Affine shadowing of renormalizations for GIETs satisfying a certain Zygmund smoothness condition

Begmatov Abdumajid Safarovich
National University of Uzbekistan named after M.Ulugbek, Tashkent, Uzbekistan
Department of Natural and Mathematical Sciences, Turin Polytechnic University in Tashkent, Uzbekistan
abdumajidb@gmail.com

Abstract. Consider generalized interval exchange transformations (GIET) with irrational rotation number of periodic type. We will show that there exists a vector that shadows (with respect to accelerated height cocycle) the logarithm of mean non-linearity of renormalizations of GIET satisfying a certain Zygmund type smoothness conditions.

MSC (2020): 37E05, 37E10, 37E20, 37C15, 37A05.

Key words: interval exchange transformations, renormalizations, cocycle, rotation numbers.

Introduction

Generalized interval exchange transformations (GIETs) form an important class of one-dimensional dynamical systems that extend classical interval exchange transformations (IETs) by allowing the exchanged subintervals to be related by orientation-preserving homeomorphisms rather than rigid translations. They appear naturally as first return maps to Poincaré sections of smooth flows on surfaces, measured foliations, Teichmuller dynamics and piecewise smooth circle homeomorphisms. Despite their simple definition, these maps exhibit rich dynamical behavior and have been extensively studied over the past several decades.

The study of circle diffeomorphisms is a classical topic in dynamical systems, initiated by Poincare's invention of the rotation number followed by Denjoy's important distortion estimates and Arnold's introduction of KAM theory methods to the topic. Then the theory was further developed to establish the regularity of the map conjugating most minimal circle diffeomorphisms to their linear model. Extending these results to higher genus surfaces, and thus to generalized interval exchange transformations, has been the focus of investigations in the seminal works of Forni [6] and Marmi, Moussa and Yoccoz [9]-[10]. Similar to the classical Poincaré classification theorem for circle maps, a typical GIET with no periodic points is semi-conjugated to a standard interval exchange transformation and this semi-conjugacy is actually a conjugacy if and only if the GIET is minimal. Moreover, for a typical IET T , the set of GIETs smoothly conjugated to T defines a smooth submanifold of positive finite codimension [10]. As a continuation of these developments, the rigidity problem for GIETs (smoothness of conjugating map between two GIETs) has been investigated by Ghazouani and Ulcigrai in [7], and by Berk and Trujillo in [2].

A fundamental tool for analyzing one-dimensional dynamical systems is renormalization, which studies induced transformations on smaller subintervals and characterizes the asymptotic structure of the dynamics. The renormalization approach used in [8] is particularly natural in the spirit of Herman's theory. Within this framework, the regularity of a conjugacy can be deduced from the convergence properties of the renormalizations of sufficiently smooth circle diffeomorphisms. In particular, the renormalizations of a smooth circle diffeomorphism converge exponentially fast to a family of linear maps with slope one. This exponential convergence, together with a suitable condition on the rotation number (e.g., of Diophantine type), ensures the smoothness of the conjugacy. More generally, results on the regularity of conjugacies between two topologically equivalent maps can be obtained from the convergence properties of the renormalizations of the corresponding maps [2], [7].

In this paper we investigate affine shadowing properties of renormalizations for generalized interval exchange transformations with low regularity, assuming certain Zygmund-type smoothness conditions. Affine

Received: 11.03.2026, **Accepted:** 23.03.2026

shadowing refers to the process by which the renormalizations of a given dynamical system can be approximated by a sequence of affine models determined by the combinatorial structure of the map. Understanding these approximations is important for analyzing the asymptotic geometry of renormalizations and for investigating rigidity properties of the underlying GIETs.

Preliminaries

Interval exchange transformation(IET). Let $I = [0, 1)$ be the unit interval. A *standard interval exchange transformation*, or simply an *interval exchange transformation* (IET), is a bijective, right-continuous function $T : I \to I$, with a finite number of discontinuities whose restriction to any subinterval of continuity is given by a translation. We say that T is an IET *on d intervals* if there exists a partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, where the indexes belong to some finite alphabet $\mathcal{A}$ with $d \geq 2$ symbols such that T is continuous when restricted to I_α for each $\alpha \in \mathcal{A}$. Notice that T is simply exchanging the order of the intervals in the partition.

