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ON THE CONSTRUCTION OF GEODESIC MAPPINGS FOR SURFACES OF REVOLUTION

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RESUME

This work is devoted to the study of geodesic mappings of surfaces of revolution. A mapping is constructed that sends the geodesics of a given surface of revolution to the geodesics of another surface of revolution, and it is proven that the constructed mapping is a geodesic mapping.

Key words: Geodesic mapping, surfaces of revolution, Christoffel symbols, deformation tensor, Riemannian manifolds, paraboloid of revolution.

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1. Introduction

The theory of geodesic lines and geodesic mappings is interesting from an applied standpoint and for modern research. This is because the motion of many types of mechanical systems, as well as bodies or particles in gravitational and electromagnetic fields within a continuous medium, often occurs along trajectories that can be regarded as geodesic lines in certain spaces of three or more dimensions. These spaces are defined by the energy regimes under which the processes occur. On this basis, two spaces that admit a geodesic mapping onto each other describe processes that proceed under equivalent external loads along the same "trajectories,"but under different energy regimes. Consequently, one of these processes can be modeled using the other.

The geodesic mapping problem was first raised by Beltrami in 1865—though not in its full generality, but specifically for the case in which a surface (a two-dimensional Riemannian manifold) is mapped onto the Euclidean plane. His results may be viewed as an initial impetus for the later recognition and development of Non-Euclidean Geometry, founded by Lobachevsky, Bolyai, and Gauss. Further research in this direction, based on the methods of the Mikes school, has been carried out. In particular, the theory of geodesic mappings of Riemannian and pseudo-Riemannian spaces, affine-connected spaces, and manifolds endowed with additional geometric structures has been systematically developed [1]. Geodesic mappings of pseudosymmetric spaces—classes of spaces more general than spaces of constant curvature and symmetric spaces—are studied in [2]. In [3], some results concerning almost geodesic curves and geodesic mappings and transformations are presented. It is proved that any mapping that sends all almost geodesic curves to almost geodesic curves is itself geodesic. Since under geodesic mappings and transformations almost geodesic curves are also preserved, the present work is devoted to the study of geodesic mappings of surfaces of revolution. A mapping is constructed that sends the geodesics of a given surface of revolution to the geodesics of another surface of revolution, and it is proven that the constructed mapping is a geodesic mapping.

The main objective is to analyze the properties and classify geodesic mappings, that is, mappings which transform the geodesic lines of one surface into the geodesic lines of another. Surfaces of revolution and their geodesic lines play an important role in differential geometry, mathematical physics, and related applied fields such as the theory of relativity, optics, and cartography.

Let M and N be smooth manifolds of dimension n with affine connections.

Definition 1 [4]. *A geodesic mapping f of a manifold M onto N is a one-to-one correspondence between their points such that every geodesic line of the manifold M is mapped to a geodesic line of the manifold N.*

Let us consider these manifolds in a general coordinate system $x^1, x^2, \dots, x^n$ with respect to the mapping f. Denote the components of the connection objects of the manifolds M and N at the corresponding points $Q(x)$ and $\bar{Q}(x)$ by $\Gamma^h_{ij}(x)$ and $\bar{\Gamma}^h_{ij}(x)$, assuming they are symmetric, and set

$$\bar{\Gamma}^h_{ij}(x) = \Gamma^h_{ij}(x) + P^h_{ij}(x) \quad (h, i, j = 1, 2, \dots, n) \tag{1}$$

where P_{ij}^h is the deformation tensor.

