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THE DYNAMICS OF PIECEWISE-CONTINUOUS VOLTERRA QSO ON S^2

USMONOV JAVOKHIR BAKHODIR UGLI
NATIONAL UNIVERSITY OF UZBEKISTAN. TASHKENT, UZBEKISTAN
javohir0107@gmail.com

RESUME

In this article, we study a class of piecewise-continuous quadratic stochastic operators (QSOs) of Volterra type defined on the two-dimensional simplex S^2. The operator is defined by two Volterra operators V_1 and V_2 acting on disjoint subsets of the simplex. We describe the fixed points of these operators for various parameter configurations and analyze the asymptotic behavior of their trajectories. In particular, we prove that for specific parameter choices, the limit set of any interior point of the simplex lies on the boundary, and in the symmetric case $a = b = c \neq 0$, all trajectories converge to the vertex $(1, 0, 0)$.

Key words: Quadratic stochastic operator, Volterra operator, piecewise-continuous operator, fixed points, limit set.

1. Introduction

Quadratic stochastic operators (QSOs) have been extensively studied in population genetics, evolutionary dynamics, and mathematical biology, where they describe the evolution of probabilities of different species or traits over discrete generations. Among QSOs, *Volterra-type operators* are particularly important due to their structured interactions and biological interpretability. In [2], it is shown that the limit set of a dynamical system generated by a Volterra operator is either a singleton or infinite.

In [6], the dynamical system of a discontinuous Volterra operator mapping S^2 was studied. Despite the absence of periodic points for continuous quadratic stochastic operators of the Volterra type, in that article, it was showed the existence of 3-periodic points for certain parameter values. Besides, the behaviour of the dynamics of discontinuous Volterra operators is more different than the continuous ones(see [1,3-7]).

In this paper, we focus on a class of *piecewise-continuous Volterra QSOs* defined on the two-dimensional simplex S^2, where the operator switches between two Volterra forms V_1 and V_2 depending on the relation between coordinates y and z. This setup allows us to study a richer dynamical behavior compared to standard continuous operators.

Our main goal is to characterize the *fixed points*, *limit sets*, and asymptotic dynamics of these operators. We provide a detailed analysis for various parameter configurations, including cases where only one parameter is non-zero and the symmetric case $a = b = c$. Our results show that the dynamics is largely boundary-attracting, with interior points converging to the simplex boundary or a specific vertex depending on the parameters. These findings extend previous studies on Volterra QSOs and contribute to the understanding of *piecewise dynamics in stochastic systems.*

2. Piecewise-continuous QSO Volterra on S^2.

Let S^{m-1} be the simplex:

$$S^{m-1} = \{\mathbf{x} = (x_1, \ldots, x_m) \in \mathbb{R}^m : x_i \geq 0, \ \sum_{i=1}^{m} x_i = 1\}.$$

The vertices of the simplex are $\mathbf{x}_1 = (1, 0, \ldots, 0)$, $\mathbf{x}_2 = (0, 1, 0, \ldots, 0)$, …, $\mathbf{x}_m = (0, 0, \ldots, 0, 1)$. We denote the interior of the simplex by

$$\operatorname{int} S^{m-1} = \{\mathbf{x} = (x_1, x_2, \ldots, x_m) \in S^{m-1} : x_i > 0 \text{ for all } i = 1, 2, \ldots, m\},$$

and its boundary by

$$\partial S^{m-1} = S^{m-1} \setminus \operatorname{int} S^{m-1}.$$

Let $V : S^{m-1} \to S^{m-1}$ be a mapping defined by

$$V(x_k) = \sum_{i,j=1}^{m} P_{ij,k}\, x_i x_j, \quad k = 1, 2, \dots, m,$$

where $P_{ij,k}$ are hereditary coefficients satisfying

$$P_{ij,k} \geq 0, \quad P_{ij,k} = P_{ji,k}, \quad \sum_{k=1}^{m} P_{ij,k} = 1, \quad i, j = 1, 2, \dots, m.$$

Such a mapping V is called a *Quadratic Stochastic Operator (QSO)*.