An IET T of d intervals can be encoded by a pair (λ, π) corresponding to a *combinatorial datum* $\pi = (\pi_0, \pi_1)$, consisting of bijections $\pi_0, \pi_1 : \mathcal{A} \to 1, ..., d$ describing the order of the intervals before and after T is applied, and a *lengths vector* $\lambda = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}}$ in the simplex $\Delta_d = \left\{\nu \in \mathbb{R}_+^{\mathcal{A}} : \sum_{\alpha\in\mathcal{A}} \nu_\alpha = 1\right\}$ which corresponds to the lengths of the intervals in the partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ associated to T. We always assume that the datum $\pi = (\pi_0, \pi_1)$ is *irreducible*, i.e.

$$\pi_1 \circ \pi_0^{-1}(\{1, 2, ..., k\}) = \{1, 2, ..., k\} \Rightarrow k = d.$$

We call $\pi_1 \circ \pi_0^{-1} : \{1, 2, ..., d\} \to \{1, 2, ..., d\}$ the *monodromy invariant* of π.

A combinatorial datum $\pi = (\pi_0, \pi_1)$ is said to be of *rotation type* if its monodromy invariant verifies

$$\pi_1 \circ \pi_0^{-1}(i) - 1 = i + k \pmod 1$$

for some $k \in \{1, 2, ..., d-1\}$ and for all $i \in \{1, 2, ..., d\}$. Similarly, we say that an IET is of *rotation type* if its combinatorial datum is of rotation type. Notice that any IET of rotation type induces a well-defined circle rotation on the circle $\mathbb{S}^1$.

Let $I_\alpha = [\ell_\alpha, r_\alpha]$ where ℓ_α and r_α are the left and right endpoints of I_α, respectively. We say that an IET $T = (\lambda, \pi)$ satisfies the *Keane condition* if $T^m(\ell_\alpha) \neq \ell_\beta$ for all $m \geq 1$ and all $\alpha, \beta \in \mathcal{A}$ with $\pi_0(\beta) \neq 1$. This condition is also called the infinite distinct orbit condition. Note that any IET verifying Keane's condition is irreducible and minimal.

Rauzy-Veech induction. Let $T = (\lambda, \pi)$ be an IET on d intervals. Denote $\alpha(\varepsilon) = \pi_\varepsilon^{-1}(d)$ for $\varepsilon = 0, 1$. The letters $\alpha(0)$ and $\alpha(1)$ correspond to the 'last' intervals in the partitions $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ and $\{T(I_\alpha)\}_{\alpha\in\mathcal{A}}$, respectively. If $\lambda_{\alpha(0)} \neq \lambda_{\alpha(1)}$, by comparing the lengths of these intervals, we define the *type* of T as $\varepsilon(\lambda, \pi) = 0$ if $\lambda_{\alpha(0)} > \lambda_{\alpha(1)}$, $\varepsilon(\lambda, \pi) = 1$ if $\lambda_{\alpha(0)} < \lambda_{\alpha(1)}$. The longest of these two intervals is sometimes referred to as the *winner* and the shortest as the *loser*. Notice that $\alpha(\varepsilon) := \alpha(\varepsilon, T)$ and $\alpha(1-\varepsilon) := \alpha(1-\varepsilon, T)$ correspond to the symbols of the winner and the loser intervals, respectively. We will sometimes refer to types 0 and 1 as *top* and *bottom*, respectively.

The *Rauzy-Veech induction* of T with type ε, which we denote by $\mathcal{RV}(T)$, is defined as the first return map of T to the subinterval $I^{(1)} = I \setminus I_{\alpha(1-\varepsilon)}$. The *Rauzy-Veech renormalization* of T is obtained by rescaling linearly $I^{(1)}$ to the unit interval I. The renormalized map is an IET with the same number of subintervals as T. This induction/renormalization procedure can be iterated infinitely many times if and only if T verifies the Keane's condition.

Denote by $\mathcal{G}_d$ the set of *irreducible combinatorial data* $\pi = (\pi_0, \pi_1)$ with d symbols. Also, denote X_d the set of IETs verifying Keane's condition. Given $\pi, \pi' \in \mathcal{G}_d$ the permutation π' is said to be a *successor* of π if there exist $\lambda, \lambda' \in \Delta_d$ such that $(\lambda', \pi') = \mathcal{RV}(\lambda, \pi)$. We denote this relation by $\pi \to \pi'$. Notice that any successor of an irreducible permutation is also irreducible. The relation '$\to$' defines an oriented graph structure on the set of irreducible permutations $\mathcal{G}_d$. We call *Rauzy classes* the connected components of the oriented graph $\mathcal{G}_d$ with respect to the successor relation.

