UDC 519.214.5

# Central limit theorem for autoregressive processes with values in $L_p[0,1]$

**Muxtorov Ibrohim G‘aybulla o‘g‘li**
V.I.Romanovskiy Institute of Mathematics, Uzbekistan Academy of Sciences, Tashkent, Uzbekistan
igmuxtorov@gmail.com

**RESUME**

In this paper, first-order autoregressive processes with values in the function space $L_p[0,1]$ are examined. Under two weak-dependence assumptions imposed on the innovations, a central limit theorem for these AR(1)-processes is established.

***Key words:*** Central limit theorem, autoregressive process, mixing condition.

## 1. Introduction

Autoregressive dynamics in infinite–dimensional settings have been studied extensively over the past years, and various aspects of such models have been explored in a number of works; see, for instance, [1]-[6]. Earlier research, including [3] and [4], focused on AR(1)-processses under independence or under weakly orthogonality assumptions. Later, the authors of [5] investigated the asymptotic behavior of the empirical mean and covariance operator when the noise variables exhibit weak dependence. Models of AR(1) in the space $L_p[0,1]$ naturally arise in functional data analysis, where observations themselves are functions rather than real numbers. Several motivating examples and applications illustrating this setting can be found in reference [3].

In the present work, we consider an AR(1)-process $\{X_n,\, n\in\mathbb{Z}\}$ defined through the recursion

$$X_n - m = T\left(X_{n-1}-m\right)+\varepsilon_n,\ n\in\mathbb{Z}$$

where $T: L_p[0,1]\to L_p[0,1]$ is a bounded linear operator, $m\in L_p[0,1]$ is a fixed element (interpreted as the mean), and $\{\varepsilon_n,\ n\in\mathbb{Z}\}$ is a sequence of innovations. Throughout this paper, $\|T\|$ denotes the norm of the linear operator $T: L_p[0,1]\to L_p[0,1]$, and $I: L_p[0,1]\to L_p[0,1]$ stands for the identity operator.

Throughout the paper we assume that the innovation sequence satisfies certain mixing conditions. Let $\{X_n(t),\, n\in\mathbb{Z}\}$ be an $L_p[0,1]$–valued stochastic process, and let $\mathcal{F}_r^s$ denote the $\sigma$-algebra generated by the random vector $(X_r(t),\dots,X_s(t))$. The classical mixing coefficients are defined as follows:

$$\alpha(n)=\sup\left\{|P(GH)-P(G)P(H)|:\ G\in\mathcal{F}_{-\infty}^{k},\ H\in\mathcal{F}_{k+n}^{\infty},\ k\in\mathbb{Z}\right\}$$

$$\rho(n)=\sup\left\{\frac{|E\left(\xi-E\xi\right)\left(\eta-E\eta\right)|}{E^{1/2}\left(\xi-E\xi\right)^2E^{1/2}\left(\eta-E\eta\right)^2}:\ \xi\in L_2\left(\mathcal{F}_{-\infty}^{k}\right),\ \eta\in L_2\left(\mathcal{F}_{k+n}^{\infty}\right),\ k\in\mathbb{Z}\right\}$$

where $L_2\left(\mathcal{F}_a^b\right)$ is family of square integrable $\mathcal{F}_a^b$-measurable random variables.

The sequence $\{X_n(t),\ n\in\mathbb{Z}\}$ is called $\alpha$–mixing or $\rho$–mixing if $\alpha(n)\to 0$ or $\rho(n)\to 0$ as $n\to\infty$, respectively.

The author in [7] introduced modifications to the coefficients discussed above. For the $\{X_n(t),\ n\in\mathbb{Z}\}$, the modification of $\alpha$-mixing coefficients is defined as follows:

$$\alpha_m(n)=\sup_{\Pi_m}\sup\left\{|P(GH)-P(G)P(H)|: G\in\mathcal{F}_{-\infty}^{k}(m), H\in\mathcal{F}_{k+n}^{\infty}(m),\ k\in\mathbb{Z}\right\}$$

where $\mathcal{F}_r^s(m)$ is the $\sigma$-algebra generated by random vector $(\Pi_m X_r(t),...,\Pi_m X_s(t))$ and $\Pi_m: L_p[0,1]\to\mathbb{R}^m$ is a projective operator, i.e.,

$$\Pi_m X_j(t)=\left(X_j(t_1),...,X_j(t_m)\right),\ t_j\in[0,1].$$

The coefficients $\rho_m(n)$ are defined analogously.

The sequence $\{X_n(t),\ n\in\mathbb{Z}\}$ is called $\alpha_m$–mixing or $\rho_m$–mixing if $\alpha_m(n)\to 0$ or $\rho_m(n)\to 0$ as $n\to\infty$, respectively, for any fixed $m=1,2,....$ It is important to note that, in general, $\alpha_m$-mixing does not necessarily imply $\alpha$-mixing.