Recall that a QSO $V : S^{m-1} \to S^{m-1}$ is called *Volterra* if

$$P_{ij,k} = 0 \quad \text{whenever } k \notin \{i, j\}.$$

Consider the following sequence for an initial point $x^{(0)} \in S^{m-1}$

$$x^{(0)}, \quad x^{(1)} = V(x^{(0)}), \quad x^{(2)} = V^2(x^{(0)}), \dots, x^{(n)} = V^n(x^{(0)}), \dots$$

where $V^n(x)$ is n-times composition of V to itself. Last sequence is called a discrete dynamical system generated by the operator V or the orbit(trajectory) of $x^{(0)}$.

The set of limit points of the trajectory starting at $\mathbf{x}^{(0)}$ is denoted by $\omega(\mathbf{x}^{(0)})$. One of the main problems in discrete dynamical systems for a given operator V is to analyze the asymptotic behavior of the trajectory

$$x^{(n)} = V^n(\mathbf{x}^{(0)}), \quad n \geq 1,$$

and, in particular, to characterize its limit points.

Define a QSO as follows:

$$V(\mathbf{x}) = \begin{cases} V_1(\mathbf{x}), & y \geq z \\ V_2(\mathbf{x}), & y < z \end{cases} \tag{1}$$

where $\mathbf{x} = (x, y, z) \in S^2$, $a, b, c \in [0, 1]$ and

$$V_1(\mathbf{x}) = \begin{cases} x' = x(1 - az + cy) \\ y' = y(1 - bz - cx) \\ z' = z(1 + ax + by) \end{cases} \qquad V_2(\mathbf{x}) = \begin{cases} x' = x(1 - ay + cz) \\ y' = y(1 + bz + ax) \\ z' = z(1 - by - cx). \end{cases}$$

If $a^2 + b^2 + c^2 = 0$, then the operator (1) is trivial. So we are not interested in this case. The operator is discontinuous at the median of the triangle at the vertex $\mathbf{x}_1$ in Fig.1.

3. Fixed points.

According to the definition of the operator V, the domain consists of two non-intersecting subsets of S^2. They are:

$$H_1 = \{\mathbf{x} \in S^2 \;\mid\; \mathbf{x} = (x, y, z), \quad y \geq z\};$$
$$H_2 = \{\mathbf{x} \in S^2 \;\mid\; \mathbf{x} = (x, y, z), \quad y < z\};$$

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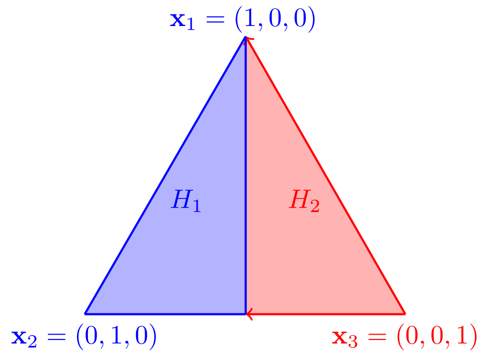


Рис. 11: The sets of H_1 and H_2 on the two-dimensional simplex

The point $\mathbf{x}$ is *a periodic point* of period p if $V^p(\mathbf{x}) = \mathbf{x}$. The least positive p for which $V^p(\mathbf{x}) = \mathbf{x}$ is called the prime period of $\mathbf{x}$. If $p = 1$ then $\mathbf{x}$ is called *a fixed point* of the operator. We denote the set of all fixed points by $\mathrm{Fix}(V)$ and the set of all periodic points of (not necessarily prime) period p by $\mathrm{Per}_p(V)$.

Let's find fixed points. We need to solve the equation $V_1(\mathbf{x}) = \mathbf{x}$ in H_1 and $V_2(\mathbf{x}) = \mathbf{x}$ in H_2. So the following lemma holds.

Lemma 1. *For the set of fixed points, the following holds:*

1) *if* $b \cdot c \neq 0$, *then* $\mathrm{Fix}(V) = \{\boldsymbol{x}_1, \boldsymbol{x}_2, \boldsymbol{x}_3\}$;

2) *if* $a = c = 0,\ b \neq 0$, *or* $a \cdot b \neq 0,\ c = 0$, *then* $\mathrm{Fix}(V) = \{\boldsymbol{x} = (x, y, z) \ \mid\ y \cdot z = 0\}$;

3) *if* $b = 0,\ a \cdot c \neq 0$ *or* $b = a = 0$ *and* $c \neq 0$, *then* $\mathrm{Fix}(V) = \{\boldsymbol{x} = (x, y, z) \ \mid\ x = 0\} \cup \{\boldsymbol{x}_1\}$;

4) *if* $b = c = 0,\ a \neq 0$, *then* $\mathrm{Fix}(V) = \{\boldsymbol{x} = (x, y, z) \ \mid\ x \cdot y \cdot z = 0\}$.

