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THE CAUCHY PROBLEM FOR A HIGH-ORDER ORDINARY DIFFERENTIAL EQUATION INVOLVING THE BESSEL OPERATOR AND LOWER-ORDER TERMS

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RESUME

This paper investigates the Cauchy problem for a high-order ordinary differential equation involving the Bessel operator with a spectral parameter. This type of problem presents significant challenges and has received limited attention in the literature due to the lack of appropriate analytical tools. The main objective of the study is to solve the Cauchy problem by employing a transmutation operator. As the transmutation operator, the generalized Erd'elyi-Kober fractional operator is utilized. When this operator is applied, the considered problem is transformed into an equation without degeneration and without a lower-order term. A key advantage of the proposed approach is that it leads to an explicit solution of the formulated problem. Despite the significant progress in modern computational techniques, obtaining exact solutions for boundary value problems of ordinary differential equations remains an important and relevant challenge. Such solutions provide a deeper understanding of the qualitative behavior of the described processes and phenomena, reveal the intrinsic properties of the underlying mathematical models, and can also serve as benchmark examples for asymptotic and numerical methods.

Key words: high-order ordinary differential equation, Cauchy problem, generalized Erdelyi-Kober fractional operator, function of Mittag-Liffler.

1. Introduction

In this article, we consider the following ordinary differential equation

$$\left(A_{a,b}^{m}-c^{2}\right)y\left(x\right)\equiv\left(\frac{d^{2}}{dx^{2}}+\frac{a}{x}\frac{d}{dx}+b^{2}\right)^{m}y\left(x\right)-c^{2}y\left(x\right)=f\left(x\right),\ \ x>0\,,\ b,\,c\in R\ , \tag{1}$$

with an integer power of the Bessel operator

$$A_{a,b}^{m}\equiv\left(B_{a}+b^{2}\right)^{m},a\in R,\quad a>0 \tag{2}$$

where $B_a=\frac{d^2}{dx^2}+\frac{a}{x}\frac{d}{dx}$ is the Bessel operator, $m\in N, f\left(x\right)$ is a given function, $A_{0,0}=\frac{d^2}{dx^2},\ A_{a,b}^0=E,\ E$ is identity operator, $A_{a,b}^m=A_{a,b}^{m-1}A_{a,b}$ is the m-th power of the operator in (2).

In the theory of differential equations, problems involving the Bessel operator hold special significance, as they appear in various fields of physics, engineering, and applied mathematics. The Bessel operator typically arises in problems with radial symmetry, such as the wave equation, heat conduction, beam vibrations, and quantum mechanics [14,15].

The entire range of equations with Bessel operators was most fully studied by the Voronezh mathematician I.A. Kipriyanov and his students [1,2,3]. These studies are closely connected to, and often based upon, transmutation theory [4,6,7]. Also methods and results from fractional calculus are very useful, [3,5,6,7,8], including special types of fractional operators that are important in applications to differential equations, namely, the Erdelyi-Kober [2,5,7,8,9], Buschman-Erdelyi [3,10] ones and fractional type operators with Gauss and Legendre kernels [11]. Some problems for products of Bessel-type operators are considered in [12], for products of Sturm-Liouville operators in [13]. We highlight in particular the remarkable paper [14], which presents many original and valuable findings. Among them problems for 3 types of integer powers of the following operators were explicitly solved in terms of integral operators with hypergeometric function kernels

$$D_1 = \left(\frac{1}{x}\frac{d}{dx}\right)^m, \ D_2 = \left(\frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx}\right)^k, \ D_3 = \left(\frac{1}{x}\frac{d}{dx}\right)^m \left(\frac{d^2}{dx^2} + \frac{\gamma}{x}\frac{d}{dx}\right)^k$$

with additional conditions at $x = 1$ on solution and its derivatives. The choice of $x = 1$ avoids the complications associated with the singularity at $x = 0$. It should be noted that we consider a more complex case with additional conditions imposed precisely at $x = 0$. For differential equations for Bessel type operators with conditions on lines of singularities as in (1) special conditions are needed, namely Kipriyanov's evenness conditions, [12, p.235], [3, p.33]. Regarding the difficulties with conditions on singularity lines for Bessel-type equations [15].