GIETs and combinatorial rotation number. A generalized interval exchange transformation (GIET) is a piecewise smooth bijective, right-continuous function $f : I \to I$ with a finite number of discontinuities, whose derivative is non-negative and extends to the closure of any subinterval where the function is smooth. Similar to the case of IETs, to any GIET f, we can associate a partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$ of I such that, for every $\alpha \in \mathcal{A}$, the restriction $f_{I_\alpha} : I_\alpha \to f(I_\alpha)$ is smooth, as well as a permutation π describing the order in which these intervals are exchanged.

Rauzy-Veech renormalization and Keane's condition, initially defined only for IETs, extend trivially to GIETs. Given a GIET f with associated permutation π and exchanged intervals $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, we denote by $\mathcal{RV}(f)$ its Rauzy-Veech renormalization whenever it is well defined, that is, if $|I_{\alpha(0)}| \neq |f(I_{\alpha(1)})|$, where $|I|$ denotes the length of the interval I. We say that f is *infinitely renormalizable* if and only if $\mathcal{RV}^n(f)$ is well defined for all $n \in \mathbb{N}$. Similar to the IET setting, if f verifies Keane's condition, then it is infinitely renormalizable. The infinite path $\gamma(f)$ on the Rauzy diagram defined by infinitely renormalizable GIET f is called the *combinatorial rotation number* or simply the *rotation number* of f. An infinite path on the Rauzy diagram is called ∞-*complete* if each letter in $\mathcal{A}$ wins infinitely many times. We say that a GIET is *irrational* if it is infinitely renormalizable and its rotation number is irrational(∞-complete).

A GIET f is called of *periodic type* if it has no connections and its combinatorial rotation number $\gamma(T)$ is irrational and periodic, i.e. there exists a $p > 0$ such that $\gamma_{n+p} = \gamma_n$ for every $n \in \mathbb{N}$. The minimal p with such property will be called the period of $\gamma(f)$. Throughout the paper, we will consider GIETs with irrational rotation number of periodic type.

The length cocycle. Given $T = (\lambda, \pi)$ such that $\lambda_{\alpha(0)} \neq \lambda_{\alpha(1)}$, we define the *Rauzy-Veech matrix* $A(T) : \mathbb{R}^d \to \mathbb{R}^d$ associated to T as

$$A(T) = I_d + E_{\alpha(\varepsilon),\, \alpha(1-\varepsilon)}, \tag{1}$$

where I_d denotes the identity matrix on $\mathbb{R}^d$ and $E_{\alpha,\,\beta}$ is the matrix whose entries are 1 at the position $(\alpha,\, \beta)$ and 0 otherwise.

Given $T = (\lambda,\, \pi)$ verifying Keane's condition, denote

$$A^n(T) = A(T)\cdots A(\mathcal{RV}^{n-1}(T)), \quad A_{m,n}(T) = A(\mathcal{RV}^{n-1}(T))\cdots A(\mathcal{RV}^{m}(T)),\ n > m \geq 0.$$

Rauzy-Veech matrix encodes the change of the length vector after one step of Rauzy-Veech induction. Namely, the lengths vector $\lambda^{(n)}$ of $\mathcal{RV}^n(T)$ verifies

$$\lambda^{(n)} = A^n(T)^{-1}\lambda, \quad \lambda^{(n)} = A_{m,n}(T)^{-1}\lambda^{(m)}, \ \text{ where } \ \lambda = \lambda(T) = \{\lambda_\alpha\}_{\alpha\in\mathcal{A}}.$$

The map

$$A^{-1} : X_d \longrightarrow SL(d,\, \mathbb{Z}), \qquad T \mapsto A(T)^{-1},$$

is a cocycle over $\mathcal{R}$, known as the *Rauzy-Veech cocycle* or *lengths cocycle*.

The height cocycle. Let $T = (\lambda,\, \pi)$ be an IET satisfying Keane's condition. Notice that the transformation $\mathcal{RV}^n(T)$ is defined as the first return map of T to some subinterval $I^{(n)} \subset I$. This interval admits a decomposition $I^{(n)} = \sqcup_{\alpha\in\mathcal{A}} I^{(n)}_\alpha$ such that the return time to $I^{(n)}$ on each subinterval $I^{(n)}_\alpha$ is constant. Denote by $q^{(n)}_\alpha$ the return time for the interval $I^{(n)}_\alpha$, that is $T^{q^{(n)}_\alpha}(x) \in I^{(n)}_\alpha,\ x \in I^{(n)}_\alpha$.