The properties of mixing coefficients were thoroughly analyzed in [8]-[10].

**2. Main results**

It is assumed in Theorems 1, 3 and 5 that there exist the real numbers $u>0$ and $0<v<1$ satisfying

$$\left\|T^j\right\|\le uv^j,\ j\ge 0. \tag{6}$$

**Theorem 1.** *Let $\{X_n,\ n\in\mathbb{Z}\}$ be an AR(1)-process and $\{\varepsilon_n,\ n\in\mathbb{Z}\}$ be a $\rho_m$-mixing strictly stationary centered sequence of $L_p$-valued $1<p<2$ random variables satisfying $E\left\|\varepsilon_1\right\|^2<\infty$. If the following conditions hold*

$$E\left|\varepsilon_1(t)\right|^2<\infty,\ t\in[0,1]$$

$$\sum_{n=1}^{\infty}\rho_m(2^n)<\infty,\ \rho_m(1)<1,\ m=1,2,...$$

*$E\left|\varepsilon_1(t+h)-\varepsilon_1(t)\right|^2\le f(h)$ for $0\le h<1$, $0\le t\le 1-h$, for some function $f(\cdot)$ such that $f(h)\to 0$ as $h\to 0$. Then the following weak convergence holds*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_i\Rightarrow (I-T)^{-1}N_K(t)$$

*where $N_K(t)$ is $L_p[0,1]$-valued Gaussian random variable with mean zero and covariance function $K(x,y)=\lim\limits_{n\to\infty}cov\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i(x),\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i(y)\right)$, where $x,y\in[0,1]$.*

***Proof of Theorem 1.*** We begin by simplifying the recursion through a centering. Since subtracting the mean does not affect weak convergence, we may assume without loss of generality that $m=0$. Hence, the process $\{X_n,\ n\in\mathbb{Z}\}$ satisfies

$$X_n=T(X_{n-1})+\varepsilon_n,\ n\in\mathbb{Z}.$$

A key decomposition, which appears for instance in [4], is given by

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}X_k=(I-T)^{-1}\frac{1}{\sqrt{n}}\sum_{k=1}^{n}\varepsilon_k-\frac{1}{\sqrt{n}}(I-T)^{-1}\sum_{i=1}^{n}T^i(\varepsilon_{n-i+1})+\frac{1}{\sqrt{n}}\sum_{k=1}^{n}T^k(X_0). \tag{7}$$

Since the assumption (1) we can estimate

$$\left\|\sum_{k=1}^{n}T^k(X_0)\right\|\le\|X_0\|\sum_{k=1}^{\infty}uv^k<\infty.$$

Consequently,

$$\frac{1}{\sqrt{n}}\sum_{k=1}^{n}T^k(X_0)\longrightarrow 0\qquad\text{in probability.} \tag{8}$$

For the second term in (2), we use the triangle inequality:

$$\left\|\frac{1}{\sqrt{n}}\sum_{i=1}^{n}T^i(\varepsilon_{n-i+1})\right\|\le\frac{1}{\sqrt{n}}\max_{1\le i\le n}\|\varepsilon_i\|\sum_{i=1}^{n}\|T^i\|.$$

The innovations satisfy $E\|\varepsilon_1\|^2<\infty$, hence

$$\frac{1}{\sqrt{n}}\max_{1\le i\le n}\|\varepsilon_i\|\longrightarrow 0\qquad\text{almost surely.} \tag{9}$$

Since $\sum_{i=1}^{\infty} \|T^i\| < \infty$ and (4), it follows that

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} T^i(\varepsilon_{n-i+1}) \longrightarrow 0 \qquad \text{in probability.} \tag{10}$$

Theorem 2 given below guarantees the weak convergence

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} \varepsilon_k \;\Rightarrow\; N_K(t), \tag{11}$$

where $N_K(t)$ is an $L^p[0,1]$–valued Gaussian random element with covariance function

$$K(x,y) = \lim_{n\to\infty} \operatorname{cov}\left( \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \varepsilon_k(x), \frac{1}{\sqrt{n}} \sum_{k=1}^{n} \varepsilon_k(y) \right).$$

Combining (3), (5), and (6) with the representation (2), the continuous mapping theorem yields

$$\frac{1}{\sqrt{n}} \sum_{k=1}^{n} X_k \;\Rightarrow\; (I-T)^{-1} N_K(t).$$

**Theorem 2.** [11] *Let $\{\varepsilon_n(t),\ n \in \mathbb{Z}\}$ be a strictly stationary centered sequences of random variables with values in $L_p[0,1]$, $1 < p < 2$. If the following conditions hold*

$$E\,|\varepsilon_1(t)|^2 < \infty,\ t \in [0,1]$$

$$\sum_{n=1}^{\infty} \rho_m(2^n) < \infty,\ \rho_m(1) < 1,\ m = 1,2,...$$

*$E\,|\varepsilon_1(t+h) - \varepsilon_1(t)|^2 \le f(h)$ for $0 \le h < 1$, $0 \le t \le 1-h$, for some function $f(\cdot)$ such that $f(h) \to 0$ as $h \to 0$. Then the following weak convergence holds*

