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**TO THE PROPERTIES OF THE SOLUTIONS OF A NON-DIVERGENT  
NONLINEAR PARABOLIC SYSTEM DESCRIBING THE PROCESSES OF  
COMBUSTION**

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**RESUME**

In this paper, the asymptotic behavior of self-similar solutions to the Cauchy problem for a system of nonlinear parabolic equations in non-divergence form is obtained, and estimates for the subsolution are derived.

**Key words:** mathematical modeling, self-similar solutions, parabolic equations, non-divergence form, subsolution, asymptotic behavior.

**1. Introduction**

In this work, in the domain  $Q = \{(t, x) : t > 0, x \in R\}$  we consider the Cauchy problem for a system of nonlinear parabolic equations not in divergence form:

$$\begin{cases} \frac{\partial u}{\partial t} = u^{\alpha_1} \frac{\partial}{\partial x} (u^{m-1} \frac{\partial u}{\partial x}) - v^{\beta_1} \\ \frac{\partial v}{\partial t} = v^{\alpha_2} \frac{\partial}{\partial x} (v^{m-1} \frac{\partial v}{\partial x}) - u^{\beta_2} \end{cases} \quad (1)$$

$$\begin{cases} u(0, x) = u_0(x) \geq 0 \\ v(0, x) = v_0(x) \geq 0 \end{cases} \quad (2)$$

where  $\beta_1 > 1, \beta_2 > 1, m > 1, \alpha_1 > 1, \alpha_2 > 1$  - numerical parameters,  $u = u(t, x) \geq 0, v = v(t, x) \geq 0$  - are the solutions.

In this paper, we study a nonlinear parabolic system of equations not in divergence form with absorption. Such systems naturally appear in models of heat transfer, combustion and chemical reactions, where nonlinear diffusion plays an essential role. The considered system describes the temperature variation and reactant concentration during the combustion process, capturing the essential features of nonlinear diffusion and heat generation. By applying a self-similar solution, we investigate qualitative properties of solutions, subsolutions and analyze their asymptotic behavior.

In [1], the author proved the asymptotic behaviour of the finite blow-up points solution  $u$  of  $u_t = \Delta u^m$  in  $\widehat{\Omega} \times (0, \infty)$ ,  $u(a_i, t) = \infty, \forall i = 1, \dots, i_0, t > 0, u(x, 0) = u_0(x)$  in  $\widehat{\Omega}$  and  $u = f$  on  $\partial\Omega \times (0, \infty)$ , as  $t \rightarrow \infty$ . Also author construct finite blow-up points solution in bounded cylindrical domain with appropriate lateral boundary value such that the finite blow-up points solution oscillates between two given harmonic functions as  $t \rightarrow \infty$  and proved the existence of the minimal solution of  $u_t = \Delta u^m$  in  $\widehat{\Omega} \times (0, \infty)$ ,  $u(x, 0) = u_0(x)$  in  $\widehat{\Omega}$ ,  $u(a_i, t) = \infty, \forall t > 0, i = 1, 2, \dots, i_0$  and  $u = \infty$  on  $\partial\Omega \times (0, \infty)$ .

In article [2], the authors studied the Cauchy problem for a class of coupled nonlinear parabolic systems and analyzed the asymptotic behavior of their solutions. A Fujita-type blow-up theorem was established using integral estimates and appropriate supersolutions. Furthermore, the critical Fujita exponent, determined by the diffusion and spatial dimension, was identified.

Reference [3] focuses on parabolic problems with anisotropic nonlinearities under nonstandard growth conditions. The existence and uniqueness of weak solutions are established in variable exponent Sobolev spaces. To analyze blow-up phenomena, energy methods are employed, showing that solutions may blow up for both negative and positive initial energies. The roles of variable exponents and coefficients are shown to be critical in Fujita-type blow-up behavior.

In article [4], the authors investigated the blow-up behavior of solutions to nonlinear parabolic systems of non-divergent equations with variable density. The study addressed key aspects in solving Cauchy problems for partial differential equations using the difference method B – including the selection of the approximation order with respect to spatial coordinates, the choice of an initial approximation function, and the development of an efficient algorithm for solving the resulting systems of difference equations. A high-accuracy difference scheme for a nonlinear system in non-divergent form was constructed in the article.

Reference [5] investigates some properties of the blow-up solutions of a nonlinear parabolic system of non-divergent form with cross-diffusion. By constructing suitable auxiliary functions and employing analytical techniques, we derive sufficient conditions for the existence and behavior of blow-up solutions. The obtained results contribute to the deeper understanding of nonlinear diffusion dynamics and provide a foundation for further studies in multidimensional and more complex systems.