Theorem [4]. *For a mapping f of an affine connection manifold M onto an affine connection manifold N to be geodesic, it is necessary and sufficient that the connection deformation tensor P_{ij}^h of the mapping f can be represented in the form*

$$P_{ij}^h(x) = \psi_i(x)\delta_j^h + \psi_j(x)\delta_i^h, \tag{2}$$

where δ_i^h are the Kronecker symbols, and ψ_i is some covariant tensor. Conditions (2) are tensorial and therefore invariant with respect to the choice of the common coordinate system $x^1, x^2, \ldots, x^n$ for the mapping f. Based on these conditions, equations (1) take the form

$$\bar{\Gamma}_{ij}^h(x) = \Gamma_{ij}^h(x) + \psi_i(x)\delta_j^h + \psi_j(x)\delta_i^h. \tag{3}$$

From (3), it is clear that the inverse mapping f^{-1}, which is the inverse of the geodesic mapping f of manifold M onto manifold N, is itself a geodesic mapping, and it corresponds to the tensor $-\psi_i$, i.e.,

$$\Gamma_{ij}^h(x) = \bar{\Gamma}_{ij}^h(x) - \psi_i(x)\delta_j^h - \psi_j(x)\delta_i^h.$$

Remark. *When the covector $\psi_i(x)$ is identically zero, the geodesic mapping f is called trivial .*

Example. Let the surface $\bar{F}$ be the image of the surface F under a homothety in three-dimensional space, and let k be the homothety coefficient. Then the metric tensors of the surfaces are related by $\bar{g}_{ij} = k^2 g_{ij}$. Consequently, the Christoffel symbols coincide, $\bar{\Gamma}_{ij}^k = \Gamma_{ij}^k$, and a geodesic ℓ on S is mapped to a geodesic $\bar{\ell}$ on $\bar{S}$, where the parameters s on ℓ and $\bar{s}$ on $\bar{\ell}$ are related by $\bar{s} = ks + \text{const}$.

2. Preliminary notions and a proof of Theorem 1.

Having the ability to use the theory of geodesic mappings of Riemannian manifolds, we turn our attention to a special type of Riemannian manifolds - surfaces of revolution.

Consider a surface of revolution F with the parametric equation

$$\vec{r}(u,v) = \{r(u)\cos v,\ r(u)\sin v,\ z(u)\}. \tag{4}$$

The first fundamental form of a surface of revolution is:

$$ds^2 = g_{11}(u)du^2 + g_{22}(u)dv^2 \tag{5}$$

where $g_{11}(u)$ and $g_{22}(u)$ are non-zero functions, and

$$g_{11}(u) = r_u^2(u) + z_u^2(u) \quad \text{and} \quad g_{22}(u) = r^2(u).$$

Consider a curve γ, lying on the surface of revolution F, given by the equation

$$\vec{r}(u) = \{r(u)\cos v(u),\ r(u)\sin v(u),\ z(u)\}.$$

Then the arc length of the curve γ is:

$$s = \int_{t_1}^{t_2} \sqrt{g_{11}(u) + g_{22}(u)(v_u)^2}\, du.$$

Geodesics are curves that provide an extremum for the length functional and satisfy the Euler–Lagrange equation[5, p.451]:

$$\left\{\frac{d}{du}\frac{\partial}{\partial v_u} - \frac{\partial}{\partial v}\right\}\sqrt{g_{11}(u) + g_{22}(u)\left(\frac{dv}{du}\right)^2} = 0,$$

Since $\sqrt{g_{11}(u) + g_{22}(u)\left(\frac{dv}{du}\right)^2}$ does not depend explicitly on v, we obtain:

$$\frac{g_{22}\frac{dv}{du}}{\sqrt{g_{11}+g_{22}\left(\frac{dv}{du}\right)^2}}=c, \tag{6}$$

where c is a constant. From (6) it follows:

$$\frac{dv(u)}{du}=\pm c\frac{\sqrt{g_{11}}}{\sqrt{g_{22}\left(g_{22}-c^2\right)}};$$

$$v(u)=v_0\pm c\int\sqrt{\frac{g_{11}}{g_{22}\left(g_{22}-c^2\right)}}\,du. \tag{7}$$