Another significant section of the paper [14] introduced and outlined explicit constructions for fractional powers of the Bessel operator (2). Further developments of these results can be found in [3,6,16.17].

It should also be noted that for operators of type (1) with a spectral parameter, a special class of transmutations was introduced by S.M. Sitnik. They were named Vekua-Erdelyi-Lowndes (VEL) transmutations after authors of first known special cases, for any operator A and a constant λ these VEL transmutations T are defined by a property

$$T(A + \lambda) = AT,$$

see [18,19]. In this paper, we use such VEL transmutations introduced by Lowndes [20]. Also our main method is to use as transmutations the generalized Erdelyi-Kober fractional operator [5,8,9].

1.1 The Cauchy Problem

In the domain $\Omega = \{x \in R : x > 0\}$ we look for the solution $y(x) \in C^{2m}[0; +\infty)$ of the equation (1) satisfying the following initial conditions.

$$y^{(j)}(0) = 0, \quad j = 0, 1, 2, \ldots, 2m - 1. \tag{3}$$

We now turn to the main properties of the generalized Erdelyi-Kober fractional operator.

1.2. Generalized Erdelyi-Kober fractional operator with Bessel functions in the Kernel

In the works of Erdelyi and Kober, the following modification of fractional integration was introduced:

$$I_{\gamma,\alpha}\varphi(x) = \frac{2x^{-2(\gamma+\alpha)}}{\Gamma(\alpha)} \int\limits_0^x \left(x^2 - t^2\right)^{\alpha-1} t^{2\gamma+1} \varphi(t)\, dt \tag{4}$$

$$K_{\gamma,\alpha}\varphi(x) = \frac{2x^{2\gamma}}{\Gamma(\alpha)} \int\limits_x^{+\infty} \left(x^2 - t^2\right)^{\alpha-1} t^{1-2(\gamma+\alpha)} \varphi(t) dt, \tag{5}$$

where $\alpha > 0, \ \gamma > 0, \ \varphi(x) \in L(R^+)$.

Operators (4), (5) and their generalizations are called Erdelyi-Kober operators [5].

The results of Erdelyi and Kober are generalized in the works of J.S. Lowndes [20], who introduced and studied generalized Erdelyi-Kober operators of the form:

$$J_\lambda(\gamma,\,\alpha)\varphi(x) = 2^\alpha \lambda^{1-\alpha} x^{-2\alpha-2\eta} \int\limits_0^x t^{2\gamma+1} (x^2 - t^2)^{(\alpha-1)/2} J_{\alpha-1}\left(\lambda\sqrt{x^2 - t^2}\right) \varphi(t) dt \tag{6}$$

$$R_\lambda(\gamma,\,\alpha)\varphi(x) = 2^\alpha \lambda^{1-\alpha} x^{2\gamma} \int\limits_x^{+\infty} t^{1-2\alpha-2\gamma} (t^2 - x^2)^{(\alpha-1)/2} J_{\alpha-1}\left(\lambda\sqrt{t^2 - x^2}\right) \varphi(t) dt, \tag{7}$$

where $\gamma,\, \alpha, \lambda \in R$ and $\alpha > 0, \ \gamma \geq -1/2$, $J_\nu(z)$ is the Bessel function of the first kind. It is clear that, when $\lambda \to 0$ da (6) and (7) operators coincide with the classical operators (4) and (5): $J_0(\gamma, \alpha) = I_{\gamma,\alpha}$, $R_0(\gamma,\, \alpha) = K_{\gamma,\alpha}$.