In article [6], the critical curves of a doubly nonlinear parabolic equation in non-divergent form with a source term and nonlinear boundary flux are studied. Both the critical global existence curve and the critical Fujita curve are derived. A distinctive feature of equation (1) is that it degenerates at points where  $u = 0$  and  $\nabla u = 0$ . As a result, classical solutions may not exist, and therefore, we consider only weak solutions in the generalized sense described.

Another systems [7,17] was studied through the method of nonlinear splitting, known previously for nonlinear parabolic equations, and systems of equations in divergence form, asymptotic theory and asymptotic methods based on different transformations. Asymptotic representation of self-similar solutions for the nonlinear parabolic system of equations not in divergence form is constructed.

In [8], the authors studied a doubly nonlinear degenerate parabolic system with nonlinear source and absorption terms, which are not located in a homogeneous medium. The system satisfies zero Dirichlet boundary conditions in a smooth bounded domain. To investigate the problem, the comparison principle and a self-similar approach were applied. Moreover, the nonlinear splitting method was employed to establish the existence of both global and finite-time blow-up solutions. It was shown that the interplay between the nonlinear source and nonlinear absorption terms plays a crucial role in determining the existence or non-existence of solutions. These results contribute to a broader understanding of nonlinear parabolic systems.

References [9,12] investigates the asymptotic behavior of self-similar solutions for a degenerate parabolic system not in divergence form, considering both the slow and fast diffusion cases. Using the comparison method, the finite speed of perturbation distribution (FSPD) property of the Cauchy problem for a cross-diffusion parabolic system not in divergence form is established. Based on the obtained asymptotic behavior of the solutions, suitable initial approximations are proposed for the iterative process in both the slow and fast diffusion regimes, depending on the values of the numerical parameters.

In [10] and [11], the authors studied the asymptotic behavior of self-similar solutions of a parabolic system. In particular, Aripov and Matyakubov [10] studied the problem of constructing ZeldovichB–Barenblatt type solution for the cross system equation with a source. Using comparison methods, the property of FSPD of the Cauchy problem for a cross-diffusion parabolic system not in divergence form is established. The asymptotic behavior is discussed for a solution of the cross-diffusion parabolic system equations in non-divergence form for slow and fast diffusion cases depending on the value of the numerical parameters. On the basis of the asymptotic of solutions, suitable initial approximations are offered for the iterative process in the cases of the slow and fast diffusions, depending on the numerical parameter values.

Also, some books and scientific articles contain more equations and methods used in the field than any other book currently available. Included in the handbook are exact, asymptotic, approximate analytical, numerical, symbolic, and qualitative methods that are used for solving and analyzing linear and nonlinear equations. The authors also present formulas for the effective construction of solutions and many different equations arising in various applications, such as heat transfer, elasticity, hydrodynamics, and more. This extensive handbook is the perfect resource for engineers and scientists searching for a comprehensive reservoir of information on ordinary differential equations [12-17].

In this paper, using the self-similar approach, we obtain a particular solution of system (1) and prove that this solution describes the asymptotic behavior of compactly supported solutions. The main purpose of this paper is to establish conditions for the existence of subsolution to problem (1)-(2) based on the self-similar analysis [1-13].

**2. The self-similar analysis of the problem.**

Below, we employ the method of nonlinear splitting [1,7,8,13] to derive the self-similar form of the equation. For construction of the self-similar solutions of the system (1), we consider the functions  $u(t, x), v(t, x)$  in the form:

$$\begin{aligned} u(t, x) &= \bar{u}(t)w(x, \tau(t)), \\ v(t, x) &= \bar{v}(t)\varphi(x, \tau(t)). \end{aligned} \tag{3}$$

Let  $\bar{u}(t), \bar{v}(t)$  be in the form:

$$\bar{u}(t) = A_1(T - t)^{\frac{\beta_1+1}{1-\beta_1\beta_2}}, \quad \bar{v}(t) = A_2(T - t)^{\frac{\beta_2+1}{1-\beta_1\beta_2}},$$

where  $A_i = (\frac{\beta_i+1}{\beta_1\beta_2-1})^{\frac{1}{\beta_1\beta_2-1}} (\frac{\beta_{3-i}+1}{\beta_1\beta_2-1})^{\frac{\beta_i}{\beta_1\beta_2-1}}, i = 1, 2.$