Proof. Now we need to solve the equation $V_1(\mathrm{x}) = \mathrm{x}$ for $\mathrm{x} \in H_1$.

$$\begin{cases} x(1 - az + cy) = x \\ y(1 - bz - cx) = y \\ z(1 + ax + by) = z \end{cases} \Rightarrow \begin{cases} x(-az + cy) = 0 \\ y(bz + cx) = 0 \\ z(ax + by) = 0 \end{cases}$$

- $x = 0$. Then $byz = 0$. If $b \neq 0$, then we have the solution $x_1 = (0, 1, 0)$. If $b = 0$, then the solution $(0, y, 1 - y)$ where $\frac{1}{2} \leq y \leq 1$.
- $x \neq 0,\ y = 0$. Then $az = 0$. If $a \neq 0$, then we have the solution $x_1 = (1, 0, 0)$. If $a = 0$, then the solution $(x, 0, 1 - x)$ does not belong to H_1.
- $x \neq 0,\ y \neq 0,\ z = 0$. If $c = 0$, then the solution is $(x, 1 - x, 0)$, where $0 \leq x \leq 1$.

Since $a, b, c \in [0, 1]$, there is no solution to the equation $V_1(\mathrm{x}) = \mathrm{x}$ when $xyz \neq 0$. For the case H_2, the results are proven as above. □

4. The limit sets.

4.1 The case $b = c = 0$ and $a \neq 0$. If $b = c = 0$ and $a \neq 0$, then the form of the operator (1) has

$$V_a(\mathbf{x}) = \begin{cases} V_{1,a}(\mathbf{x}), & y \geq z \\ V_{2,a}(\mathbf{x}), & y < z \end{cases} \tag{2}$$

and

$$V_{1,a} = \begin{cases} x' = x(1 - az) \\ y' = y \\ z' = z(1 + ax) \end{cases} \qquad V_{2,a} = \begin{cases} x' = x(1 - ay) \\ y' = y(1 + ax) \\ z' = z. \end{cases}$$

Lemma 2. *There does not exist any $A \subset int\, S^2$ such that $V_a(A) \subset A$.*

Proof. Let us assume conversely, i.e., there exists $A \subset int\, S^2$ such that $V_a(A) \subset A$. Since $A \subset int\, S^2$, we can find $x^*, y^*, z^* \in (0,1)$ such that $x > x^*$, $y > y^*$, and $z > z^*$ for all $\mathbf{x} = (x, y, z) \in A$(it is easy to see that if we take the subset A with $x \geq x^*$, then $V_a(\mathbf{x}^*) \notin A$, because $x^{(n)}$ strictly decreasing). Since $x^{(n)}$ is strictly decreasing $int\, S^2$, we show there exists $\overline{\mathbf{x}} = (\overline{x}, \overline{y}, \overline{z}) \in A$ such that $V_a(\overline{\mathbf{x}}) \notin A$. Let $\overline{x} = x^* + \frac{1}{n} \in A$ for $n \geq 1$. We show that there exists $n_0 \in \mathbb{N}$ such that $V_a(\overline{x}) > x^*$ for all $n > n_0$.

$$(x^* + \frac{1}{n})(1 - a\min\{\overline{y}, \overline{z}\}) \leq x^* \Rightarrow x^* + \frac{1}{n} \leq \frac{x^*}{1 - a\min\{\overline{y}, \overline{z}\}} \Rightarrow$$

$$n \geq \frac{1 - a\min\{\overline{y}, \overline{z}\}}{ax^* \min\{\overline{y}, \overline{z}\}}$$

If we take $n_0 \geq \frac{1 - a\min\{\overline{y}, \overline{z}\}}{ax^* \min\{\overline{y}, \overline{z}\}}$, then $\overline{\mathbf{x}} \in A$ and $V_a(\overline{\mathbf{x}}) \notin A$ for all $n \geq n_0$. □