Using the Rauzy-Veech matrix $A(T)$ given by equation (1), we define a cocycle over IETs, known as the *heights cocycle*, by

$$A^T : X_d \longrightarrow SL(d,\, \mathbb{Z}), \qquad T \mapsto A(T)^T,$$

that encodes the change of the return times vector after one step of Rauzy-Veech induction. The vector of return times $q^{(n)} = (q^{(n)}_\alpha)_{\alpha\in\mathcal{A}}$ verifies

$$q^{(n)} = A^n(T)^T\overline{1}, \quad q^{(n)} = A_{m,n}(T)^T q^{(m)},\ n > m,$$

where $\overline{1} \in \mathbb{N}^{\mathcal{A}}$ is the vector whose entries are all equal to 1.

Dynamical partitions. Given an IET $T = (\pi, \lambda)$ verifying Keane's condition, we can associate a sequence of *dynamical partitions* and *Rohlin towers* as follows. We define the *dynamical partition* $\mathcal{P}^{(n)}$ of I *at level* n as

$$\mathcal{P}^{(n)} = \bigcup_{\alpha\in\mathcal{A}} \mathcal{P}^{(n)}_\alpha, \ \text{ where } \ \mathcal{P}^{(n)}_\alpha = \left\{ I^{(n)}_\alpha,\, T(I^{(n)}_\alpha), ..., T^{q^{(n)}_\alpha - 1}(I^{(n)}_\alpha) \right\}.$$

One can verify that $\mathcal{P}^{(n)}$ is a partition of $[0,\, 1)$ into subintervals and that, for each $\alpha \in \mathcal{A}$, the collection $\mathcal{P}^{(n)}_\alpha$ is a Rohlin tower of height $q^{(n)}_\alpha$. Notice that if $n > m$, then $\mathcal{P}^{(n)}$ is a refinement of $\mathcal{P}^{(m)}$.

Zorich acceleration. If T is of periodic type with period p, it is natural to consider the renormalization operator $\mathcal{R} = \mathcal{RV}^p$, viewed as an acceleration of $\mathcal{RV}$. In this case, for every $k \in \mathbb{N}$ we consider $\mathcal{R}^k(T) = \mathcal{RV}^{kp}(T)$.

Another important acceleration of $\mathcal{RV}$ is the *Zorich acceleration*, denoted by $\mathcal{Z}$. It is obtained by grouping together all consecutive elementary steps of the Rauzy-Veech induction that are of the same type (top or bottom). More precisely, the Zorich map is given by

$$\mathcal{Z}(T) = \mathcal{RV}^{z(T)}(T),$$

where $z(\pi, \lambda)$ is the largest integer such that $T, \mathcal{RV}(T), ..., \mathcal{RV}^{z(T)-1}(T)$ all have the same type. Using $z : X_d \to \mathbb{N}$ as the *accelerating map*, define the *accelerated lengths and heights cocycles*

$$B^{-1} : X_d \longrightarrow SL(d, \mathbb{Z}), \quad B^T : X_d \longrightarrow SL(d, \mathbb{Z}),$$

by setting

$$B^{-1}(T) = A^{z(T)}(T)^{-1}, \quad B^T(T) = A^{z(T)}(T)^T.$$

These cocycles are related to the transformation of lengths and heights under the action of $\mathcal{Z}$. Note that if T is of periodic type with period p, then the associated cocycle is given by $B(T) = A^p(T)$.

Affine interval exchange transformations An affine interval exchange transformation (AIET) is a GIET for which the restriction to each subinterval of continuity is a linear map. Notice that f will change the order of these intervals and linearly modify their lengths. Given an AIET f on d intervals with associated partition $\{I_\alpha\}_{\alpha\in\mathcal{A}}$, we define its *log-slope* as the logarithm of the slope of f in each interval of continuity, namely, the vector $\omega = (\omega_\alpha)_{\alpha\in\mathcal{A}}$, where $\omega_\alpha = \log Df(x),\ x \in I_\alpha$. The Rauzy-Veech induction and the Zorich acceleration extend naturally to the space of AIETs, as well as all the notions introduced above in the IET setting, such as combinatorial rotation number, dynamical partitions, incidence matrices, etc.

Height and length cocycles for GIET. It is important to note that while the height and length cocycles extend trivially to the GIET setting (since they depend only on the combinatorial rotation number), their roles differ. The height cocycle continues to represent the return times for the induced transformation. However, the length cocycle no longer describes the lengths of the intervals in the partition of the induced transformation.