Consider a paraboloid of revolution F given by the equations

$$\overrightarrow{r}(u,v)=\left\{\sqrt{u}\cos v,\ \sqrt{u}\sin v,\ u\right\}.$$

The first fundamental form of the paraboloid of revolution F is:

$$ds^2=\left(1+\frac{1}{4u}\right)du^2+u\,dv^2.$$

Using formula (7), we find the differential equation for the geodesic lines on the paraboloid of revolution:

$$v=v_0\pm\arcsin\left(\frac{u-c^2}{u(1+4c^2)}\right)^{\frac{1}{2}}+2c\ln\left(\frac{2\sqrt{u-c^2}+\sqrt{4u+1}}{\sqrt{1+4c^2}}\right). \tag{8}$$

For function (8), the point $\{u=c^2, v=v_0\}$ is a turning point. Function (8) can be considered as describing one geodesic line that descends down the paraboloid, turns at the turning point, and then ascends again, intersecting itself infinitely many times.

Let $\gamma(u)=\{\sqrt{u}\cos v(u),\sqrt{u}\sin v(u),u\}$ be a curve on the paraboloid of revolution, where u is some parameter. Then the equation of the geodesic lines has the form:

$$\gamma(u):\begin{cases} x=\sqrt{u}\cos\left(v_0\pm\left[\arcsin\sqrt{\dfrac{u-c^2}{u(1+4c^2)}}+2c\ln\left(\dfrac{2\sqrt{u-c^2}+\sqrt{4u+1}}{\sqrt{1+4c^2}}\right)\right]\right)\\ y=\sqrt{u}\sin\left(v_0\pm\left[\arcsin\sqrt{\dfrac{u-c^2}{u(1+4c^2)}}+2c\ln\left(\dfrac{2\sqrt{u-c^2}+\sqrt{4u+1}}{\sqrt{1+4c^2}}\right)\right]\right)\\ z=u. \end{cases}$$

Let us now consider the structure of the geodesic mapping f of the paraboloid of revolution F, and let us denote the image of this paraboloid under this geodesic mapping by $\overline{F}$. It is proven in works [6, 7] that the images of surfaces of revolution under a geodesic mapping remain surfaces of revolution. Following from this, we can find the first fundamental form of the surface of revolution $\overline{F}$ of the form:

$$d\bar{s}^2=\frac{\left(1+c^2\right)\left(1+\frac{1}{4u}\right)}{\left(1+u\right)^2}du^2+\frac{\left(1+c^2\right)u}{1+u}dv^2, \tag{9}$$

$$v=v_0+\arcsin\left(\frac{u-c^2}{u\left(1+4c^2\right)}\right)^{\frac{1}{2}}+2c\ln\left(\frac{2\sqrt{u-c^2}+\sqrt{4u+1}}{\sqrt{1+4c^2}}\right).$$

Expression (9) can be taken as the first fundamental form of a surface of revolution $\overline{F}$ with equations:

$$\overrightarrow{\bar{r}}(u,v)=\left\{\bar{r}(u)\cos v,\bar{r}(u)\sin v,\bar{z}(u)\right\},$$

where u and v are the same parameters as u on the surface F. Where

$$\begin{cases} \bar{r}_u^2(u) + \bar{z}_u^2(u) = \dfrac{\left(1+c^2\right)\left(1+\dfrac{1}{4u}\right)}{\left(1+u\right)^2}, \\ \bar{r}^2(u) = \dfrac{\left(1+c^2\right)u}{1+u}. \end{cases}$$

Considering that $\bar{\mathbf{r}}(u, v(u)) = \bar{\mathbf{r}}(u) = \{\bar{r}(u)\cos v(u), \bar{r}(u)\sin v(u), \bar{z}(u)\}$, we obtain:

$$\bar{r}(u) = \sqrt{\frac{\left(1+c^2\right)u}{1+u}},$$

$$\bar{z}(u) = \sqrt{1+c^2}\int \frac{1}{2\left(1+u\right)}\sqrt{\frac{5+4u}{1+u}}\,du = \sqrt{1+c^2}\left[2\ln\sqrt{5+4u} + 2\sqrt{1+u} - \sqrt{\frac{5+4u}{1+u}}\right].$$