Lemma 3. *Let V_a be a QSO given by (2). Then for any $\mathbf{x}^{(0)} \in S^2$, it holds $\omega(\mathbf{x}^{(0)}) \subset \partial S^2$.*

Proof. Since $V_a(\partial S^2) \subset \partial S^2$ we consider where an initial point is in $int\, S^2$. Let $\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \in H_1 \cap int\, S^2$. According to the Lemma 2 there exists $n_1 \in \mathbb{N}$ such that $y^{(n)} \geq z^{(n)}$ for all $n < n_1$ and $x^{(n)} > x^{(n+1)}$, $y^{(n)} = const$ and $z^{(n)} < z^{(n+1)}$. For $n \geq n_1$ there exists $n_2 \in \mathbb{N}$ such that $y^{(n)} < z^{(n)}$ for all $n_1 \leq n < n_2$ and $x^{(n)} > x^{(n+1)}$, $y^{(n)} < y^{(n+1)}$ and $z^{(n)} = const$. For $n \geq n_2$ there exists $n_3 \in \mathbb{N}$ such that $y^{(n)} \geq z^{(n)}$ for all $n_2 \leq n < n_3$ and $x^{(n)} > x^{(n+1)}$, $y^{(n)} = const$ and $z^{(n)} < z^{(n+1)}$ and so on. It should be noted that $x^{(n)}$ is always decreasing. Let us consider the sequence

$$v_n = \min\{y^n, z^n\}: \quad z^{(0)} < z^{(1)} < \ldots < z^{(n_1 - 1)} < y^{(n_1)} < \ldots < y^{(n_2 - 1)} < z^{(n_2)} \ldots$$

Since v_n is increasing we obtain that $u_n = \frac{x^{(n+1)}}{x^{(n)}} = 1 - a\min\{y^{(n)}, z^{(n)}\}$ is decreasing. Then

$$0 < \frac{x^{(n+1)}}{x^{(0)}} = \prod_{m=1}^{n} \Big(1 - a\min\{y^{(m)}, z^{(m)}\}\Big) < (1 - az^{(0)})^n. \tag{3}$$

We get the limit from the inequality (3)

$$\lim_{n\to\infty} \frac{x^{(n+1)}}{x^{(0)}} = \lim_{n\to\infty} (1 - az^{(0)})^n = 0 \Rightarrow \lim_{n\to\infty} x^{(n+1)} = 0.$$

□

Theorem 1. *Let V_a QSO defined by (2) and $\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \in int\, S^2$.*

(i) If $y^{(0)} \leq \frac{1}{2}$ and $z^{(0)} \leq \frac{1}{2}$, then $\omega(\mathbf{x}^{(0)}) = \{(0, \frac{1}{2}, \frac{1}{2})\}$;

(ii) If $y^{(0)} > \frac{1}{2}$, then $\omega(\mathbf{x}^{(0)}) = \{(0, y^{(0)}, 1 - y^{(0)})\}$;

(iii) If $z^{(0)} > \frac{1}{2}$, then $\omega(\mathbf{x}^{(0)}) = \{(0, 1 - z^{(0)}, z^{(0)})\}$.

Proof. (i) Let $\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \in S^2 \cap int\, S^2$ and $y^{(0)} \leq \frac{1}{2}$, $z^{(0)} \leq \frac{1}{2}$. $y^{(n)} z^{(n)}$ is an increasing sequence and bounded. So, it is convergent. For $y^{(n)}$ and $z^{(n)}$ we have $y^{(n)} \leq y^{(n+1)}$ and $z^{(n)} \leq z^{(n+1)}$. Besides they are bounded above. Then $y^{(n)}$ and $z^{(n)}$ are convergent. If $y^{(n)}$ and $z^{(n)}$ are convergent then $\frac{y^{(n)}}{z^{(n)}}$ is also convergent.

$$\lim_{n\to\infty} \frac{y^{(n)}}{z^{(n)}} = \lim_{n\to\infty} \frac{1}{1 + ax^{(n)}} = \lim_{n\to\infty} (1 + ax^{(n)}) = 1.$$

Since $y^{(n)} + z^{(n)} \to 1$ we obatain $y^{(n)} \to \frac{1}{2}$ and $z^{(n)} \to \frac{1}{2}$.