The following representation of operator (6) will be required later:

$$J_\lambda(\gamma,\,\alpha)\varphi(x)=\frac{2x^{-2(\alpha+\gamma)}}{\Gamma(\alpha)}\int\limits_0^x\left(x^2-t^2\right)^{\alpha-1}\bar{J}_{\alpha-1}\left(\lambda\sqrt{x^2-t^2}\right)t^{2\gamma+1}\varphi(t)dt,\tag{8}$$

where $\bar{J}_\nu(z)$ is the normalized Bessel function [6] defined by

$$\bar{J}_\nu(z)=\Gamma(\nu+1)(z/2)^{-\nu}J_\nu(z)=\sum_{k=0}^\infty\frac{\left(-z^2/4\right)^k}{(\nu+1)_k k!}.$$

The function $\bar{J}_\nu(z)$ is even and infinitely differentiable. Moreover, $\left|\bar{J}_\nu(z)\right|\le 1$ for $\nu>-1/2$ and it satisfies the following equation:

$$B_\gamma^x\bar{J}_\gamma(\lambda x)+\lambda^2\bar{J}_\gamma(\lambda x)=0,$$

with the initial conditions

$$\bar{J}_\nu(0)=1,\,(d/dx)\bar{J}_\nu(\lambda x)\big|_{x=0}=0.$$

The inverse operator to (8) is given by [5]:

$$J_\lambda^{-1}(\gamma,\alpha)\varphi(x)=\frac{2x^{-2\gamma}}{\Gamma(p-\alpha)}\left(\frac{1}{2x}\frac{d}{dx}\right)^p\int\limits_0^x\frac{\bar{I}_{p-\alpha-1}\left(\lambda\sqrt{x^2-s^2}\right)}{\left(x^2-s^2\right)^{\alpha-p+1}}s^{2(\gamma+\alpha)+1}\varphi(s)ds,\tag{9}$$

where $p=[\alpha]+1$, $\bar{I}_\nu(z)=\bar{J}_\nu(iz)=\Gamma(\nu+1)(z/2)^{-\nu}I_\nu(z)=\sum\limits_{k=0}^\infty\frac{\left(z^2/4\right)^k}{(\nu+1)_k k!}$, $I_\nu(z)$ is the Bessel function with imaginary argument.

From [21,22] we cite the following theorem for (2):

Theorem 1. *Let* $\alpha>0$, $\gamma\ge-\frac{1}{2}$, $\varphi(x)\in C^{2m}(0,b)$, $b>0$, *and the function* $x^\gamma\left[A_{a,0}^x\right]^{k+1}\varphi(x)$, *be integrable at zero and* $\lim\limits_{x\to 0}x^{2\gamma+1}\frac{d}{dx}\left[A_{a,0}^x\right]^{k+1}\varphi(x)=0$, $k=0,1,\ldots,m-1$. *Then the following equality holds:*

$$A_{\gamma+a,b}^m J_b\left(\frac{\gamma-1}{2},\frac{a}{2}\right)\varphi(x)=J_b\left(\frac{\gamma-1}{2},\frac{a}{2}\right)A_{\gamma,0}^m\varphi(x).$$

In particular, if $b=0$, *then*

$$A_{\gamma+a,0}^m I_{\frac{\gamma-1}{2},\frac{a}{2}}\varphi(x)=I_{\frac{\gamma-1}{2},\frac{a}{2}}A_{\gamma,0}^m\varphi(x).$$

Moreover, for $\gamma=0$ *we have:*

$$A_{a,b}^m J_b\left(-\frac{1}{2},\frac{a}{2}\right)\varphi(x)=J_b\left(-\frac{1}{2},\frac{a}{2}\right)A_{0,0}^m\varphi(x).\tag{10}$$

In the subsequent calculations, we utilize of these special functions.