Using (3) the system (1) can be reduced to the following form:

$$\begin{cases} \frac{\partial w}{\partial \tau} = w^{\alpha_1} \frac{\partial}{\partial x} (w^{m-1} \frac{\partial w}{\partial x}) + d_1 \tau^{-1} (\varphi^{\beta_1} + w), \\ \frac{\partial \varphi}{\partial \tau} = \varphi^{\alpha_2} \frac{\partial}{\partial x} (\varphi^{m-1} \frac{\partial \varphi}{\partial x}) + d_2 \tau^{-1} (w^{\beta_2} + \varphi) \end{cases} \tag{4}$$

where

$$\tau(t) = \begin{cases} A_1^{\alpha_1+m-1} \frac{(T-t)^{n_1(\alpha_1+m-1)+1}}{n_1(\alpha_1+m-1)+1}, & \text{at } n_1(\alpha_1+m-1)+1 < 0, \\ -A_1^{\alpha_1+m-1} \ln(T-t), & \text{at } n_1(\alpha_1+m-1)+1 = 0. \end{cases}$$

$$n_2(\alpha_2+m-1) = n_1(\alpha_1+m-1), \quad l = \frac{A_2^{\alpha_2+m-1}}{A_1^{\alpha_1+m-1}},$$

$$d_1 = \frac{n_1}{n_1(\alpha_1+m-1)+1}, \quad d_2 = \frac{n_2 l}{n_2(\alpha_2+m-1)+1}.$$

We consider the problem for the case  $n_1(\alpha_1+m-1)+1 < 0.$

We seek the self-similar solution of system (4) in the following form:

$$w(\tau, x) = f(\xi), \quad \psi(\tau, x) = \phi(\xi), \quad \xi = \frac{x}{\tau^{\frac{1}{2}}} \tag{5}$$

By substituting (5) into system (4), we obtain the following self-similar solution:

$$\begin{cases} f^{\alpha_1} \frac{d}{d\xi} \left( f^{m-1} \frac{df}{d\xi} \right) + \frac{\xi}{2} \frac{df}{d\xi} + d_1 (\phi^{\beta_1} + f) = 0 \\ \phi^{\alpha_2} \frac{d}{d\xi} \left( \phi^{m-1} \frac{d\phi}{d\xi} \right) + \frac{\xi}{2} \frac{d\phi}{d\xi} + d_2 (f^{\beta_2} + \phi) = 0 \end{cases} \tag{6}$$

**3. A subsolution of the problem.**

Suppose that for system of equations (6) the following conditions are fulfilled:

$$f'(0) = 0, \quad \phi'(0) = 0.$$

Then, the functions

$$\bar{f}(\xi) = B_1(a - \xi^2)^{\frac{1}{\alpha_1+m-1}}, \quad \bar{\phi}(\xi) = B_2(a - \xi^2)^{\frac{1}{\alpha_2+m-1}} \tag{7}$$

where,  $a > 0$ ,  $B_i = (\frac{c_i+d_i}{4mc^2-2c})^{c_i}$ ,  $c_i = \frac{1}{\alpha_i+m-1}$  ( $i=1,2$ ), satisfy system of equations (6). Then, the Subsolution theorem holds.

**Theorem 1.** Let

1.  $n_i(\alpha_i + m - 1) + 1 < 0$ ,
2.  $B_i^{\alpha_i+m-1}(mc_i - 1) < \frac{1}{4}$ , ( $i = 1, 2$ ),
3.  $u_0(x) \geq u_-(0, x)$ ,  $v_0(x) \geq v_-(0, x)$ ,  $x \in \mathbb{R}$ .

Then, for problem (1)-(2), a global solution exists in  $Q$  and the following estimate is satisfied:  $u(t, x) \geq u_-(t, x)$ ,  $v(t, x) \geq v_-(t, x)$ ,  $x \in \mathbb{R}$ , where

$$\begin{aligned} u_-(t, x) &= \bar{u}(t) \bar{f}(\xi), \\ v_-(t, x) &= \bar{v}(t) \bar{\phi}(\xi). \end{aligned} \tag{8}$$

**Proof.** To prove the theorem we use the method of comparison of solutions. Hence, comparing solution methods [14] it is taken the functions  $u_-(t, x)$ ,  $v_-(t, x)$ . Substituting (8) in (1) the following inequality can be obtained:

$$\begin{cases} \bar{f}^{\alpha_1} \frac{d}{d\xi} \left( \bar{f}^{m-1} \frac{d\bar{f}}{d\xi} \right) + \frac{\xi}{2} \frac{d\bar{f}}{d\xi} + d_1 (\bar{\phi}^{\beta_1} + \bar{f}) \geq 0 \\ \bar{\phi}^{\alpha_2} \frac{d}{d\xi} \left( \bar{\phi}^{m-1} \frac{d\bar{\phi}}{d\xi} \right) + \frac{\xi}{2} \frac{d\bar{\phi}}{d\xi} + d_2 (\bar{f}^{\beta_2} + \bar{\phi}) \geq 0 \end{cases} \tag{9}$$