(ii) Let $\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \in S^2 \cap int\, S^2$ and $y^{(0)} > \frac{1}{2}$. Since $y^{(0)} > \frac{1}{2}$ we have $V_a(\mathbf{x}^{(n)}) = V_{1,a}(\mathbf{x}^{(n)})$ and $y^{(n)} = y^{(0)} = const$ for all $n \geq 0$. Then $z^{(n)} \to 1 - y^{(0)}$, because $x^{(n)} + y^{(n)} + z^{(n)} = 1$ and $x^{(n)} \to 0$.

(iii) This part is proved like (ii). □

Remark 1. *If $a = c = 0$ and $b \neq 0$, i.e., consider a operator V_b, then the operator (1) was studied in [4].*
4.2 The case $a = b = 0$ and $c \neq 0$. If $a = b = 0$ and $c \neq 0$, then the form of the operator (1) has

$$V_c(\mathbf{x}) = \begin{cases} V_{1,c}(\mathbf{x}), & y \geq z \\ V_{2,c}(\mathbf{x}), & y < z \end{cases} \tag{4}$$

and

$$V_{1,c} = \begin{cases} x' = x(1 + cy) \\ y' = y(1 - cx) \\ z' = z \end{cases} \qquad V_{2,c} = \begin{cases} x' = x(1 + cz) \\ y' = y \\ z' = z(1 - cx). \end{cases}$$

Lemma 4. *Let V_c be QSO given by (4). Then for any $\mathbf{x}^{(0)} \in S^2$ it holds $\omega(\mathbf{x}^{(0)}) \subset \partial S^2$.*

Proof. It is easy to see that if $\mathbf{x}^{(0)} \in \partial S^2$ then $\omega(\mathbf{x}^{(0)})$ consists of the fixed point. So, let $\mathbf{x}^{(0)} \in int\, S^2$. Then $\mathbf{x}^{(n)} < \mathbf{x}^{(n+1)}$, $\mathbf{y}^{(n)} \geq \mathbf{y}^{(n+1)}$ and $\mathbf{z}^{(n)} \geq \mathbf{z}^{(n+1)}$. Let us consider the continuous function $\varphi : S^2 \to \mathbb{R}$ given by

$$\varphi(\mathbf{x}) = yz, \quad \mathbf{x} = (x, y, z).$$

It is clear that if we take $\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \in S^2$ such that $y^{(0)} = 0$ or $z^{(0)} = 0$ then $\varphi(\mathbf{x}^{(0)}) = \varphi(V_a(\mathbf{x}^{(0)})) = 0$. And if $x^{(0)} = 0$ then this point $\mathbf{x}^{(0)}$ is a fixed point. If we take arbitrary $\mathbf{x}^{(0)} \in intS^2 \Rightarrow \varphi(V_a(\mathbf{x}^{(0)})) = (1 - cx)\varphi(\mathbf{x}^{(0)})$. Since $(1 - cx^{(0)}) \in (0, 1)$ it holds $\varphi(V_a(\mathbf{x}^{(0)})) < \varphi(\mathbf{x}^{(0)})$. Thus, we have shown that φ is Lyapunov function. Now we prove $\lim\limits_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = 0$ for any $\mathbf{x}^{(0)} \in intS^2$

Since $\varphi(\mathbf{x}^{(n)})) < \varphi(V_a(\mathbf{x}^{(n-1)})$ we obtain

$$0 < \frac{\varphi(\mathbf{x}^{(n)}))}{\varphi(V_a(\mathbf{x}^{(0)})} = \prod_{m=0}^{n-1} \left(1 - cx^{(m)}\right). \tag{5}$$

Since $x^{(n)}$ is increasing we have $\prod\limits_{m=0}^{n-1} \left(1 - cx^{(m)}\right) \leq \left(1 - cx^{(0)}\right)^n$ and the inequality (5) has the following form

$$0 < \frac{\varphi(\mathbf{x}^{(n)})}{\varphi(\mathbf{x}^{(0)})} < (1 - cx^{(0)})^n. \tag{6}$$

Due to $x^{(0)} \in (0, 1)$ we get the limit from the inequality (6)

$$\lim_{n\to\infty} \frac{\varphi(\mathbf{x}^{(n)})}{\varphi(\mathbf{x}^{(0)})} = \lim_{n\to\infty} (1 - cx^{(0)})^n = 0 \quad \Rightarrow \quad \lim_{n\to\infty} \varphi(\mathbf{x}^{(n)}) = 0.$$

