e_t,\\ \{e_2,x\} &= (\beta_{1,1}-\alpha_{1,2})e_2+\beta_{2,2m}e_{2m},\\ \{e_{2j},x\} &= ((2j-1)\beta_{1,1}-\alpha_{1,2})e_{2j}+\beta_{1,2m-2j+1}e_{2m},\\ \{e_{2j+1},x\} &= (2j\beta_{1,1}-\alpha_{1,2})e_{2j+1}-\beta_{1,2m-2j+1}e_{2m},\\ \{e_{2m},x\} &= ((2m-1)\beta_{1,1}-2\alpha_{1,2})e_{2m}, \end{cases}$$

where $2 \leq i \leq n, \quad 2 \leq j \leq m-1$.

Proof. By using derivations properties we obtain the solvable extension of $\mathcal{L}_n$

$$\begin{cases} [e_1, x] = \alpha_{1,1}e_1 + \sum\limits_{t=2}^{n} \alpha_{1,t}e_t, \\ [e_i, x] = ((i-2)\alpha_{1,1} + \alpha_{2,2})e_i + \sum\limits_{t=i+1}^{n} \alpha_{2,t-i+2}e_t, \quad 2 \leq i \leq n. \end{cases}$$

Setting $x' = x + \alpha_{2,3}e_1 - \sum\limits_{i=3}^{n} \alpha_{1,t}e_{t-1}$, we get $[e_1, x] = \alpha_{1,1}e_1 + \alpha_{1,2}e_2$.

Applying the description of $Der(\mathcal{Q}_n)$ we derive the products

$$\begin{cases} \{e_1, x\} = \sum\limits_{t=1}^{2m} \beta_{1,t}e_t, \\ \{e_2, x\} = (\beta_{1,1} + \beta_{1,2})e_2 + \sum\limits_{t=1}^{m-1} \beta_{2,2t+1}e_{2t+1} + \beta_{2,2m}e_{2m}, \\ \{e_{2j}, x\} = ((2j-1)\beta_{1,1} + \beta_{1,2})e_{2j} + \sum\limits_{t=1}^{m-j} \beta_{2,2t+1}e_{2t+2j-1} + \beta_{1,2m-2j+1}e_{2m}, \\ \{e_{2j+1}, x\} = (2j\beta_{1,1} + \beta_{1,2})e_{2j+1} + \sum\limits_{t=1}^{m-j-1} \beta_{2,2t+1}e_{2t+2j} + (-\beta_{1,2m-2j+1} + \beta_{2,2m-2j+1})e_{2m}, \\ \{e_{2m}, x\} = ((2m-1)\beta_{1,1} + 2\beta_{1,2})e_{2m}, \end{cases}$$

where $1 \leq j \leq m-1, \ n = 2m$.

Using the condition (i) in Proposition 1 we obtain the products $[[-,-]] = [-,-] + \{-,-\}$.

Jacobi identity		Constraints
$L(e_1, e_{2j-1}, x) = 0,$	$\Rightarrow$	$\alpha_{1,2} = -\beta_{1,2},$
$L(e_1, e_{2j}, x) = 0,$	$\Rightarrow$	$\beta_{2,2m-2j+1} = 0, \quad 1 \leq j \leq m-1,$

Therefore, we obtain the multiplications table of the algebra $\mathcal{R}$.

Due to the result [5], $\mathcal{L}_n$ admits solvable compatible Lie extensions of dimension less than two. Consequently, we conclude that $\mathcal{N}$ possesses only one-dimensional solvable compatible Lie extensions via non-nilpotent derivations of $\mathcal{L}_n$.

Let now consider the solvable extensions of $\mathcal{N}$ by means of non-nilpotent derivations from $Der(\mathcal{L}_n) \cap Der(\mathcal{Q}_n)$.

Theorem 3. Let $\mathcal{R} = \mathcal{N} \oplus Span\{x\}$ be a one-dimensional solvable extension of $\mathcal{N}$ by means of non-nilpotent derivations of $Der(\mathcal{L}_n) \cap Der(\mathcal{Q}_n)$. Then $\mathcal{R}$ admits one of the following two multiplications tables:

$$\begin{cases} [e_1, x] = e_1, \quad [e_i, x] = (i-1)e_i, \quad 2 \leq i \leq 2m, \\ \{e_1, x\} = \beta_{1,1}e_1 + \sum\limits_{t=3}^{2m} \beta_{1,t}e_t, \\ \{e_2, x\} = \beta_{1,1}e_2 + \sum\limits_{t=1}^{m-1} \beta_{2,2t+1}e_{2t+1} + \beta_{2,2m}e_{2m}, \\ \{e_{2j}, x\} = (2j-1)\beta_{1,1}e_{2j} + \sum\limits_{t=1}^{m-j} \beta_{2,2t+1}e_{2t+2j-1} + \beta_{1,2m-2j+1}e_{2m}, \\ \{e_{2j+1}, x\} = 2j\beta_{1,1}e_{2j+1} + \sum\limits_{t=1}^{m-j-1} \beta_{2,2t+1}e_{2t+2j} + (-\beta_{1,2m-2j+1} + \beta_{2,2m-2j+1})e_{2m}, \end{cases}$$

