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On the solvability of the Cauchy problem for the Biharmonic equation

Sharipova Salibxan Arislanbaevna
National University of Uzbekistan. Tashkent, Uzbekistan
sharipovasalibkhan@mail.ru

RESUME

The present research is devoted to studying the solvability of the Cauchy problem for Biharmonic equation. It is well known that the Cauchy problem for higher-order elliptic equations is ill-posed. We construct an explicit representation of the solution using the Fourier transform. Furthermore, we obtain several a priori estimates for the solution and prove the existence and uniqueness of the solution in the class of analytic functions.

Key words: Biharmonic equation, Cauchy problem, ill-posed problem, existence, uniqueness.

1. Introduction

The Cauchy problem for the Laplace equation has been studied in a large number of mathematical publications, starting with the classical work of Hadamard [3]. It is well known that this problem is ill-posed (for this reason, see [4],[9]). In most works, the problems of stability and regularization methods were studied (see [4]-[6], [12]).

The prominent Soviet mathematicians, academicians A.N. Tikhonov and M.M. Lavrentev, their disciples and followers proved that the Cauchy problem is conditionally well-posed for the Laplace equation and ill-posed for other problems (see [6],[8]), and they suggested the regularization method for these ill-posed problems. Numerical methods for solving ill-posed problems of mathematical physics were also discussed in [12] (see also [2]).

The main monograph on ill-posed boundary value problems for the biharmonic equation was presented in [13]. In it, three essentially ill-posed internal boundary value problems for the biharmonic equation and the Cauchy problem for the abstract biharmonic equation were studied.

The Cauchy problem for the biharmonic equation was also studied by T.Sh. Kal'menov and U.A. Iskakova in [7]. However, they did not show class of functions. But, in this paper we have shown exact class of functions which ensure the existence and uniqueness of the solution.

It is known that the existence and uniqueness of the solution to the Cauchy problem for elliptic equations has been less studied. Sh.A. Alimov and A.K. Qudaybergenov obtained some important a priori estimates for hyperbolic functions in [16], which help to show the existence and uniqueness of the solution of the Cauchy problem for elliptic equations, and they proved the existence and uniqueness of the solution (see [14]-[16]).

It is well known that the Cauchy problem for the given equation is an ill-posed problem. Therefore, to show the existence and uniqueness of the solution to the given problem, we will use some known estimates that were proved in [16].

2. Statement of the problem

Set

$$\Omega = \{(x, y) \in \mathbb{R}^2 \ : x \in \mathbb{R},\ 0 < y < h\}.$$

Consider the following eqaution in Ω

$$\frac{\partial^4 u}{\partial x^4} + 2\frac{\partial^4 u}{\partial x^2 \partial y^2} + \frac{\partial^4 u}{\partial y^4} = 0, \tag{1}$$

with boundary conditions

$$u(x, 0) = \phi(x),\ u_y(x, 0) = u_{yy}(x, 0) = u_{yyy}(x, 0) = 0. \tag{2}$$

where $\phi(x)$ belongs to the $L_2(\mathbb{R})$.

The solution of the problem (1)-(2) is the function $u \in C^4(\Omega)$, which satisfies in the domain Ω equation (1), and satisfies the boundary conditions (2) in the following sense:

$$\lim_{y\to 0^+} \int\limits_{-\infty}^{\infty} |u(x,y) - \phi(x)|^2 \, dx \ = \ 0, \tag{3}$$

and

$$\lim_{y\to 0^+} \int\limits_{-\infty}^{\infty} \left| \frac{\partial^m u}{\partial y^m}(x,y) - 0 \right|^2 \, dx \ = \ 0, \ m = 1,2,3. \tag{4}$$

Problem A. *For a given function $\phi \in L_2(\mathbb{R})$, find the values on the upper border of Ω of the solution $u(x,y)$ to the problem (1)-(2).*

The solution of the problem A is the function $\chi \in L_2(\partial\Omega)$ satisfying the condition

$$\lim_{\epsilon\to 0} \int\limits_{-\infty}^{\infty} |u(x, h-\epsilon) - \chi(x)|^2 \, dx \ = \ 0. \tag{5}$$