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i(t) \Rightarrow N_K(t)$$

*$N_K(t)$ is $L_p[0,1]$-valued Gaussian random variable with mean zero and covariance function $K(x,y) = \lim_{n\to\infty} cov\,(S_n(x),\ S_n(y))$, where $x, y \in [0,1]$.*

□

In the next theorem, we consider the case $L_p$, $2 \le p < \infty$.

**Theorem 3.** *Let $\{X_n,\ n \in \mathbb{Z}\}$ be an AR(1)-process and $\{\varepsilon_n,\ n \in \mathbb{Z}\}$ be a $\rho_m$-mixing strictly stationary centered sequence of $L_p$-valued $2 \le p < \infty$ random variables satisfying $E\,\|\varepsilon_1\|^2 < \infty$. Assume that the following conditions hold for some $\alpha > 0$*

$$E\,|\varepsilon_1(t)|^2 < \infty,\ t \in [0,1],$$

$$\sum_{n=1}^{\infty} \rho_m{}^{\frac{2}{p+\alpha}}(2^n) < \infty,\ \rho_m(1) < 1,\ m = 1,2,...$$

*$E\,|\varepsilon_1(t+h) - \varepsilon_1(t)|^{p+\alpha} \le f(h)$ for $0 \le h < 1$, $0 \le t \le 1-h$, for some function $f(\cdot)$ such that $f(h) \to 0$ as $h \to 0$.*
*Then the following weak convergence holds*

$$\frac{1}{\sqrt{n}} \sum_{i=1}^{n} X_i \Rightarrow (I-T)^{-1} N_K(t)$$

*where $N_K(t)$ is $L_p[0,1]$-valued Gaussian random variable with mean zero and covariance function $K(x,y) = \lim_{n\to\infty} cov\left(\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i(x),\ \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \varepsilon_i(y)\right)$, where $x, y \in [0,1]$.*

***Proof of Theorem 3.*** The proof of Theorem 3 is analogous to that of Theorem 1, except that Theorem 4 given below is used in place of Theorem 2; therefore, we omit the details.

**Theorem 4.** [11] *Let $\{\varepsilon_n(t),\ n\in\mathbb{Z}\}$ be a strictly stationary centered sequences of random variables with values in $L_p[0,1]$, $2\le p<\infty$. If for some $\alpha>0$*

$$E\left|\varepsilon_1(t)\right|^2<\infty,\ t\in[0,1],$$

$$\sum_{n=1}^{\infty}\rho_m{}^{\frac{2}{p+\alpha}}(2^n)<\infty,\ \rho_m(1)<1,\ m=1,2,...$$

$E\left|\varepsilon_1(t+h)-\varepsilon_1(t)\right|^{p+\alpha}\le f(h)$ *for* $0\le h<1$, $0\le t\le 1-h$, *for some function* $f(\cdot)$ *such that* $f(h)\to 0$ *as* $h\to 0$.

*Then the following weak convergence holds*

$$S_n(t)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i(t)\Rightarrow N_K(t)$$

*where $N_K(t)$ is $L_p[0,1]$-valued Gaussian random variable with mean zero and covariance function* $K(x,y)=\lim\limits_{n\to\infty}cov\left(S_n(x),\ S_n(y)\right)$, *where* $x,y\in[0,1]$. □

In the following theorem, the case where the sequence of innovations is $\alpha_m$-mixing is considered.

**Theorem 5.** *Let $\{X_n,\ n\in\mathbb{Z}\}$ be an AR(1)-process and $\{\varepsilon_n,\ n\in\mathbb{Z}\}$ be an $\alpha_m$-mixing strictly stationary centered sequence of $L_p$-valued $1\le p\le 2$ random variables satisfying $E\left\|\varepsilon_1\right\|^2<\infty$. Assume that the following conditions hold for some $\delta>0$*

$$\sum_{n=1}^{\infty}\left(\alpha_m(n)\right)^{\frac{\delta}{2+\delta}}<\infty,\ m=1,2,...$$

$E\left|\varepsilon_1(t+h)-\varepsilon_1(t)\right|^{2+\delta}\le f(h)$ *for* $0\le h<1$, $0\le t\le 1-h$, *for some function* $f(\cdot)$ *such that* $f(h)\to 0$ *as* $h\to 0$.