1.3 Hypergeometric Functions

The Gauss hypergeometric function, inside the circle |z| < 1, is defined as the sum of the hypergeometric series [23,24]:

$${}_2F_1\left(a,\;b;\;c;\;z\right)\;=\;F\left(a,\;b,\;c;\;z\right)\;=\;\sum_{k=0}^\infty\frac{(a)_k(b)_k z^k}{(c)_k\;k!},\tag{11}$$

for $|z|<1$, the function is defined by the series expansion. For $|z|\ge 1$, it is defined by the analytic continuation of this series. In expression (10), the parameters a,b,c and the variable z may be complex, where $c\ne 0,-1,-2,\ldots$. The product $(a)_k$ is the Pochhammer symbol:

$$(a)_n\;=\;a\left(a\;+\;1\right)...\left(a\;+\;n\;-\;1\right),$$

$$n\;=\;1,\;2,\;...,\;(a)_0\;\equiv\;1.$$

In the following computations, the needed relations and transformations include:

$$ {}_2F_1(a,b;c;z) = \frac{\Gamma(c)}{\Gamma(b)\Gamma(c-b)} \int\limits_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt \qquad (12)$$

$$ {}_2F_1(a,b;c;z) = (1-zt)^{c-a-b} {}_2F_1(c-a,c-b;c;z) \qquad (13)$$

$$ F\left(a,b,a-b+1;z\right) = \frac{1}{(1-z)^a} F\left(\frac{a}{2},\frac{a+1}{2}-b;a-b+1;-\frac{4z}{(1-z)^2}\right). \qquad (14)$$

The function $F_3\left(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma;x,y\right)$ is Appell's hypergeometric function of two variables [23,24], and its series representation is given by

$$ F_3\left(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma;x,y\right) = \sum_{m,n=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_n(\beta_1)_m(\beta_2)_n}{(\gamma)_{m+n}m!n!}(x)^n(y)^m, \ \max\left[|x|,|y|\right] < 1, \qquad (15)$$

it also satisfies the formula

$$ F_3\left(\alpha_1,\alpha_2,\beta_1,\beta_2,\gamma;x,y\right) = \sum_{n=0}^{\infty} \frac{(\alpha_1)_n(\beta_1)_n}{(\gamma)_n n!}(x)^n {}_2F_1\left(\alpha_2,\beta_2;\gamma+n;y\right), \qquad (16)$$

where ${}_2F_1\left(\alpha_2,\beta_2;\gamma+n;y\right)$ is the Gauss hypergeometric function. Moreover,

$$ F_3\left(a,a',b,b',a'+b;x,\frac{x}{x-1}\right) = (1-x)^{-a} {}_2F_1\left(a',a+b';a'+b;\frac{x}{x-1}\right) \qquad (17)$$

$$ \Xi_2\left(a,b,c;x,y\right) = \sum_{n=0}^{\infty} \frac{y^n}{(c)_n n!} {}_2F_1\left(a,b;c+n;x\right), \ \ |x|<1, \qquad (18)$$

where $\Xi_2\left(a,b,c;x,y\right)$ is Humbert's confluent hypergeometric function [24].

The general triple hypergeometric function introduced by Srivastava [25].

$$ F^{(3)}(x,y,z) = F^{(3)}\left[\begin{array}{c} (a)::(b);\ (b');\ (b'');\ (c);\ (c');\ (c'') \\ (e)::(g);\ (g');\ (g'');\ (h);\ (h');\ (h'') \end{array}\middle| x,y,z\right] = $$
$$ = \sum_{m,n,k=0}^{\infty} \wedge(m,n,k)\,\frac{x^m}{m!}\frac{y^n}{n!}\frac{z^k}{k!}, \qquad (19)$$