From the last limit $\varphi(\mathbf{x})$ is Lyapunov function and it is proved that $\omega(\mathbf{x}^{(0)}) \subset \partial S^2$. □

Theorem 2. *Let $a = b = 0$ and V_c be QSO given by (4). Then for any $\mathbf{x}^{(0)} \in int\, S^2$ it holds $\omega(\mathbf{x}^{(0)}) = (1, 0, 0)$.*

Proof. Let $\mathbf{x}^{(0)} = (x^{(0)}, y^{(0)}, z^{(0)}) \in int\, S^2$ and $y^{(0)} \geq z^{(0)}$. According to the operator (4) the sequences $y^{(n)}$ and $z^{(n)}$ has the following forms

$$y^{(n)}: \quad \underbrace{y^{(0)} > \ldots > y^{(n_1-1)}}_{n_1} > \underbrace{y^{(n_1)} = \ldots = y^{(n_2-1)}}_{n_2-n_1} = \underbrace{y^{(n_2)} > \ldots > y^{(n_3-1)}}_{n_3-n_2} \ldots$$

$$z^{(n)}: \quad \underbrace{z^{(0)} = \ldots = y^{(n_1-1)}}_{n_1} = \underbrace{z^{(n_1)} > \ldots > z^{(n_2-1)}}_{n_2-n_1} > \underbrace{z^{(n_2)} = \ldots = z^{(n_3-1)}}_{n_3-n_2} \ldots$$

In general, one can assume that $y^{(n)} \to 0$ and $z^{(n)} \to a$ $(a > 0)$ as $n \to \infty$. Note that $y^{(n)}z^{(n)} \to 0$. Since $z^{(n)} \geq z^{(n+1)}$ there is no $n_0 \in \mathbb{N}$ such that $z^{(n)} > a$ for all $n > n_0$. If we consider that $y^{(n)} \to 0$ then after several iterations only $V_{2,c}$ acts at points in the trajectory. But, on the other hand $y^{(n)} = const$ at these points. It is contradiction to $y^{(n)} \to 0$ as $n \to \infty$. Consequently, $z^{(n)} \to 0$. □

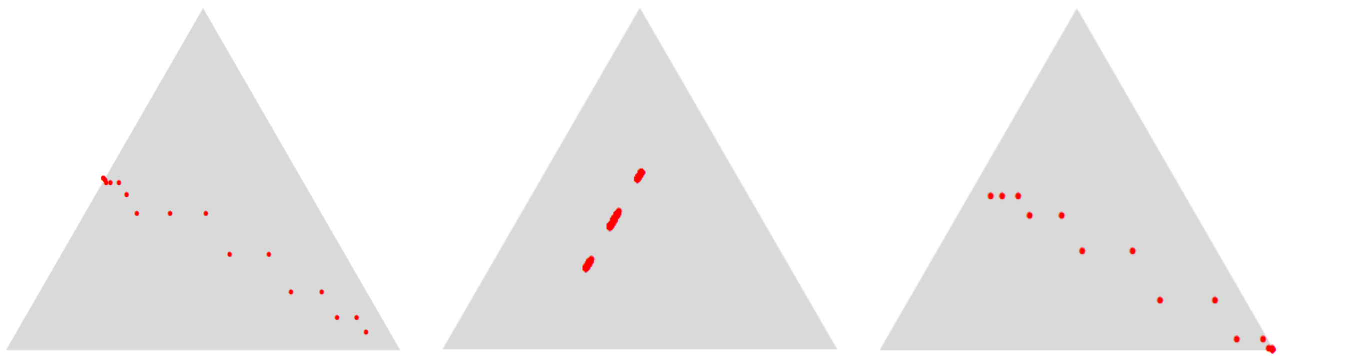
4.3 The case $a = 0$ and $bc \neq 0$. If $a = 0$ and $bc \neq 0$, then the form of the operator (1) has

$$V_{bc}(\mathbf{x}) = \begin{cases} V_{1,bc}(\mathbf{x}), & y \geq z \\ V_{2,bc}(\mathbf{x}), & y < z \end{cases} \tag{7}$$

and

$$V_{1,bc} = \begin{cases} x' = x(1 + cy) \\ y' = y(1 - bz - cx) \\ z' = z(1 + by) \end{cases} \qquad V_{2,bc} = \begin{cases} x' = x(1 + cz) \\ y' = y(1 + bz) \\ z' = z(1 - by - cx). \end{cases}$$