The equations of the geodesic lines on the surface $\overline{F}$ can be written in a similar way:

$$\bar{\gamma}(u): \begin{cases} \bar{x} = \bar{r}(u)\cos v(u) = \sqrt{\dfrac{\left(1+c^2\right)u}{1+u}}\cos v(u), \\ \bar{y} = \bar{r}(u)\sin v(u) = \sqrt{\dfrac{\left(1+c^2\right)u}{1+u}}\sin v(u), \\ \bar{z} = \sqrt{1+c^2}\left[2\ln\sqrt{5+4u} + 2\sqrt{1+u} - \sqrt{\dfrac{5+4u}{1+u}}\right]. \end{cases}$$

Since the function $v(u)$ is the same for the surfaces of revolution F and $\overline{F}$, we can establish a geodesic mapping between $\gamma(u)$ and $\bar{\gamma}(u)$:

$$\bar{\mathbf{f}}(u): \begin{cases} \bar{x} & = \sqrt{\frac{(1+c^2)}{1+x^2+y^2}}\ x \\ \bar{y} & = \sqrt{\frac{(1+c^2)}{1+x^2+y^2}}\ y \\ \bar{z} & = \sqrt{1+c^2}\left[2\ln\left(\sqrt{5+4z} + 2\sqrt{1+z}\right) - \sqrt{\frac{5+4z}{1+z}}\right]. \end{cases} \tag{10}$$

Using this mapping, we can find the parametric equation for the image of the paraboloid of revolution F, i.e., the surface $\overline{F}$:

$$\vec{\bar{r}}(u,v) = \left(\sqrt{\frac{(1+c^2)u}{1+u}}\cos v,\ \sqrt{\frac{(1+c^2)u}{1+u}}\sin v,\ \sqrt{1+c^2}\left[2\ln\left(\sqrt{5+4u} + 2\sqrt{1+u}\right) - \sqrt{\frac{5+4u}{1+u}}\right]\right).$$

Taking into account the existence of a geodesic mapping between surfaces of revolution, the following theorem holds:

Theorem 1. *Mapping (10) is a non-trivial geodesic mapping which sends every geodesic on the paraboloid of revolution F to a geodesic on the surface of revolution $\overline{F}$.*

Proof. To prove that the obtained mapping f is a non-trivial geodesic mapping of the surfaces F and $\overline{F}$, we use the necessary and sufficient condition from namely:

$$P_{ij}^h = \bar{\Gamma}_{ij}^h - \Gamma_{ij}^h = \psi_i\delta_j^h + \psi_j\delta_i^h,$$

where $\bar{\Gamma}_{ij}^h(x)$ and $\Gamma_{ij}^h(x)$ are the Christoffel symbols of the surfaces of revolution $\overline{F}$ and F, respectively, and have the following form:

For surface $\overline{F}$:

$$\bar{\Gamma}^1_{11} = -\frac{1}{2}\left(\frac{8u^2+3u+1}{u(1+4u)(1+u)}\right); \bar{\Gamma}^1_{12} = \bar{\Gamma}^1_{21} = 0; \bar{\Gamma}^1_{22} = -\frac{2}{1+4u}; \bar{\Gamma}^2_{11} = 0; \bar{\Gamma}^2_{12} = \frac{1}{2u(1+u)}; \bar{\Gamma}^2_{22} = 0.$$

For surface F:

$$\Gamma^1_{11} = -\frac{1}{2}\frac{1}{u(1+u)}; \Gamma^1_{12} = \Gamma^1_{21} = 0; \Gamma^1_{22} = -\frac{2}{1+4u}; \Gamma^2_{11} = 0; \Gamma^2_{12} = \frac{1}{2u(1+u)}; \Gamma^2_{22} = 0.$$