We can define an equivalent of a length cocycle for GIETs. Let f be a GIET and let $T_0 = (\lambda, \pi)$ be an IET such that $\gamma(f) = \gamma(T_0)$. Let $A^{(n)} := A(\lambda, \pi)\cdots A(\mathcal{RV}^{n-1}(\lambda, \pi))$. For any $n \in \mathbb{N}$ consider a $d \times d$ matrix $A^{(n)}(f)$ defined as follows

$$A^{(n)}_{\alpha\beta}(f) := \sum_{i=1}^{A^{(n)}_{\alpha\beta}} \frac{|f^{m_i(\alpha,\beta)}(I_\alpha(\mathcal{RV}^n(f))|}{|I_\alpha(\mathcal{RV}^n(f)|},$$

where $m_i(\alpha, \beta)$ is the i-th return time of the interval $I_\alpha(\mathcal{RV}^n(f))$ to the interval $I_\beta(f)$ via f. The importance of the matrices defined above follows from the fact that

$$(|I_\alpha(f)|)_{\alpha\in\mathcal{A}} = A^{(n)}(|I_\alpha(\mathcal{RV}^n(f))|)_{\alpha\in\mathcal{A}}.$$

Main results

GIET with a certain Zygmund smoothness condition. Now we define a class of GIETs satisfying a Zygmund condition. Consider the function $\mathcal{Z}_\gamma : [0, 1) \to (0, +\infty)$, defined as

$$\mathcal{Z}_\gamma(x) = |\log x|^{-\gamma}, \quad \text{for} \quad x \in (0, 1)$$

and $\mathcal{Z}_\gamma(0) = 0$, where $\gamma > 0$.

Let $J = [a, b]$ be a finite interval and consider a differentiable function $K : J \mapsto K(J)$. Denote by $\Delta^2 K(\xi, \tau)$ the *second symmetric difference* of K on J, that is,

$$\Delta^2 K(\xi, \tau) = K'(\xi + \tau) + K'(\xi - \tau) - 2K'(\xi)$$

where $\xi \in J$ and $\tau \in [0, |J|/2]$ such that $\xi - \tau, \xi + \tau \in J$.

Suppose that there exists a constant $C > 0$ such that the following inequality holds:

$$\|\Delta^2 K(\cdot, \tau)\|_{L^\infty(J)} \le C\tau\mathcal{Z}_\gamma(\tau). \tag{2}$$

Note that the class of real valued functions satisfying (2) with $\mathcal{Z}_\gamma(\tau) \equiv 1$ is called the Zygmund class and denoted by Λ_* . The class Λ_* plays a key role to investigate trigonometric series. The class Λ_* was applied to the theory of circle homeomorphisms for the first time by Jun Hu and Sullivan. They extended the classical Denjoy's theorem to the class Λ_*. Generally speaking, the function satisfying (2) does not imply the boundedness of total variation of its and the reverse also is not true. In this work we study the GIETs satisfying (2) which have bounded variations.

Denote by $X_d^{1+\mathcal{Z}_\gamma}$ the set of GIETs f of d intervals that satisfy the following conditions:

(i) has irreducible combinatorics and has no connections;

(ii) has combinatorial irrational rotation number of periodic type;

(iii) derivatives Df have bounded variation and satisfy the inequality (2) on the closer of each interval of continuity.

Previous Results and the Statements of Main Theorems. We define a metric d_{C^r} on the space C^r of piecewise smooth homeomorphisms with d branches and with a fixed combinatorial data, where the distance $d_{C^r}(f,\, g)$ of any two maps f and g in this class, is given by

$$\max_{\alpha\in\mathcal{A}}\left\{\|\Xi(f|_{I_\alpha(f)})-\Xi(g|_{I_\alpha(g)})\|_{C^r}+\|I_\alpha(f)-I_\alpha(g)\|+\|f(I_\alpha(f))-g(I_\alpha(g))\|\right\}$$

where Ξ is the zoom operator, which rescales any homeomorphism between two bounded intervals, via affine transformations, to a homeomorphism of the unit interval. More precisely, if $g: I\to J$ is an homeomorphism between two closed bounded intervals, then

$$\Xi(g)=A_1\circ g\circ A_2,$$

where $A_1: J\to[0,1],\ A_2:[0,1]\to I$ are bijective orientation preserving homeomorphisms. Whenever necessary, we will use $D^m f$ instead of the m^{th} derivative of f.