If the specific form (7) is given for the functions  $\bar{f}(\xi)$ ,  $\bar{\phi}(\xi)$  inequality (9) can be rewritten as follows:

$$\begin{cases} 4B_1^{\alpha_1+m} c_1 (mc_1 - 1) - B_1 c_1 a + d_1 B_2^{\beta_1} (a - \xi^2)^{c_2 \beta_1 - c_1 + 1} \geq 0, \\ 4B_2^{\alpha_2+m} c_1 (mc_2 - 1) - B_2 c_2 a + d_2 B_1^{\beta_2} (a - \xi^2)^{c_1 \beta_2 - c_2 + 1} \geq 0. \end{cases} \tag{10}$$

It is known that  $\bar{f}(\xi)$ ,  $\bar{\phi}(\xi)$  are positive functions and  $d_1, d_2 > 0$ , therefore  $B_i^{\alpha_i+m-1}(mc - 1) \geq \frac{1}{4}$ ,  $i=1,2$ .

Then, according to the theorem, for the comparison of the solutions to problem (1)-(2), there exists a global solution in  $Q$  and the following estimate is satisfied:  $u(t, x) \geq u_-(t, x)$ ,  $v(t, x) \geq v_-(t, x)$ .

The proof of the theorem is completed.

#### 4. Asymptotic of the self-similar solutions.

Next, the asymptotic behavior of the self-similar solutions of the system (6) is studied. Self-similar solution of system equations (6) will be searched for in the form:

$$f(\xi) = \bar{f}(\xi)y(\eta), \quad \varphi(\xi) = \bar{\varphi}(\xi)z(\eta), \quad \eta = -\ln(a - \xi^2), \tag{11}$$

where

$$\bar{f}(\xi) = A_1(a - \xi^2)_+^{\frac{1}{\alpha_1+m-1}}, \quad \bar{\varphi}(\xi) = A_2(a - \xi^2)_+^{\frac{1}{\alpha_2+m-1}}, \text{ and } y_1(\eta), y_2(\eta) \text{ are the new functions.}$$

Then, substituting (11) into (6) for the functions the following system of nonlinear equations is obtained:

$$\begin{aligned} y^{\alpha_1} \frac{dL_1(y)}{d\eta} - a_1 y^{\alpha_1} L_1(y) + a_2 \left( \frac{dy}{d\eta} - \frac{y}{\alpha_1 + m - 1} \right) + a_3 z^{\beta_1} + a_4 y &= 0 \\ z^{\alpha_2} \frac{dL_2(z)}{d\eta} - b_1 z^{\alpha_2} L_2(z) + b_2 \left( \frac{dz}{d\eta} - \frac{z}{\alpha_2 + m - 1} \right) + b_3 y^{\beta_2} + b_4 z &= 0 \end{aligned} \tag{12}$$

Here:

$$\begin{aligned}
 a_1(\eta) &= \frac{1 - \alpha_1}{\alpha_1 + m - 1} + \frac{e^{-\eta}}{2(a - e^{-\eta})}, & a_2(\eta) &= \frac{1}{4}B_1^{1-\alpha_1-m}, & a_3(\eta) &= \frac{d_1B_2^{\beta_1}}{4B_1^{\alpha_1+m}} \frac{e^{-\eta h_1}}{(a - e^{-\eta})} & a_4(\eta) &= \frac{d_1B_1^{1-\alpha_1-m}e^{-\eta}}{4(a - e^{-\eta})}, \\
 b_1(\eta) &= \frac{1 - \alpha_2}{\alpha_2 + m - 1} + \frac{e^{-\eta}}{2(a - e^{-\eta})}, & b_2(\eta) &= \frac{1}{4}B_2^{1-\alpha_2-m}, & b_3(\eta) &= \frac{d_2B_1^{\beta_2}}{4B_2^{\alpha_2+m}} \frac{e^{-\eta h_2}}{(a - e^{-\eta})} & b_4(\eta) &= \frac{d_2B_2^{1-\alpha_2-m}e^{-\eta}}{4(a - e^{-\eta})}. \\
 c_1 &= \frac{\beta_1}{\alpha_1 + m - 1} - \frac{1 - \alpha_1}{\alpha_1 + m - 1} + 1, & c_2 &= \frac{\beta_2}{\alpha_2 + m - 1} - \frac{1 - \alpha_2}{\alpha_2 + m - 1} + 1 \\
 L_1(y) &= y^{m-1} \left( \frac{dy}{d\eta} - \frac{y}{\alpha_1 + m - 1} \right), & L_2(z) &= z^{m-1} \left( \frac{dz}{d\eta} - \frac{z}{\alpha_2 + m - 1} \right)
 \end{aligned}$$