We are looking for the solution of the problem (1)-(2) in the following form

$$u(x,y) = \frac{1}{2\pi} \int\limits_{-\infty}^{\infty} \widehat{v}(s,y) e^{isx} ds.$$

Then we reduce the following equation

$$\widehat{v}^{(4)}(s,y) \ - 2s^2 \widehat{v}^{(2)}(s,y) \ + \ s^4 \ \widehat{v}(s,y) = 0. \tag{6}$$

Then the solution will be as follows

$$\widehat{v}(s,y) \ = [A + By] \cosh |s|y \ + \ [C + Dy] \sinh |s|y. \tag{7}$$

Then

$$\widehat{v}'(s,y) \ = B \cosh |s|y \ + \ |s|[A + By] \sinh |s|y \ + D \sinh |s|y \ + \ |s|[C + Dy] \cosh |s|y, \tag{8}$$

and

$$\widehat{v}''(s,y) \ = 2B|s| \sinh |s|y \ + \ |s|^2[A + By] \cosh |s|y \ + 2D|s| \cosh |s|y \ + \ |s|^2[C + Dy] \sinh |s|y, \tag{9}$$

and

$$\widehat{v}'''(s,y) \ = 3B|s|^2 \cosh |s|y \ + \ |s|^3[A + By] \sinh |s|y \ + 3D|s|^2 \sinh |s|y \ + \ |s|^3[C + Dy] \cosh |s|y, \tag{10}$$

and

$$\widehat{v}^{(4)}(s,y) \ = \ 4B|s|^3 \sinh |s|y \ + \ |s|^4[A + By] \cosh |s|y \ + 4D|s|^3 \cosh |s|y \ + \ |s|^4[C + Dy] \sinh |s|y. \tag{11}$$

According to (7)-(11), we obtain

$$A = \frac{\widehat{\phi}}{2}, \quad B = C = 0, \quad D = -\frac{\widehat{\phi} \cdot |s|}{2}.$$

Consequently,

$$u(x,y)=\int_{-\infty}^{\infty}\widehat{v}(s,y)e^{isx}ds=\int_{-\infty}^{\infty}\widehat{\phi}\left(\cosh|s|y-\frac{|s|y}{2}\sinh|s|y\right)e^{isx}ds. \tag{12}$$

Denote by A_σ the set of functions $f(z)$ which are holomorphic on the stripe

$$S_\sigma \;=\; \{z\in\mathbb{C}\;:\;|\mathrm{Im}\,z|<\sigma\}, \tag{13}$$

and satisfy conditions:

$$\|f\|_\sigma^2 \;=\; \sup_{|y|\le\sigma}\int_{-\infty}^{\infty}|f(x+iy)|^2\,dx \;<\; +\infty. \tag{14}$$

Consider the Fourier expand

$$f(x) \;=\; \int_{-\infty}^{\infty}\widehat{f}(s)e^{isx}ds. \tag{15}$$

Theorem 2.1. *For any function $f\in A_\sigma$ the following inequalities*

$$\pi\|f\|_\sigma^2 \;\le\; \int_{-\infty}^{\infty}|\widehat{f}(s)|^2\cosh 2|s|\sigma ds \;\le\; 2\pi\|f\|_\sigma^2 \tag{16}$$

are valid. (see [16])

3. Existence of the solution of the problem

The following statements are true.

Theorem 3.1. *Let the function ϕ belong to class A_σ for some $\sigma>h$. Then the solution of the problem A exists and is unique.*

Lemma 3.1. *The solution of the equation (6) satisfies the following estimates*

$$|\widehat{v}^{(m)}(s,y)|\le\;|\widehat{\phi}|\cosh|s|y\;\left||s|^{m+1}y+|s|^m\right|,\quad m=1,2,3,4. \tag{17}$$

Proof. Indeed, if we take derivative from function $\widehat{v}(s,y)$, then we get required estimates.