Let $\mathcal{M}_N$ be a Möbius transformation $\mathcal{M}_N:[0,\,1]\mapsto[0,\,1]$ such that $\mathcal{M}_N(0)=0$, $\mathcal{M}_N(1)=1$ and

$$\mathcal{M}_N(x)=\frac{xN}{1+x(N-1)}.$$

Note that if $\gamma>1$ then second derivative of f exists on each continuity intervals of f. We define a new quantity as follows:

$$m_\alpha^{(n)}=\exp\left\{-\sum_{i=0}^{q_\alpha^{(n)}-1}\int_{I_\alpha^{(n)}}\frac{f''(t)}{2f'(t)}dt\right\}.$$

Theorem 1.(see [1]) Let $f\in X_d^{1+\mathcal{Z}_\gamma},\ \gamma>1$ be a GIET with combinatorial data of rotation type. Then there exists a constant $C=C(f)>0$ such that for all $\alpha\in\mathcal{A}$ the following bounds hold:

$$\|\Xi(\mathcal{R}\mathcal{V}^n(f))|_{I_\alpha^{(n)}}-M_{m_\alpha^{(n)}}\|_{C^1[0,1])}\le\frac{C}{n^\gamma},$$

$$\|\Xi(D^2\mathcal{R}\mathcal{V}^n(f))|_{I_\alpha^{(n)}}-D^2M_{m_\alpha^{(n)}}\|_{C^0[0,1])}\le\frac{C}{n^{\gamma-1}}.$$

Theorem 1 implies the following

Corollary 1. For a.e. $T=(\lambda,\,\pi)\in\mathcal{G}_d\times\Delta_d$ and for any GIET f of class $X_d^{1+\mathcal{Z}_\gamma},\ \gamma>0$ with $\mathcal{N}(f)=0$ and $\gamma(f)=\gamma(T)$, it holds

$$\max_{\alpha\in\mathcal{A}}\max_{x,y\in I_\alpha^n}\frac{Df^{q_\alpha^n}(x)}{Df^{q_\alpha^n}(y)}=1+O(\frac{1}{n^\gamma}).$$

Now we assume that f is a infinitely renormalizable generalized interval exchange map of periodic type. Thus, there exists a $p>0$ such that the rotation number is periodic with period p, namely if $n=kp+r$ for some $k\in\mathbb{N}$ and $0\le r<p$ then $\gamma(\mathcal{R}\mathcal{V}^{kp+r}(f))=\gamma(\mathcal{R}\mathcal{V}^{kp+r}(f))$. In this case, we will use, as renormalization

operator, the acceleration of Rauzy-Veech induction which corresponds to the period p of the rotation number $\gamma(f)$, namely the operator given by $\mathcal{R}^n(f) = (\mathcal{RV}^p)^n(f)$. Consider the cocycle corresponding to $\mathcal{R}^n(T)$:

$$B(\mathcal{R}(f)) = A(f)\cdots A(\mathcal{RV}^p(f)), \quad B_{m,n}(f) = B(\mathcal{R}^m(f))\cdots B(\mathcal{R}^m(f)),\ n > m \geq 0.$$

Let us write $\mathbb{R}^d = E^s \oplus E^c \oplus E^u$ for the splitting of $\mathbb{R}^d$ into respectively the stable space E^s, the central space E^c and the unstable space E^u for the action of A on $\mathbb{R}^d$ (corresponding to eigevectors with norm respectively smaller, equal and greater than 1). We will use the following hyperbolic properties of Rauzy-Veech cocycle restricted to the $K-$bounded combinatorics(which also hold in the periodic case).

Proposition 1(see [4]). For each K there exists $\mu = \mu(K) > 1$ and $C > 0$ with the following property

(1) For every $n \geq 1$ and $v \in E^u$, we have $\|^T B_{0,n} v\| \geq C_1 \mu^n \|v\|$.

(2) For every $n \geq 1$ and $v \in E^s$, we have $\|(^T B_{0,n})^{-1} v\| \geq C_1 \mu^n \|v\|$.

Proposition 2(see [4]). For all vectors $v \in E^c$ and for all $n \geq 1$, there exists $C > 0$ such that

$$C^{-1}\|v\| \leq \|^T B_{0,n} v\| \leq C\|v\|.$$

The following lemma can be easily verified.