where

$$ \wedge(m,n,k) = \frac{\prod\limits_{j=1}^{A}(a_j)_{m+n+k}\prod\limits_{j=1}^{B}(b_j)_{m+n}\prod\limits_{j=1}^{B'}(b'_j)_{n+k}\prod\limits_{j=1}^{B''}(b''_j)_{m+k}\prod\limits_{j=1}^{C}(c_j)_m\prod\limits_{j=1}^{C'}(c'_j)_n\prod\limits_{j=1}^{C''}(c''_j)_k}{\prod\limits_{j=1}^{E}(e_j)_{m+n+k}\prod\limits_{j=1}^{G}(g_j)_{m+n}\prod\limits_{j=1}^{G'}(g'_j)_{n+k}\prod\limits_{j=1}^{G''}(g''_j)_{m+k}\prod\limits_{j=1}^{H}(h_j)_m\prod\limits_{j=1}^{H'}(h'_j)_n\prod\limits_{j=1}^{H''}(h''_j)_k}. $$

$F^{(3)}(x,y,z)$ is Srivastava's generalized triple hypergeometric function .

2. Application of the generalized Erdelyi-Kober operator to the solution of the Cauchy problem

Assuming that the solution to problem (1), (3) exists, we seek it in the form

$$ y\left(x\right) = J_b\left(-\frac{1}{2},\frac{a}{2}\right)z\left(x\right) = \frac{2x^{1-a}}{\Gamma\left(a/2\right)} \int\limits_0^x \left(x^2-t^2\right)^{\frac{a}{2}-1} \bar{J}_{\frac{a}{2}-1}\left(b\sqrt{x^2-t^2}\right) z\left(t\right)dt, \qquad (20)$$

where $z(x)$ is an unknown function, and we assume that $z(x)$ is sufficiently smooth.

Equation (1), according to equality (10), can be expressed as follows:

$$J_b\left(-\frac{1}{2},\frac{a}{2}\right)z^{(2m)}(x)-c^2J_b\left(-\frac{1}{2},\frac{a}{2}\right)z(x)=f(x). \tag{21}$$

For the case $0<\frac{a}{2}<1$, $p=\left[\frac{a}{2}\right]+1=1$ in equation (21), we apply (9) in the form $J_b^{-1}\left(-\frac{1}{2},\frac{a}{2}\right)$, and using the initial conditions (3), we obtain the following equation for the function $z(x)\in C^{2m}(R^+)$ to

$$z^{(2m)}(x)-c^2z(x)=F(x),\ \ x>0 \tag{22}$$

with the initial condition

$$z^{(j)}(0)=0,\ \ j=0,1,...,2m-1, \tag{23}$$

where

$$F(x)=J_\lambda^{-1}\left(-\frac{1}{2},\frac{a}{2}\right)f(x)=\frac{1}{\Gamma(1-\frac{a}{2})}\frac{d}{dx}\int\limits_0^x\frac{\bar{I}_{-\frac{a}{2}}\left(\lambda\sqrt{x^2-\eta^2}\right)}{(x^2-\eta^2)^{\frac{a}{2}}}\eta^af(\eta)\,d\eta. \tag{24}$$

The unique solution of equation (22) is obtained as follows (see [26, p. 245]):

$$z(x)=\sum_{k=1}^{2m}B_k(x-A)^{2m-k}E_{2m,2m-k+1}\left[c^2(x-A)^{2m}\right]+\int\limits_0^x(x-s)^{2m-1}E_{2m,2m}\left[c^2(x-s)^{2m}\right]F(s)\,ds, \tag{25}$$

where $E_{2m,2m}\left[c^2(x-s)^{2m}\right]=\sum\limits_{k=0}^{\infty}\frac{c^{2k}(x-s)^{2mk}}{\Gamma(2mk+2m)}$ is the Mittag - Leffler function.