It is supposed that  $\xi \in [\xi_0, \xi_1)$ ,  $0 < \xi_0 < \xi_1$ ,

$$\xi_1 = a^{-\frac{1}{2}}.$$

Therefore, the function  $\eta(\xi)$  has the properties:

$$\begin{aligned}
 \eta'(\xi) &> 0 \quad \text{at} \quad \xi \in [\xi_0, \xi_1), \\
 \eta_0 &= \eta(\xi_0) > 0, & \lim_{\xi \rightarrow \xi_1} \eta(\xi) &= +\infty.
 \end{aligned}$$

It follows from self-similar system of equations (12) the following limitations

$$\lim_{\eta \rightarrow +\infty} a_i(\eta) = a_i^0, \quad \lim_{\eta \rightarrow +\infty} b_i(\eta) = b_i^0 \quad (i = 1, 2, 3, 4),$$

which exist, and are finite and nonzero, that is

$$0 < |a_i^0| < +\infty, 0 < |b_i^0| < +\infty.$$

Further, we consider the solutions of the system (12), in a certain neighborhood of  $+\infty$ , which satisfy the inequalities

$$y_i(\eta) > 0, \quad z_i(\eta) > 0, \quad y' + a_{i0}(\eta)y \neq 0, \quad z' + b_{i0}(\eta)z \neq 0.$$

Here, we introduce the notations:

$$\begin{aligned}
 c_{i1} &= \frac{1 - \alpha_i}{(\alpha_i + m - 1)^2}, & c_{i2} &= \frac{B_i^{1-\alpha_i-m}}{4(\alpha_i + m - 1)}, & c_{i3} &= -\frac{d_i B_{3-i}^{\beta_i}}{4B^{\alpha_i+m}}, \\
 h_i &= \frac{\beta_i}{\alpha_{3-i} + m - 1} - \frac{1}{\alpha_i + m - 1} + 1, & b_i &= A_i B_i \quad (i = 1, 2)
 \end{aligned}$$

Let  $y(\eta) = y^0 + o(1)$ ,  $z(\eta) = z^0 + o(1)$  as  $\eta \rightarrow +\infty$  and the equality  $n_2(\alpha_2 + m - 1) = n_1(\alpha_1 + m - 1)$  is performed.

Then the following theorems hold.

**Theorem 2.** Let  $h_1 > 0, h_2 > 0$ . Then, the self-similar solution of system (1) has the asymptotic at  $x \rightarrow a^{-\frac{1}{2}}(t - T)^\gamma$ :

$$\begin{aligned}
 u_A(t, x) &\simeq b_1(T - t)^{-n_1} \left( a - \frac{\gamma x^2}{A_1^{\alpha_1+m-1}(T - t)^\gamma} \right)^{\frac{1}{\alpha_1+m-1}} (y^0 + o(1)), \\
 v_A(t, x) &\simeq b_2(T - t)^{-n_2} \left( a - \frac{\gamma x^2}{A_2^{\alpha_2+m-1}(T - t)^\gamma} \right)^{\frac{1}{\alpha_2+m-1}} (z^0 + o(1)),
 \end{aligned} \tag{13}$$

where  $0 < y^0 < +\infty$ ,  $0 < z^0 < +\infty$  and  $y^0, z^0$  are the solutions  $w_1, w_2$  for the system of nonlinear algebraic equations:

$$\begin{aligned}
 c_{11}w_1^{\alpha_1+m} + c_{12}w_1 + c_{13}w_2^{\beta_1} &= 0, \\
 c_{21}w_2^{\alpha_2+m} + c_{22}w_2 + c_{23}w_1^{\beta_2} &= 0.
 \end{aligned} \tag{14}$$

**Theorem 3.** Let  $h_1 = 0, h_2 > 0$ . Then, the self-similar solution of system (1) has the asymptotic at  $x \rightarrow a^{-\frac{1}{2}}(t - T)^\gamma$  form (13), where  $0 < y^0 < +\infty, 0 < z^0 < +\infty$  and  $y^0, z^0$  are the solutions  $w_1, w_2$  for the system of nonlinear algebraic equations:

$$\begin{aligned} c_{11}w_1^{\alpha_1+m} + c_{12}w_1 &= 0, \\ c_{21}w_2^{\alpha_2+m} + c_{22}w_2 + c_{23}w_1^{\beta_2} &= 0. \end{aligned} \tag{15}$$