Let us find the components of the deformation tensor P^h_{ij}:

$$\begin{aligned}
P^1_{11} &= \bar{\Gamma}^1_{11} - \Gamma^1_{11} = \psi_1\delta^1_1 + \psi_1\delta^1_1 = 2\psi_1;\\
P^1_{12} &= P^1_{21} = \bar{\Gamma}^1_{12} - \Gamma^1_{12} = \psi_1\delta^1_2 + \psi_2\delta^1_1 = \psi_2;\\
P^1_{22} &= \bar{\Gamma}^1_{22} - \Gamma^1_{22} = \psi_2\delta^1_2 + \psi_2\delta^1_2 = 0;\\
P^2_{11} &= \bar{\Gamma}^2_{11} - \Gamma^2_{11} = \psi_1\delta^2_1 + \psi_1\delta^2_1 = 0;\\
P^2_{12} &= P^2_{21} = \bar{\Gamma}^2_{12} - \Gamma^2_{12} = \psi_1\delta^2_2 + \psi_2\delta^2_1 = \psi_1;\\
P^2_{22} &= \bar{\Gamma}^2_{22} - \Gamma^2_{22} = \psi_2\delta^2_2 + \psi_2\delta^2_2 = 2\psi_2.
\end{aligned}$$

From this it follows that

$$\psi_1 = \frac{1}{2}(\bar{\Gamma}^1_{11}(u) - \Gamma^1_{11}(u)) = -\frac{1}{2(u+1)}, \quad \psi_2 = \bar{\Gamma}^1_{12}(u) - \Gamma^1_{12}(u) = 0.$$

According to [6, p. 297], if M and $\overline{M}$ are Riemannian manifolds, then ψ_i can also be computed by the formula:

$$\psi_i = \frac{1}{n+1}(\bar{\Gamma}^a_{ia} - \Gamma^a_{ia}) = \frac{1}{1+n}(\partial_i \ln(\sqrt{\bar{\Delta}}) - \partial_i \ln(\sqrt{\Delta})) = \frac{1}{2(n+1)}\partial_i \ln\left|\frac{\bar{\Delta}}{\Delta}\right|,$$

where $\Delta = \det(g_{ij})$ and $\bar{\Delta} = \det(\bar{g}_{ij})$.

In our case $n = 2$, and g_{ij} and $\bar{g}_{ij}$ are the coefficients of the first fundamental forms of the surfaces of revolution F and $\overline{F}$, respectively. $\bar{\Delta} = \det(\bar{g}_{ij})$ and

$$\Delta = \det(g_{ij});$$

$$\bar{\Delta} = \frac{(1+c^2)^2(1+4u)}{4(1+u)^3} \quad \text{and} \quad \Delta = \frac{(1+4u)}{4}.$$

For $i = 1$:

$$\psi_1 = \frac{1}{2(2+1)}\frac{\partial}{\partial u}\ln\left|\frac{(1+c^2)^2}{(1+u)^3}\right| = -\frac{1}{2(u+1)},$$

And for $i = 2$:

$$\psi_2 = \frac{1}{2(2+1)}\frac{\partial}{\partial v}\ln\left|\frac{(1+c^2)^2}{(1+u)^3}\right| = 0,$$

since the determinants do not depend on the variable v.

Thus, by an independent method, we have obtained that $\psi_1 = -\frac{1}{2(u+1)}$, $\psi_2 = 0$, which coincides with the values found earlier. The theorem is proven.

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REZYUME

Ushbu ish aylanma sirtlarining geodezik akslantirishlarini o‘rganishga bag‘ishlangan bo’lib, berilgan aylanma sirtnining geodezik chiziqlarini boshqa bir aylanma sirtning geodezik chiziqlariga akslantiruvchi akslantirish qurilgan va qurilgan akslantirish geodezik akslantirish ekanligi isbotlangan.

Kalit so‘zlar: geodezik akslantirish, aylanma sirtlar, Kristoffel simvollari, deformatsiya tenzori, riman ko‘pxilliklari, aylanma paraboloid.

РЕЗЮМЕ

Данная работа посвящена изучению геодезических отображений поверхностей вращения. Построено отображение, которое переводит геодезические линии заданной поверхности вращения в геодезические линии другой поверхности вращения, и доказано, что построенное отображение является геодезическим отображением.

Ключивые слова: геодезическое отображение, поверхности вращения, символы Кристоффеля, тензор деформации, римановы многообразия, параболоид вращения.