Expression (25), according to the initial condition (23), takes the following form

$$z(x)=\int\limits_0^x(x-s)^{2m-1}E_{2m,2m}\left[c^2(x-s)^{2m}\right]F(s)\,ds, \tag{26}$$

Substituting (24) into (26), we have

$$z(x)=\frac{1}{(1-\frac{a}{2})}\sum_{k=0}^{\infty}\frac{c^{2k}}{\Gamma(2mk+2m)}\int\limits_0^x(x-s)^{2mk+2m-1}\frac{d}{ds}G(s)\,ds, \tag{27}$$

where

$$G(s)=\int\limits_0^s\left(s^2-\eta^2\right)^{-\frac{a}{2}}\bar{I}_{-\frac{a}{2}}\left(\lambda\sqrt{s^2-\eta^2}\right)\eta^af(\eta)\,d\eta. \tag{28}$$

The following lemma is applicable to expression (28):

Lemma. *If the function $f(\eta)$ is continuous, then* $\lim\limits_{s\to0}G(s)=0$.

Proof. We make the change of variable $\eta=st$ in equation (28).

$$G(s)=\int\limits_0^1s^{-a}\left(1-t^2\right)^{-\frac{a}{2}}\bar{I}_{-\frac{a}{2}}\left(\lambda s\sqrt{1-t^2}\right)s^at^af(st)\,sdt$$

$$=s\int\limits_0^1\left(1-t^2\right)^{-\frac{a}{2}}\bar{I}_{-\frac{a}{2}}\left(\lambda s\sqrt{1-t^2}\right)t^af(st)\,dt$$

$$\lim_{s\to 0} G(s) = 0\int_0^1 \left(1-t^2\right)^{-\frac{a}{2}} \bar{I}_{-\frac{a}{2}}(0)\, t^a f(0)\, dt$$

$$= 0 f(0)\int_0^1 \left(1-t^2\right)^{-\frac{a}{2}} t^a dt = 0.$$

The proof is complete. □

By integrating (27) by parts and applying Lemma 1, we obtain:

$$z(x) = \frac{2mk+2m-1}{\left(1-\frac{a}{2}\right)} \sum_{k=0}^{\infty} \frac{c^{2k}}{\Gamma(2mk+2m)} H(x), \tag{29}$$

where

$$H(x) = \int_0^x (x-s)^{2mk+2m-2} G(s)\, ds. \tag{30}$$

We substitute expression (28) into integral (30) and, after changing the order of integration and performing some simplifications, we have

$$H(x) = \int_0^x \eta^a f(\eta)\, h(x,\eta)\, d\eta, \tag{31}$$

where

$$h(x,\eta) = \int_\eta^x (x-s)^{2mk+2m-2} \left(s^2-\eta^2\right)^{-\frac{a}{2}} \bar{I}_{-\frac{a}{2}}\left(b\sqrt{s^2-\eta^2}\right) ds. \tag{32}$$

We perform the substitution of variable in integral (32) as$s = \eta + (x-\eta)\tau$ and, using formulas (12), (13), (14), and (18), we get

$$h(x,\eta) = \left(\frac{x^2-\eta^2}{2x}\right)^{2mk+2m-1} \left(x^2-\eta^2\right)^{-\frac{a}{2}} \frac{\Gamma\left(1-\frac{a}{2}\right)\Gamma(2mk+2m-1)}{\Gamma\left(2mk+2m-\frac{a}{2}\right)}$$

$$\times \Xi_2\left(mk+m-\frac{1}{2}, mk+m; 2mk+2m-\frac{a}{2}; \frac{x^2-\eta^2}{x^2}, \frac{b^2}{4}\left(x^2-\eta^2\right)\right). \tag{33}$$

Equation (33) is substituted into (31), and the resulting expression is then substituted into (29). After simplification, we obtain the following result:

$$z(x) = \int_0^x \eta^a \left(\frac{x^2-\eta^2}{2x}\right)^{2m-1} \left(x^2-\eta^2\right)^{-\frac{a}{2}} E_{2m,2m-\frac{a}{2}}\left[c^2\left(\frac{x^2-\eta^2}{2x}\right)^{2m}\right]$$

$$\times \Xi_2\left(mk+m-\frac{1}{2}, mk+m; 2mk+2m-\frac{a}{2}; \sigma, \omega\right) f(\eta)\, d\eta. \tag{34}$$