Theorem 3. *Let $a = 0$, $bc \neq 0$ and V_{bc} be QSO given by (7). Then for any $\mathbf{x}^{(0)} \in int\, S^2$ it holds $\omega(\mathbf{x}^{(0)}) = (1, 0, 0)$.* The theorem follows directly from the monotonicity of the sequence $x^{(n)}$.



(a) $V_a^{(n)}(x_0)$, $x_0 = (0.89, 0.05, 0.06)$ (b) $V_b^{(n)}(x_0)$, $x_0 = (0.25, 0.4, 0.35)$ (c) $V_c^{(n)}(x_0)$, $x_0 = (0.06, 0.45, 0.49)$

Рис. 12: The trajectories of the operator (1) with one non-zero parameter.

4.4 The case $a = b = c \neq 0$.

If $a = b = c \neq 0$, then the form of the operator (1) has

$$V_p(\mathbf{x}) = \begin{cases} V_{1,p}(\mathbf{x}), & y \geq z \\ V_{2,p}(\mathbf{x}), & y < z \end{cases} \tag{8}$$

and

$$V_{1,p} = \begin{cases} x' = x(1 - p(z - y)) \\ y' = y(1 - p(x + z)) \\ z' = z(1 + p(x + y)) \end{cases} \qquad V_{2,p} = \begin{cases} x' = x(1 - p(y - z)) \\ y' = y(1 + p(x + z)) \\ z' = z(1 - p(x + y). \end{cases}$$

where $p = a = b = c > 0$.

Theorem 4. *Let V_p be the quadratic stochastic operator defined in (8) and assume $a = b = c = p \in (0, 1]$. Then for every initial point $x^{(0)} \in intS^2$, $\omega(x^{(0)}) = \{(1, 0, 0)\}$.*

Proof. Write $\mathbf{x} = (x, y, z) \in S^2$. Under the hypothesis $a = b = c = p$ the two branches $V_{1,p}$ and $V_{2,p}$ of V_p combine into the following single useful identity for the first coordinate:

$$x' = x\big(1 + p|y - z|\big),$$

so that

$$x' - x = px|y - z| \geq 0. \tag{9}$$

Therefore the sequence (x_n) defined by $x_{n+1} = (V_p(\mathbf{x}^{(n)}))$ is monotone nondecreasing. Because $0 \leq x_n \leq 1$ it has a limit

$$a := \lim_{n \to \infty} x_n \in [0, 1].$$

Let $s_n := y_n + z_n = 1 - x_n$. From (9) we obtain the exact relation

$$s_{n+1} = s_n - px_n|y_n - z_n|. \tag{10}$$

Since $x_n \to a$, the telescoping/nonnegativity in (10) shows that s_n is monotone nonincreasing and converges to $1-a$. Moreover

$$s_{n+1} - s_n = -px_n|y_n - z_n| \to 0,$$

and because $x_n \to a$, we deduce

$$a\,|y_n - z_n| \to 0 \quad (n \to \infty). \tag{11}$$

We now show $a = 1$. Suppose, to the contrary, that $a < 1$. Then $s_n \to s_* := 1 - a > 0$. Combining this with (11) yields $|y_n - z_n| \to 0$. Hence y_n and z_n converge to the same limit, say ℓ, with $2\ell = s_* > 0$.

Examine the multiplicative factors for y and z in the two branches. If $y_n \geq z_n$ then

$$y_{n+1} = y_n\big(1 - pz_n - px_n\big), \qquad z_{n+1} = z_n\big(1 + px_n + py_n\big),$$

and if $y_n < z_n$ the formulas are the analogous ones coming from $V_{2,p}$. In either case, taking limits along the subsequence where the same branch applies (or passing to the limit of the factors directly, which is legitimate because $y_n - z_n \to 0$), we obtain that the multiplicative factor for y tends to