Lemma 1. Assume $0 < \lambda < 1$ and $\gamma > 0$. Then

$$\sum_{i=0}^{n-1} \frac{\lambda^i}{(n-i)^\gamma} = O(\frac{1}{n^\gamma}), \qquad \sum_{i=0}^{n-1} \frac{\lambda^{n-i}}{(2n-i)^\gamma} = O(\frac{1}{n^\gamma}).$$

Let f be a GIET of class $X_d^{1+\mathcal{Z}_\gamma},\ \gamma > 1$ verifying $\mathcal{N}(f) = \int_0^1 D\log Df(x)dx = 0$. Define log-slope vectors $L^n = (L^n_\alpha)_{\alpha\in\mathcal{A}}$ of $\mathcal{R}^n(f)$ as

$$L^n_\alpha = \ln\left(\frac{1}{|I^{(n)}_\alpha|}\int\limits_{I^{(n)}_\alpha} Df^{q^n_\alpha}(s)ds\right),\ \ I^{(n)}_\alpha \in \mathcal{P}^{(n)}_b.$$

Now we state our main result in this article.

Theorem 2. Let $f \in X_d^{1+\mathcal{Z}_\gamma},\ \gamma > 1$ with $\mathcal{N}(f) = 0$. Then there exists a vector $\omega \in E^c(\tau, \lambda, \pi)$ such that

$$|\omega^n - L^n| = O(\frac{1}{n^\gamma}),$$

where

$$\omega^n =^T B_{0,n}(\tau, \lambda, \pi)\omega.$$

This theorem shows the existence of a vector that "shadows" the algorithm for the mean nonlinearity of the subsequent renormalizations $\mathcal{R}f$ of f, with respect to the accelerated height cocycle B associated with accelerated $\mathcal{R}$.

To prove this theorem we first show that the sequence $(L^n)_{n\geq 1}$ behaves as a pseudo-orbit with respect to the heights cocycle. The following lemma describes the relationship between L^n and L^{n+1}.

Lemma 2. We have

$$|L^{n+1} -^T B_{n,n+1}L^n| = O(\frac{1}{n^\gamma}),\ \ \gamma > 1.$$

Proof. For any $\alpha \in \mathcal{A}$ and $n \in \mathbb{N}$, let $x^n_\alpha \in I^{(n)}_\alpha(f)$ such that $L^n_\alpha = \ln Df^{q^n_\alpha}(x^n_\alpha)$. Let us fix $n \in \mathbb{N}$. Given $\alpha \in \mathcal{A}$, let

$$b^n_\alpha = \sum_{\beta\in\mathcal{A}} (^T B_{n,n+1})_{\alpha\beta}.$$

For any $\alpha \in \mathcal{A}$, we can express q^{n+1}_α uniquely as $q^{n+1}_\alpha = \sum\limits_{i=1}^{b^n_\alpha} q^n_{\zeta_i(\alpha)}$, for some $\zeta_i(\alpha) \in \mathcal{A}$, such that

$$f^{h_i}(I^{n+1}_\alpha(f)) \subset I^n_{\zeta(\alpha)}(f), \quad \text{where,} \quad h_i = \sum_{j=1}^{i-1} q^n_{\zeta_j(\alpha)}, \tag{3}$$

for $i = 1, 2, ..., b^n_\alpha$. Notice that

$$({}^T B_{n,n+1} L^n)_\alpha = \sum_{\beta \in \mathcal{A}} ({}^T B_{n,n+1})_{\alpha\beta} L^n_\beta, \quad \text{and} \quad \#\{1 \le i \le b^n_\alpha : \zeta_i(\alpha) = \beta\} = ({}^T B_{n,n+1})_{\alpha\beta},$$

for any $\alpha, \beta \in \mathcal{A}$. A simple calculations shows that

$$(L^{n+1} -^T B_{n,n+1} L^n)_\alpha = \sum_{\beta \in \mathcal{A}} \sum_{\zeta_i(\alpha)=\beta} \ln \frac{Df^{q^n_\beta}(f^{h_i}(x^n_\alpha))}{Df^{q^n_\beta}(x^n_\beta)}, \tag{4}$$

for any $\alpha \in \mathcal{A}$. Corollary 1 and the relations (3)-(4) imply that

$$\frac{|L^{n+1} -^T B_{n,n+1} L^n|}{\|^T B_{n,n+1}\|} = O(\frac{1}{n^\gamma}), \;\; \gamma > 0.$$

Therefore, the claim is a direct consequence of the previous relation and the fact that $\|^T B_{n,n+1}\|$ is bounded.

We show that this pseudo-orbit is shadowed by the iterates of a vector under the height cocycle. For any $n \ge 0$, let us decompose L^n with respect to the Oseledet's splitting at $(\tau^n, \lambda^n, \pi^n)$ as $L^n = L^{n,s} \oplus L^{n,c} \oplus L^{n,u} \in E^s_n \oplus E^c_n \oplus E^u_n$, and define $v_n =^T B^{-1}_{0,n} L^{n,c}$.