Substituting equation (34) into (20) and performing some simplifications, we have

$$y(x) = \frac{2x^{1-a}}{\Gamma\left(\frac{a}{2}\right)} \int_0^x \eta^a f(\eta)\, K(x,\eta)\, d\eta, \tag{35}$$

where

$$K(x,\eta) = \int_\eta^x \left(x^2-t^2\right)^{\frac{a}{2}-1} \bar{J}_{\frac{a}{2}-1}\left(b\sqrt{x^2-t^2}\right) \left(\frac{t^2-\eta^2}{2t}\right)^{2m-1} \left(t^2-\eta^2\right)^{-\frac{a}{2}} E_{2m,2m-\frac{a}{2}}\left[c^2\left(\frac{t^2-\eta^2}{2t}\right)^{2m}\right]$$

$$\times \Xi_2\left(mk+m-\frac{1}{2}, mk+m; 2mk+2m-\frac{a}{2}; \sigma, \omega\right) dt. \quad (36)$$

We will now perform some calculations and simplifications on expression (36), first using formula (18) and then formulas (16) and (17), ultimately obtaining the following:

$$K(x,\eta)=\Gamma\left(\frac{a}{2}\right)\left(\frac{x^2-\eta^2}{2x}\right)^{2m}\left(x^2-\eta^2\right)^{-1}\sum_{k=0}^{\infty}\frac{c^{2k}\left(\frac{x^2-\eta^2}{2x}\right)^{2mk}}{\Gamma(2mk+2m)}\sum_{n=0}^{\infty}\frac{\left(-\frac{b^2}{4}\left(x^2-\eta^2\right)\right)^n}{(2mk+2m)_n n!}$$
$$\times \Xi_2\left(m+mk, mk+m+n+\frac{a-1}{2}; 2mk+2m+n; \frac{x^2-\eta^2}{x^2}, \frac{b^2}{4}\left(x^2-\eta^2\right)\right). \quad (37)$$

By substituting expression (37) into (35) and performing simplifications, the explicit form of the solution can be written as

$$y(x)=\int\limits_0^x\left(\frac{x^2-\eta^2}{2x}\right)^{2m-1}\left(\frac{\eta}{x}\right)^a\sum_{k=0}^{\infty}\frac{c^{2k}\left(\frac{x^2-\eta^2}{2x}\right)^{2mk}}{\Gamma(2mk+2m)}\sum_{n=0}^{\infty}\frac{\left(-\frac{b^2}{4}\left(x^2-\eta^2\right)\right)^n}{(2mk+2m)_n n!}$$
$$\times \Xi_2\left(m+mk, mk+m+n+\frac{a-1}{2}; 2mk+2m+n; \sigma, \omega\right) f(\eta)\, d\eta, \quad (38)$$

where $\sigma=\frac{x^2-\eta^2}{x^2}, \omega=\frac{b^2}{4}\left(x^2-\eta^2\right)$.

If we express the last triple series under the integral in expression (38) using (19), then the solution is expressed as follows:

$$y(x)=\int\limits_0^x\left(\frac{x^2-\eta^2}{2x}\right)^{2m-1}\left(\frac{\eta}{x}\right)^a\sum_{k=0}^{\infty}\frac{c^{2k}\left(\frac{x\sigma}{2}\right)^{2mk}}{\Gamma(2mk+2m)}$$
$$\times F^{(3)}\left[\begin{matrix}\left((-):(-);\ (-);\ (mk+m+\frac{a-1}{2});\ (mk+m);\ (-);\ (-)\right)\\ \left((2mk+2m):(-);\ (-);\ (-);\ (mk+m+\frac{a-1}{2});\ (-);\ (-)\right)\end{matrix}\ \middle|\ \sigma,\omega,-\omega\right] f(\eta)\, \mathrm{d}\eta. \quad (39)$$

Consequently, the following theorem is established.