$$F_y := \lim_{n\to\infty}\big(1 - pz_n - px_n\big) = 1 - p\ell - pa,$$

while the multiplicative factor for z tends to

$$F_z := \lim_{n\to\infty}\big(1 + px_n + py_n\big) = 1 + pa + p\ell.$$

If $y_n \to \ell > 0$ and $z_n \to \ell > 0$ then both sequences (y_n) and (z_n) must have multiplicative factors tending to 1 (otherwise their limits could not be positive and finite). Thus we would require

$$1 - p\ell - pa = 1 \quad \text{and} \quad 1 + pa + p\ell = 1,$$

which immediately forces $p(a+\ell) = 0$ and $p(a+\ell) = 0$ and hence $a + \ell = 0$. But $a \geq 0$ and $\ell > 0$ give a contradiction. Therefore our assumption $a < 1$ is impossible, and so $a = 1$.

Consequently $x_n \to 1$ and $s_n = 1 - x_n \to 0$, which implies $y_n \to 0$ and $z_n \to 0$. This proves that $\mathbf{x}^{(n)} \to x_1 = (1,0,0)$ for every initial point with $x^{(0)} > 0$. □

REFERENCES

1. Abdurakhimova Sh. B., Rozikov U. A., Dynamical system of a quadratic stochastic operator with two discontinuity points. *Math. Notes.*, V.111, No.5, (2022), 676-687.

2. Ganikhodzhaev R. N., Quadratic stochastic operators, Lyapunov functions, and tournaments. *Sb. Math.*, 76(2),(1993), 489-506.

3. Khamraev A. Yu., Jumayev J. N., Dynamics of a cubic stochastic operator with one discontinuity point. *Uzbek Mathematical Journal*, Vol.69, Iss.1, (2025), 79-86.

4. Rozikov U. A., Usmonov J. B., Dynamics of a population with two equal dominated species. *Qual.Theory Dyn.Syst.*, 19, 62 (2020).

5. Usmonov J. B., On a two dimensional dynamical system generated by the foor function. *Uzbek Mathematical Journal*, 2 (2019), 127-134.

6. Usmonov J. B., On dynamics of a discontinuous Volterra operator. *Uzbek Mathematical Journal*, Vol.65, Iss.2, (2021), 164-173.

7. Usmonov J. B., Kodirova M. A., A quadratic stochastic operator with variable coefficients. *Bulletin of the Institute of Mathematics*, Iss.3, (2020), 98-107.

REZYUME

Ushbu maqolada ikki oʻlchamli S^2 simpleksda aniqlangan boʻlakli-uzluksiz Volterra tipidagi kvadratik stoxastik operatorlar (KSO) sinfi oʻrganiladi. Operator simpleksning kesishmaydigan toʻplamlarida aniqlanuvchi ikkita V_1 va V_2 Volterra operatorlari bilan taʼriflanadi. Turli parametrlar qiymatlari uchun ushbu operatorlarning qoʻzgʻalmas nuqtalari tavsiflanadi va ularning trayektoriyalarining assimptotik xarakterlari tahlil qilinadi. Xususan, maʼlum bir parametr qiymatlarida simpleksning har qanday ichki nuqtasi uchun trayektoriyaning limit nuqtalari toʻplami simpleks chegarasida joylashishini va $a = b = c \neq 0$ holatda barcha trajektoriyalar $(1, 0, 0)$ qirraga yaqinlashishini isbotlaymiz.

Kalit soʻzlar: Kvadratik stoxastik operator, Volterra operatori, boʻlakli-uzluksiz operator, qoʻzgʻalmas nuqtalar, limit nuqtlar toʻplami.

РЕЗЮМЕ

В данной статье изучается класс кусочно-непрерывных квадратичных стохастических операторов (КСО) типа Вольтерра, определённых на двумерном симплексе S^2. Оператор задаётся двумя операторами Вольтерра V_1 и V_2, действующими на непересекающихся подмножествах симплекса. Мы описываем неподвижные точки этих операторов для различных конфигураций параметров и анализируем асимптотическое поведение их траекторий. В частности, доказывается, что для определённых значений параметров множество пределов любой внутренней точки симплекса лежит на границе, а в симметричном случае $a = b = c \neq 0$ все траектории сходятся к вершине $(1, 0, 0)$.

Ключевые слова: Квадратичный стохастический оператор, оператор Вольтерра, кусочно-непрерывный оператор, неподвижные точки, множество пределов.