**Theorem 4.** Let  $h_1 > 0, h_2 = 0$ . Then, the self-similar solution of system (1) has the asymptotic at  $x \rightarrow a^{-\frac{1}{2}}(t - T)^\gamma$  form (11), where  $0 < y^0 < +\infty, 0 < z^0 < +\infty$  and  $y^0, z^0$  are the solutions  $w_1, w_2$  for the system of nonlinear algebraic equations:

$$\begin{aligned} c_{11}w_1^{\alpha_1+m} + c_{12}w_1 + c_{13}w_2^{\beta_1} &= 0, \\ c_{21}w_2^{\alpha_2+m} + c_{22}w_2 &= 0. \end{aligned} \tag{16}$$

**Theorem 5.** Let  $h_1 = 0, h_2 = 0$ . Then, the self-similar solution of system (1) has the asymptotic at  $x \rightarrow a^{-\frac{1}{2}}(t - T)^\gamma$  form (11), where  $0 < y^0 < +\infty, 0 < z^0 < +\infty$  and  $y^0, z^0$  are the solutions  $w_1, w_2$  for the system of nonlinear algebraic equations:

$$\begin{aligned} c_{11}w_1^{\alpha_1+m} + c_{12}w_1 &= 0, \\ c_{21}w_2^{\alpha_2+m} + c_{22}w_2 &= 0. \end{aligned} \tag{17}$$

**The proof.** Assuming in Eqs. (12):

$$\vartheta_1(\eta) = L_1y, \quad \vartheta_2(\eta) = L_2z, \tag{18}$$

the following identity is obtained:

$$v_1(\eta) = y^{m-1} \left( \frac{dy}{d\eta} - \frac{y}{\alpha_1 + m - 1} \right), \quad v_2(\eta) = z^{m-1} \left( \frac{dz}{d\eta} - \frac{z}{\alpha_2 + m - 1} \right). \tag{19}$$

we obtain the identity:

$$\begin{aligned} v_1'(\eta) &\equiv a_{11}(\eta)v_1(\eta) - a_{12}v_1(\eta)^{\frac{1}{m}}y^{-\alpha_1} - a_{13}z^{\beta_1}y^{-\alpha_1} + a_{14}y^{1-\alpha_1} \\ v_2'(\eta) &\equiv a_{21}(\eta)v_1(\eta) - a_{22}v_2(\eta)^{\frac{1}{m}}z^{-\alpha_2} - a_{23}y^{\beta_2}z^{-\alpha_2} + a_{24}z^{1-\alpha_2} \end{aligned} \tag{20}$$

Furthermore, consider the functions:

$$\begin{aligned} g_1(\lambda_1, \eta) &\equiv a_{11}(\eta)\lambda_1 - a_{12}\lambda_1^{\frac{1}{m}}y^{-\alpha_1} - a_{13}z^{\beta_1}y^{-\alpha_1} + a_{14}y^{1-\alpha_1}, \\ g_2(\lambda_2, \eta) &\equiv a_{21}(\eta)\lambda_2 - a_{22}\lambda_2^{\frac{1}{m}}z^{-\alpha_2} - a_{23}y^{\beta_2}z^{-\alpha_2} + a_{24}z^{1-\alpha_2}, \end{aligned} \tag{21}$$

where  $\lambda_i \in \mathbb{R}, (i = 1, 2)$ .

Let us suppose  $\frac{\beta_i}{\alpha_i+m-1} - \frac{1-\alpha_i}{\alpha_i+m-1} + 1 = 0$  ( $i=1,2$ ) then,

$$\lim_{\eta \rightarrow +\infty} a_1(\eta) = \frac{1 - \alpha_1}{\alpha_1 + m - 1}, \quad \lim_{\eta \rightarrow +\infty} a_2(\eta) = \frac{1}{4}B_1^{1-\alpha_1-m}, \quad \lim_{\eta \rightarrow +\infty} a_3(\eta) = \frac{d_1B_2^{\beta_1}}{4B_1^{\alpha_1+m}}, \quad \lim_{\eta \rightarrow +\infty} a_4(\eta) = 0,$$

$$\lim_{\eta \rightarrow +\infty} b_1(\eta) = \frac{1 - \alpha_2}{\alpha_2 + m - 1}, \quad \lim_{\eta \rightarrow +\infty} b_2(\eta) = \frac{1}{4}B_2^{1-\alpha_2-m}, \quad \lim_{\eta \rightarrow +\infty} b_3(\eta) = \frac{d_2B_1^{\beta_2}}{4B_2^{\alpha_2+m}}, \quad \lim_{\eta \rightarrow +\infty} b_4(\eta) = 0.$$

and the functions  $g_i(\lambda_i, \eta)$  ( $i = 1, 2$ ) preserve sign on some interval  $[\eta_1, +\infty) \subset [\eta_0, +\infty)$  for every fixed value  $\lambda_i$  ( $i = 1, 2$ ).