Theorem 2. *If $f(x)\in C[0,\infty)$, then the solution of the Cauchy problem can be represented in the form (39).*

Remark. *When $c=0$, equation (1) $\left(A_{a,b}^m-c^2\right)y(x)=f(x)$ transforms into the form $A_{a,b}^m y(x)=f(x)$, and its solution, when $a=2\alpha$, takes the following form and coincides with the result obtained from reference[27]*

$$y(x)=\frac{1}{\Gamma(2m)}\int\limits_0^x\left(\frac{x^2-\eta^2}{2x}\right)^{2m-1}\left(\frac{\eta}{x}\right)^{2\alpha}\sum_{n=0}^{\infty}\frac{(-\omega)^n}{(2m)_n n!}$$
$$\times \Xi_2\left(m, m+n+\alpha-\frac{1}{2}; 2m+n; \sigma, \omega\right) f(\eta)\, d\eta.$$

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REZYUME

Ushbu maqolada spektral parametrga ega bo'lgan Bessel operatori ishtirok etgan yuqori tartibli oddiy differensial tenglama uchun Koshi masalasi tadqiq qilinadi. Bunday turdagi masalalar sezilarli murakkabliklarga ega bo'lib, mos analitik vositalarning yetishmasligi sababli ilmiy adabiyotlarda kam o'rganilgan. Tadqiqotning asosiy maqsadi - almashtirish operatoridan foydalanib, Koshi masalasining yechimini olishdir. Almashtirish operatori sifatida umumlashtirilgan Erdelyi-Kober kasr operatori qo'llaniladi. Bu operator qo'llanganda, ko'rib chiqilayotgan masala singulyar koefisentsiz va pastroq tartibli hadi bo'lmagan tenglamaga keltiriladi. Taklif etilgan yondashuvning muhim afzalliklaridan biri shundaki, u qo'yilgan masalaning aniq yechimini olishga imkon beradi. Zamonaviy hisoblash texnologiyalarida katta yutuqlarga erishilganiga qaramay, oddiy differensial tenglamalarning chegara masalalari uchun aniq yechimlarni topish hanuz muhim va dolzarb masala bo'lib qolmoqda. Bunday yechimlar tasvirlanayotgan jarayon va hodisalarning sifat jihatdan xatti-harakatini chuqurroq anglashga, asosiy matematik modellarning ichki xususiyatlarini ochib berishga hamda asimptotik va sonli metodlar uchun tayanch misollar sifatida xizmat qilishga yordam beradi.

Kalit so'zlar: Yuqori tartibli oddiy differensial tenglama, Koshi masalasi, umumlashtirilgan Erdelyi-Kober kasr operatori, Mittag-Leffler funksiyasi.

РЕЗЮМЕ

В данной работе исследуется задача Коши для обыкновенного дифференциального уравнения высокого порядка с оператором Бесселя и спектральным параметром. Подобные задачи представляют собой значительные трудности и в литературе изучены лишь в ограниченном объёме из-за отсутствия соответствующих аналитических методов. Основная цель исследования заключается в решении задачи Коши с использованием оператора преобразования. В качестве оператора преобразования применяется обобщённый дробный оператор Эрдейи-Кобера. При применении этого оператора рассматриваемая задача преобразуется в уравнение без вырождения и без членов более низкого порядка. Ключевым преимуществом предлагаемого подхода является получение явного решения сформулированной задачи.

Несмотря на значительный прогресс в современных вычислительных методах, получение точных решений краевых задач для обыкновенных дифференциальных уравнений остаётся важной и актуальной задачей. Такие решения позволяют глубже понять качественное поведение описываемых процессов и явлений, выявить внутренние свойства соответствующих математических моделей, а также могут служить эталонными примерами для асимптотических и численных методов.

Ключевые слова: Обыкновенное дифференциальное уравнение высокого порядка, задача Коши, обобщённый дробный оператор Эрдейи-Кобера, функция Миттаг-Леффлера.