$$|\widehat{v}'(s,y)|=|\widehat{\phi}|\;\left|\left(\frac{|s|}{2}\sinh|s|y\;-\frac{|s|^2y}{2}\cosh|s|y\right)\right|\le$$

$$\le\;|\widehat{\phi}|\;\left(\left|\frac{|s|}{2}\sinh|s|y\right|+\left|\frac{|s|^2y}{2}\cosh|s|y\right|\right)\le|\widehat{\phi}|\;\cosh|s|y\;||s|^2y+|s||.$$

Then, for $m=2$

$$|\widehat{v}''(s,y)|=\left|\widehat{\phi}\;\frac{|s|^3y}{2}\sinh|s|y\;\right|\le\left|\widehat{\phi}\;\right|\cosh|s|y\;\left|\frac{|s|^3y}{2}\right|\le\left|\widehat{\phi}\right|\;\cosh|s|y\;\left||s|^3y+|s|^2\right|.$$

For $m=3$,

$$|\widehat{v}'''(s,y)|=\;|\widehat{\phi}|\;\left|\frac{|s|^4y}{2}\cosh|s|y\;+\frac{|s|^3}{2}\sinh|s|y\right|\le\;|\widehat{\phi}|\;\cosh|s|y\;\left||s|^4y+|s|^3\right|.$$

Then for $m=4$,

$$|\widehat{v}^{(4)}(s,y)| = |\widehat{\phi}| \left|\frac{|s|^4y+|s|^3}{2}\cosh|s|y+\frac{|s|^5y}{2}\sinh|s|y\right|^4 \le |\widehat{\phi}| \cosh|s|y \left||s|^5y+|s|^4\right|.$$

Lemma 3.1 has been proved. □

Lemma 3.2. *Let the function ϕ belongs to class A_σ . Then the function $u(x,y)$ which defined by (12) belongs to $C^4\left((-\infty,\infty)\times[0,\sigma)\right)$.*

Proof. Let us fix an arbitrary number ρ in the interval $0<\rho<\sigma$ and prove that the function $u(x,y)$ is four times differentiable with respect x and y.

Indeed, for $m=1,2,3,4$ we can write

$$u^{(4,m)}(x,y)=\int_{-\infty}^{\infty}|s|^4\widehat{v}^{(m)}(s,y)e^{isx}ds. \tag{18}$$

Then according to Lemma 3.1 and Cauchy-Bunyakovs|s|y for $\rho<\sigma$

$$|u^{(4,m)}(x,y)|\le\int_{-\infty}^{\infty}|s|^4|\widehat{v}^{(m)}(s,y)|ds\le\int_{-\infty}^{\infty}|s|^4\ |\widehat{\phi}|\ \cosh|s|y\ \left||s|^{m+1}y+|s|^m\right|ds\le$$

$$\le\int_{-\infty}^{\infty}|s|^4\ |\widehat{\phi}|\ \cosh|s|\rho\ \left||s|^{m+1}y+|s|^m\right|ds\le\int_{-\infty}^{\infty}\frac{|s|^4\ |\widehat{\phi}|\ \cosh|s|\rho\ \left||s|^{m+1}\rho+|s|^m\right|\cosh|s|\sigma}{\cosh|s|\sigma}ds\le$$

$$\le\int_{-\infty}^{\infty}|\widehat{\phi}|^2\ \cosh^2|s|\sigma\ ds\int_{-\infty}^{\infty}\frac{|s|^8\left(|s|^{m+1}\rho+|s|^m\right)^2\cosh^2|s|\rho}{\cosh^2|s|\sigma}ds\le C\|\phi\|_\sigma^2. \tag{19}$$

Lemma 3.2 has been proved. □

Lemma 3.3. *Let ϕ belongs to class A_σ for some $\sigma>h$. Then for each $y\in[0,\ h]$ the function $u(x,y)$ defined by (12) belongs to $L_2[-\infty,\infty]$ and*

$$\lim_{y\to h}\int_{-\infty}^{\infty}\left|u(x,y)-u(x,h)\right|^2dx\ =\ 0. \tag{20}$$