and

$$\begin{cases} [e_1,x]=\alpha_{1,1}e_1+\sum\limits_{t=3}^{2m}\alpha_{1,t}e_t,\\ [e_2,x]=\alpha_{1,1}e_2+\sum\limits_{t=1}^{m-1}\alpha_{2,2t+1}e_{2t+1}+\alpha_{2,2m}e_{2m},\\ [e_{2j},x]=(2j-1)\alpha_{1,1}e_{2j}+\sum\limits_{t=1}^{m-j}\alpha_{2,2t+1}e_{2t+2j-1}+\alpha_{1,2m-2j+1}e_{2m},\\ [e_{2j+1},x]=2j\alpha_{1,1}e_{2j+1}+\sum\limits_{t=1}^{m-j-1}\alpha_{2,2t+1}e_{2t+2j}+(-\alpha_{1,2m-2j+1}+\alpha_{2,2m-2j+1})e_{2m},\\ \{e_1,x\}=e_1,\quad \{e_i,x\}=(i-1)e_i,\quad 2\le i\le 2m, \end{cases}$$

where $1\le j\le m-1,\ n=2m$.

Proof. Applying derivations of $\mathcal{L}_n$ and $\mathcal{Q}_n$, we have the products in $\mathcal{R}$:

$$\begin{cases} [e_1,x]=\alpha_{1,1}e_1+\sum\limits_{t=3}^{2m}\alpha_{1,t}e_t,\\ [e_2,x]=\alpha_{1,1}e_2+\sum\limits_{t=1}^{m-1}\alpha_{2,2t+1}e_{2t+1}+\alpha_{2,2m}e_{2m},\\ [e_{2j},x]=(2j-1)\alpha_{1,1}e_{2j}+\sum\limits_{t=1}^{m-j}\alpha_{2,2t+1}e_{2t+2j-1}+\alpha_{1,2m-2j+1}e_{2m},\\ [e_{2j+1},x]=2j\alpha_{1,1}e_{2j+1}+\sum\limits_{t=1}^{m-j-1}\alpha_{2,2t+1}e_{2t+2j}+(-\alpha_{1,2m-2j+1}+\alpha_{2,2m-2j+1})e_{2m}, \end{cases}$$

and

$$\begin{cases} \{e_1,x\}=\beta_{1,1}e_1+\sum\limits_{t=3}^{2m}\beta_{1,t}e_t,\\ \{e_2,x\}=\beta_{1,1}e_2+\sum\limits_{t=1}^{m-1}\beta_{2,2t+1}e_{2t+1}+\beta_{2,2m}e_{2m},\\ \{e_{2j},x\}=(2j-1)\beta_{1,1}e_{2j}+\sum\limits_{t=1}^{m-j}\beta_{2,2t+1}e_{2t+2j-1}+\beta_{1,2m-2j+1}e_{2m},\\ \{e_{2j+1},x\}=2j\beta_{1,1}e_{2j+1}+\sum\limits_{t=1}^{m-j-1}\beta_{2,2t+1}e_{2t+2j}+(-\beta_{1,2m-2j+1}+\beta_{2,2m-2j+1})e_{2m}, \end{cases}$$

where $1\le j\le m-1,\ n=2m$.

The solvability of extension $\mathcal{R}$ leads that $(\alpha_{1,1},\beta_{1,1})\neq(0,0)$.

Let $\alpha_{1,1}\neq 0$, then by setting

$$x'=\frac{1}{\alpha_{1,1}}(x-\sum_{t=3}^{2m}\alpha_{1,t}e_{t-1}),$$

we can assume $[e_1,x]=e_1$. Then, setting

$$e_1'=e_1,\quad e_i'=e_i+\sum_{t=3}^{2m-i+2}A_te_{t+i-2},\quad 2\le i\le n,$$

with

$$A_3=-\alpha_{2,3},\qquad A_{2j+1}=-\frac{1}{2j-1}(\alpha_{2,2j+1}+\sum_{t=2}^{j}A_{2,t}\alpha_{2,2j-2t+3}),$$

$$A_{2j}=-\frac{1}{2j-2}\sum_{t=1}^{j-1}A_{2t+1}\alpha_{2,2j-2t+1},\quad A_{2m}=-\frac{1}{2m-2}(\alpha_{2,2m}+\sum_{t=1}^{m-1}A_{2t+1}\alpha_{2,2m-2t+1}),$$

where $2\le j\le m-1$, we obtain

$$[e_i,x]=(i-1)e_i,\quad 2\le i\le n.$$

Note that the basis transformation leaves the multiplication tables of $\mathcal{Q}_n$ unchanged. Similarly, when $\beta_{1,1}\neq 0$, we get

$$\{e_1,x\}=e_1,\quad \{e_i,x\}=(i-1)e_i,\quad 2\le i\le n.$$

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REZYUME

Ushbu maqolada Li komponentlari klassik filiform Li algebralari $\mathcal{L}_n$ va $\mathcal{Q}_n$ bo'lgan nilpotent kompatibl Li algebrasini maksimal nilpotent ideal sifatida o'z ichiga olgan ba'zi yechiluvchan kompatibli Li algebralari qurilgan.

Kalit so'zlar: mos keluvchi Li algebralari, differensiyallashlar, nilpotentlik, yechiluvchanlik.

РЕЗЮМЕ

В настоящей статье мы конструируем ряд разрешимых совместимых алгебр Ли, содержащих максимальный нильпотентный идеал $\mathcal{N}$, компоненты которого представляют собой классические филиформные алгебры Ли $\mathcal{L}_n$ и $\mathcal{Q}_n$.

Ключевые слова: совместимые алгебры Ли, дифференцирования, нильпотентность, разрешимость.