Therefore, the functions  $g_i(\lambda_i, \eta)$  ( $i = 1, 2$ ) for all  $\eta \in [\eta_1, +\infty)$  satisfies one of the inequalities:

$$g_i(\lambda_i, \eta) > 0 \quad \text{or} \quad g_i(\lambda_i, \eta) < 0 \quad (i = 1, 2). \tag{22}$$

Let us assume that for the functions  $v_i(\eta)$  ( $i = 1, 2$ ) the limit at  $\eta \rightarrow +\infty$  does not exist. Consider the case, where one of the inequalities (22) are fulfilled. By force of the functionB's  $v_i(\eta)$  ( $i = 1, 2$ ) oscillation, the straight line  $\bar{v}_i = \lambda_i$  ( $i = 1, 2$ ) and its graph intersects an infinite number of times in the interval  $[\eta_1, +\infty)$  ( $i = 1, 2$ ). However, this is impossible, since the interval  $[\eta_1, +\infty)$  ( $i = 1, 2$ ) is justly one of the inequalities (21) and therefore, it follows from Eq. (22), that graph of the functions  $v_i(\eta)$  ( $i = 1, 2$ ) intersects the straight line  $\bar{v}_i = \lambda_i$  ( $i = 1, 2$ ) only once in the interval  $[\eta_1, +\infty)$  ( $i = 1, 2$ ). Therefore, for the functions  $v_i(\eta)$  ( $i = 1, 2$ ) the limit at  $\eta \rightarrow +\infty$  exists.

Assuming the functions  $v_i(\eta)$  ( $i = 1, 2$ ) are defined in accordance with Eq. (18) and have a limit at  $\eta \rightarrow +\infty$ , one can show that  $y'_i(\eta)$  ( $i = 1, 2$ ) has a limit at  $\eta \rightarrow +\infty$ , which is equal to zero. Then,

$$v_1(\eta) = y^{m-1} \left( \frac{dy}{d\eta} - \frac{y}{\alpha_1 + m - 1} \right) = a_1^0 (y^0)^m + o(1),$$

$$v_2(\eta) = z^{m-1} \left( \frac{dz}{d\eta} - \frac{z}{\alpha_2 + m - 1} \right) = b_1^0 (z^0)^m + o(1),$$

and by (19) derivative of functions  $v_i(\eta)$  ( $i = 1, 2$ ) has a limit at  $\eta \rightarrow +\infty$ , which obviously equals zero. Consequently,

$$\lim_{\eta \rightarrow +\infty} (a_{11}(\eta)v_1(\eta) - a_{12}v_1(\eta)^{\frac{1}{m}}y^{-\alpha_1} - a_{13}z^{\beta_1}y^{-\alpha_1} + a_{14}y^{1-\alpha_1}) = 0,$$

$$\lim_{\eta \rightarrow +\infty} (a_{21}(\eta)v_1(\eta) - a_{22}v_2(\eta)^{\frac{1}{m}}z^{-\alpha_2} - a_{23}y^{\beta_2}z^{-\alpha_2} + a_{24}z^{1-\alpha_2}) = 0.$$

It is easy to see that the system (18) has a solution  $(y_1(\eta), y_2(\eta))$  with a finite non-zero limit at  $\eta \rightarrow +\infty$  necessary to comply with the conditions of Theorem 2, 3, 4, 5.

Then the compactly supported solution of the problem (6) has an asymptotic of the form (13) as  $\xi \rightarrow a^{\frac{1}{2}}$ . Thus, the theorem is proved.

### CONCLUSION

The Cauchy problem for a non-divergent nonlinear parabolic system with an absorption term describing combustion processes was studied using the comparison method. A subsolution corresponding to the considered problem was constructed, and the lower bounds for the main solution were obtained. Asymptotic representations of self-similar solutions for this system were then obtained, and their asymptotic behavior was analyzed. It was shown that the coefficients of the main terms of asymptotic of solution satisfy to some system of a nonlinear algebraic equation. These results highlight the essential influence of nonlinear diffusion and heat generation on the combustion dynamics and the evolution of temperature. The obtained results contribute to the theoretical development of nonlinear parabolic systems modeling heat transfer and combustion processes.

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### REFERENCES

1. Shu-Yu Hsu. Asymptotic behaviour of blow-up solutions of the fast diffusion equation. *Nonlinear Differential Equations and Applications NoDEA*, 2023, P.71-101.
2. Yan Leng, Yuanyuan Nie, Qian Zhou. Asymptotic behavior of solutions to a class of coupled nonlinear parabolic systems. *Boundary Value Problems*, 2019,P. 68-79.