Proof. Indeed according to Parseval's identity, we have

$$\|u(x,y)\|^2=\int_{-\infty}^{\infty}|\widehat{\phi}|^2\cosh^2|s|y\ \left(|s|y+1\right)^2ds\ \le\int_{-\infty}^{\infty}|\widehat{\phi}|^2\cosh2|s|\sigma\ \frac{\left(|s|y+1\right)^2\cosh2|s|h}{\cosh2|s|\sigma}ds\le$$

$$\le M_1(h)\int_{-\infty}^{\infty}|\widehat{\phi}|^2\cosh2k\sigma\le CM_1(h)\|\phi\|_\sigma^2. \tag{21}$$

where

$$M_1(h)=\max_{\sigma>h}\frac{(|s|h+1)^2\cosh2|s|h}{\cosh2|s|\sigma}.$$

Taking into account (12) and $\cosh^2x\le\cosh x$, we get

$$\int_{-\infty}^{\infty}\left|u(x,y)-u(x,h)\right|^2dx\ =\ \int_{-\infty}^{\infty}\left|\widehat{v}(s,y)-\widehat{v}(s,h)\right|^2ds\ =$$

$$= \int_{-\infty}^{\infty} |\widehat{\phi}|^2 \left| \cosh |s|y \; - \frac{|s|y}{2} \sinh |s|y \; - \cosh |s|h + \frac{|s|h}{2} \sinh |s|h \right|^2 ds \leq$$

$$\leq \int_{-\infty}^{\infty} |\widehat{\phi}|^2 \left[|\cosh |s|y - \cosh |s|h| + \frac{|s|}{2} \, |y \sinh |s|y \; - h \cos |s|h| \right]^2 ds \leq$$

$$\leq \int_{-\infty}^{\infty} |\widehat{\phi}|^2 \left[|\cosh |s|h| + |\cosh |s|h| + \frac{|s|h}{2} \left(|\sinh |s|h| \; + |\cos |s|h| \right) \right]^2 ds \leq$$

$$\leq \int_{-\infty}^{\infty} |\widehat{\phi}|^2 \; \cosh^2 |s|h \; (|s|h+2)^2 ds \; \leq \int_{-\infty}^{\infty} |\widehat{\phi}|^2 \cosh 2|s|\sigma \; \frac{(|s|h+2)^2 \cosh 2|s|h}{\cosh 2|s|\sigma} ds \leq$$

$$\leq M_2(h) \int_{-\infty}^{\infty} |\widehat{\phi}|^2 \cosh 2k\sigma \leq C M_2(h) \|\phi\|_\sigma^2 .$$

where

$$M_2(h) = \max_{\sigma > h} \frac{(|s|h+2)^2 \cosh 2|s|h}{\cosh 2k\sigma}.$$

Note that each term is a continuous function with respect to $y \in [0, h]$ and the integral is majored by a convergent integral. Therefore, according to the Weierstrass theorem (see [11], Theorem 7.10), the integral converges uniformly over $y \in [0, h]$ and is a continuous function of y. Hence, the equality (20) follows.

Lemma 3.3 has been proved. □

If we set

$$u(x, h) = \chi(x),$$

then equality (5) is proved.

Proof of Theorem 3.1 follows directly from Lemma 3.1-3.3.

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REZYUME

Ushbu tadqiqot Bigarmonik tenglama uchun Koshi masalasining yechiluvchanligini o'rganishga bag'ishlangan. Avvalo, Koshi masalasining integral shaklda ifodalangan yechimini olamiz. So'ngra berilgan masalaning yechimining mavjudligi va yagonaligi isbotlanadi.

Kalit so'zlar: Bigarmonik tenglama, Koshi masalasi, nokorrekt masala, mavjudlik, yagonalik.

РЕЗЮМЕ

Настоящее исследование посвящено изучению разрешимости задачи Коши для бигармонического уравнения. Сначала решается задача Коши, представленная в интегральной форме, и доказывается существование и единственность решения данной задачи.

Ключевые слова: Бигармоническое уравнение, задача Коши, некорректная задача, существование, единственность.