*for all* $t\in[0,1]$

$$E\left|\varepsilon_1(t)\right|^{2+\delta}<\infty,$$

$$\lim_{n\to\infty}E\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i(t)\right)^2>0.$$

*Then the following weak convergence holds*

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}X_i\Rightarrow\left(I-T\right)^{-1}N_K(t)$$

*where $N_K(t)$ is $L_p[0,1]$-valued Gaussian random variable with mean zero and covariance function* $K(x,y)=\lim\limits_{n\to\infty}cov\left(\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i(x),\ \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i(y)\right)$, *where* $x,y\in[0,1]$.

***Proof of Theorem 5.*** The proof of Theorem 5 is analogous to that of Theorem 1, except that Theorem 6 given below is used in place of Theorem 2; therefore, we omit the details.

**Theorem 6.** [11] *Let $\{\varepsilon_n(t),\ n\in\mathbb{Z}\}$ be a strictly stationary centered sequences of random variables with values in $L_p[0,1]$, $1<p\le 2$ and for some $\delta>0$*

$$\sum_{n=1}^{\infty}\left(\alpha_m(n)\right)^{\frac{\delta}{2+\delta}}<\infty,\ m=1,2,...$$

$E\left|\varepsilon_1(t+h)-\varepsilon_1(t)\right|^{2+\delta}\le f(h)$ *for* $0\le h<1$, $0\le t\le 1-h$, *for some function* $f(\cdot)$ *such that* $f(h)\to 0$ *as* $h\to 0$.

*for all* $t \in [0,1]$

$$E\left|\varepsilon_1(t)\right|^{2+\delta} < \infty,$$

$$\lim_{n\to\infty} ES_n^2(t) > 0.$$

*Then the following weak convergence holds*

$$S_n(t) = \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\varepsilon_i(t) \Rightarrow N_K(t)$$

*where* $N_K(t)$ *is* $L_p[0,1]$*-valued Gaussian random variable with mean zero and covariance function* $K(x,y) = \lim\limits_{n\to\infty} cov\left(S_n(x),\ S_n(y)\right)$*, where* $x, y \in [0,1]$.

□

## REFERENCES


1. Allam A., Mourid T., Geometric absolute regularity of Banach space-valued autoregressive processes. *Statistics and Probability Letters.* **60** (2002), 241-252.

2. Bensaber F., Mourid T., Functional autoregressive process with seasonality. *Communications in Statistics - Theory and Methods.* **60** (2022), 7131-7145.

3. Bosq D., Linear Processes in Function Spaces: Theory and Applications (book). *Springer, New York* (2000).

4. Bosq D., Estimation of mean and covariance operator of autoregressive processes in Banach spaces. *Statist. Infer. Stoch. Process.* **5** (2002), 287-306 .

5. Dehling H., Sharipov O.Sh., Estimation of mean and covariance operator for Banach space valued autoregressive processes with dependent innovations. *Statist. Infer. Stoch. Process.* **8** (2005), 137-149.

6. Parvardeh A., Mohammadi Jouzdani N., Mahmoodi S., Soltani A.R., First order autoregressive periodically correlated model in Banach spaces: Existence and central limit theorem. *J. Math. Anal. Appl.* **449** (2017), 756-768.

7. Zhurbenko I.G., On mixing conditions for random processes with values in a Hilbert space. *Soviet Math. Dokl.* **30** (1984), 465-467 (In Russian)

8. Bradley R.C., Introduction to Strong Mixing Conditions (book). *Kendrick Press, Heber City(Utah)* (2007).

9. Doukhan P., Mixing: Properties and Examples (book). *Springer, New York* (2012).

10. Ibragimov I.A., Linnik, Yu. V., Independent and stationary sequences of random variables (book). *Wolters-Noordhoff, Groningen* (1971).

11. Sharipov O.Sh., Muxtorov, I.G., Central limit theorem for weakly dependent random variables with values in $L_p[0,1]$ space. *Uzbek Mathematical Journal.* **67** (2023), 166-171.


## REZYUME


Ushbu maqolada $L_p[0,1]$ funksiyalar fazosida qiymat qabul qiluvchi birinchi tartibli avtoregressiv jarayonlar o‘rganilgan. Innovatsiyalarga qo‘yilgan ikki kuchsiz bog‘liqlik shartlarida ushbu AR(1) jarayonlar uchun markaziy limit teorema isbotlangan.



***Kalit so‘zlar:*** Markaziy limit teorema, avtoregressiv jarayon, qorishmalilik sharti.

## РЕЗЮМЕ

В данной работе рассматриваются авторегрессионные процессы первого порядка, принимающие значения в функциональном пространстве $L_p[0,1]$. При двух предположениях о слабой зависимости, наложенных на инновации, устанавливается центральная предельная теорема для этих AR(1)-процессов.


***Ключевые слова:*** Центральная предельная теорема, авторегрессионный процесс, условие перемешивания.