Lemma 3. There exists $\omega \in E^c(\tau, \lambda, \pi)$ such that $\lim_{n\to\infty} v_n = \omega$. Moreover,

$$|v_n - \omega| = O(\frac{1}{n^\gamma}), \;\; \gamma > 1.$$

Proof. By Lemma 2 and Proposition 2, we have

$$|v_{n+1} - v_n| = |^T B^{-1}_{0,n+1}(L^{n+1,c} -^T B^{-1}_{n,n+1} L^{n,c})| \le \|^T B^{-1}_{0,n+1}|_{E^c}\| L^{n+1,c} -^T B^{-1}_{n,n+1} L^{n,c}| = O(\frac{1}{n^\gamma}).$$

Therefore, v_n converges. This completes the proof.

Lemma 4. The sequence $\{L^{n,s}\}_{n\ge1}$ satisfies $\|L^{n,s}\| = O(\frac{1}{n^\gamma}), \;\; \gamma > 1$.

Proof. By Proposition 1, for all j, $n \ge 0$ and for all $v \in E^s_j$ we have

$$\|^T B_{j,n+j} v\| \le \frac{1}{C \cdot \mu^n} \|v\|.$$

Proposition 2 implies that there exists constant $\mu > \widetilde{\mu} > 1$ such that for all $n \ge 0$ and for all $v \in E^s_j$, we have $\|^T B_{0,n} v\| \le \widetilde{\mu}^{-1} \|v\|$. By Lemma 1, we obtain

$$\|L^{n,s}\| \le \frac{1}{\widetilde{\mu}} \|L^{n-1,s}\| + C \cdot \frac{1}{n^\gamma}.$$

By iterating this estimate n times, we obtain

$$\|L^{n,s}\| \le \frac{1}{\widetilde{\mu}^n} \|L^{0,s}\| + C \cdot \sum_{i=0}^{n-1} \frac{1}{\widetilde{\mu}^i} \frac{1}{(n-i-1)^\gamma}.$$

Since $\widetilde{\mu} > 1$, last relation and Lemma 1 imply the result.

Lemma 5. The sequence $\{L^u_n\}_{n\ge1}$ satisfies $\|L^u_n\| = O(\frac{1}{n^\gamma}), \;\; \gamma > 1$.

Proof. The proof is analogous to that of Lemma 4, and we again use the adapted norm. For all $n \ge 0$, we have

$$\|L^{n+1,u}\| \ge \widetilde{\mu} \|L^{n,u}\| - C \cdot \frac{1}{n^\gamma}.$$

Applying this estimate k times, we obtain

$$\|L^{n+k,u}\| \geq \widetilde{\mu}^k \|L^{n,u}\| - C \cdot \sum_{j=0}^{k-1} \widetilde{\mu}^j \frac{1}{(n+k-j-1)^\gamma},$$

and therefore

$$\|L^{n,u}\| \leq \frac{1}{\widetilde{\mu}^k} \|L^{n+k,u}\| + C \cdot \sum_{j=0}^{k-1} \widetilde{\mu}^{j-k} \frac{1}{(n+k-j-1)^\gamma},$$

Taking $n = k$, we have

$$\|L^{n,u}\| \leq \frac{1}{\widetilde{\mu}^n} \|L^{2n,u}\| + C \cdot \sum_{j=0}^{n-1} \widetilde{\mu}^{j-n} \frac{1}{(2n-j-1)^\gamma}.$$

Since the sequence $\{L^{2n,u}\}$ is uniformly bounded and $\widetilde{\mu} > 1$, last relation and Lemma 1 imply the result.

Proof of Theorem 2. It follows from Lemma 3 that

$$\|\omega^n - L^{n,c}\| = \|^T B_{0,n}\omega - L^{n,c}\| = \|^T B_{0,n}(\omega -^T Q_{0,n}^{-1} L^{n,c})\| \leq$$

$$\leq \|^T B_{0,n}|_{E^c_{0,\infty}}\| \cdot \|\omega - v_n\| = O(\frac{1}{n^\gamma}).$$

Thus, the last relation, together with Lemmas 4 and 5, completes the proof of Theorem 2.

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Cite this article

Begmatov A. S. *Affine shadowing of renormalizations for GIETs satisfying a certain Zygmund smoothness condition*, **Acta NUUz**, 2026, No 2/1, pp. 10-17.