3. Yohei Fujishima, Kazuhiro Ishige. Blowing Up Solutions for Nonlinear Parabolic Systems with Unequal Elliptic Operators. *Dynamics and Differential Equations*, 2019, P. 1219-1231.
4. Matyakubov A. S., Raupov D. R. Explicit estimate for blow-up solutions of nonlinear parabolic systems of non divergence form with variable density. *Aip Conference Proceedings*. 2023, 2781, 020055.
5. Matyakubov A. S., Raupov D. R. On Some Properties of the Blow-Up Solutions of a Nonlinear Parabolic System Non-divergent Form with Cross-Diffusion. *Scopus: Springer Nature Switzerland AG*. 2021. 289-303 p.
6. Aripov M. and Raimbekov J., "The critical curves of a doubly nonlinear parabolic equation in non-divergent form with a source and nonlinear boundary flux," *Journal of Siberian Federal University 12(1)*, 2019, 112-124.
7. Mirsaid Aripov, Alisher Matyakubov. To the qualitative properties of solution of system equations not in divergence form. *International Journal of Innovative Science, Engineering Technology*. 2016, p. 533-537.
8. Aripov M., Atabaev O., Al-Marashi A. On the behavior of solutions of a doubly nonlinear degenerate parabolic system with nonlinear sources and absorptions with variable densities. *Bulletin of the Karaganda University*, 2025, No.1(117) pp. 12-23.
9. A.S. Matyakubov. Finite Speed of the Perturbation Distribution and Asymptotic Behavior of the Solutions of a Parabolic System not in Divergence Form. *Universal Journal of Computational Mathematics 5 (3)*, 2017, pp. 57-67.
10. Aripov M., Matyakubov A. To the properties of the solutions of a cross-diffusion parabolic system not in divergence form. *Universal Journal of Computational Mathematics*, 2017, 5(1), pp. 1-7.
11. Aripov M., Matyakubov A On the asymptotic behavior solutions of nonlinear parabolic systems of equations not in divergence form. *The KazNU Journal*. 2015, 3 (86). 275-282.
12. Aripov M., Matyakubov A. S. Self-similar solutions of a cross-diffusion parabolic system with variable density: explicit estimates and asymptotic behaviour. *Nanosystems: Physics, Chemistry, Mathematics*, 2017, 8(1), pp. 5-12.
13. Арипов М., Садуллаева С. Компьютерное моделирование нелинейных процессов диффузии. *Университет, Ташкент*, 2020. 750 p.
14. Samarskii A. A., Galaktionov V. A., Kurdyumov S. P., Mikhailov A.P. Blow-up in Quasilinear Parabolic Equations. *Walter de Grueter, Berlin*, 1995, 4, 535 p.
15. Aripov M., Sadullaeva S. A. Qualitative properties of solutions of a doubly nonlinear reaction-diffusion system with a source. *J Appl Math Phys*, 2015. 3(9):1090-1099
16. Aripov M., Bobokandov M. M. Blow-up analysis for a doubly nonlinear parabolic non-divergence form equation with source term. *Bulletin of the Institute of Mathematics*, 2022. Vol. 5, №4, ISSN-2181-9483.
17. Aripov M., Matyakubov A. S., Imomnazarov BK The Cauchy problem for a nonlinear degenerate parabolic system in non-divergence form. *Math Notes NEFU*, 2020. 27(3):27-38
18. Aripov M., Matyakubov A. S., Xasanov J. O. Global solvability and explicit estimation of solutions of a cross-diffusion parabolic system in non-divergent form with a source and variable density. *Bulletin of the Institute of Mathematics*, 2022, Vol. 5, №4, ISSN-2181-9483.
19. Rakhmonov Z. R., Khaydarov A., Urunbaev J. E. (2020) Global existence and nonexistence of solutions to a cross diffusion system with nonlocal boundary conditions. *Math Stat* 8(4):404 409
20. Rakhmonov Z. R, Tillaev A. I. On the behavior of the solution of a nonlinear poly tropic filtration problem with a source and multiple nonlinearities. *Nanosyst Phys Chem Math*, 2018. 9(3):323-329

**REZYUME**

Ushbu maqolada nodivergent ko‘rinishdagi nochiziqli parabolik tenglamalar sistemasining Koshi masalasi uchun avtomodel yechimlar asimptotikasi topildi va quyi yechim uchun baholar olindi.

**Kalit so‘zlar:** matematik modellashtirish, avtomodel yechimlar, parabolik tenglamalar sistemasi, nodivergent shakl, quyi yechim, asimptotik baholash.

**РЕЗЮМЕ**

В данной статье найдена асимптотика автомодельных решений задачи Коши для системы нелинейных параболических уравнений в недивергентной форме и получены оценки для нижнего решения.

**Ключевые слова:** математическое моделирование, автомодельные решения, система параболических уравнений, недивергентная форма, нижняя оценка, асимптотическое поведение.