Acta NUUz **Exact sciences** **No 2/1, 2026, 18-23**

LOW SMOOTHNESS: RENORMALIZATIONS OF CIRCLE MAPS WITH RATIONAL ROTATION NUMBERS AGAIN BEHAVE AS MÖBIUS FUNCTIONS

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Abstract. Consider a one-parameter family of circle maps $f_t = f_0 + t (\bmod 1)$, where f_0 is a circle homeomorphism with two break points. Suppose Df_0 satisfies a certain Zygmund type smoothness condition depending on a parameter $\gamma > 0$. We prove that the renormalizations of circle homeomorphisms from this family with rational rotation number of sufficiently large rank are approximated by Mobius functions in C^{1+L_1}-norm if $\gamma \in (1/2, 1]$ and in C^2-norm if $\gamma > 1$.

MSC (2020): 37C15, 37C40, 37E10, 37F25

Key words: circle maps, rotation number, break point, renormalization, Zygmund-type smoothness.

Introduction

The study of circle maps originates from the classical works of Poincaré and Denjoy. Their contributions laid the foundation for the modern study of dynamical systems on the circle. Poincaré (1885) noticed that the orbit structure of orientation-preserving diffeomorphism f is determined by some irrational mod 1 $\rho(f)$, called the rotation number of f. For orientation preserving homeomorphisms the rotation number, an average rotation rate, is a topological invariant that determines many properties of the dynamics. If the rotation number is rational then there is at least one periodic orbit, whilst if the rotation number is irrational then there are no periodic orbits and, provided the map is sufficiently smooth (at least C^2) all orbits are dense in the circle. The dynamics of circle maps are often described through their conjugacy classes. Two maps f and g are topologically conjugate if there is a homeomorphism h such that $h \circ f = g \circ h$. If h is a diffeomorphism then the maps are differentiably conjugate. Results on the regularity of h that go back to Denjoy state that if f is an orientation-preserving C^2-diffeomorphism of the circle with irrational rotation number ρ, then f is conjugate to the linear rotation. Since then the problem of smoothness of the conjugacy h of smooth diffeomorphisms has come to be very well understood by several authors [4]-[7]. For instance, Sinai and Khanin [7] introduced ideas that later evolved into the renormalization method, which studies asymptotical behaviour of the induced transformations on smaller subintervals of the circle.

An important generalization of circle diffeomorphisms consists of maps with break-type singularities, that is, circle homeomorphisms that are smooth away from finitely many points (the *breaks*), where the derivative has jump discontinuities. Circle diffeomorphisms with break points were introduced by Khanin and Vul [8], who also studied their renormalizations. Their results have many applications in various areas of one-dimensional dynamics, such as investigations of the invariant measures, nontrivial scalings and prevalence of periodic trajectories in one parameter families.

In one-parameter families of circle maps, rational rotation numbers typically correspond to mode-locking intervals, where the rotation number remains constant. Using the convexity of renormalizations of circle maps with a break and rational rotation numbers, Khanin and Vul [8] also showed that the Lebesgue measure of the set of parameters corresponding to rational rotation numbers is zero. This result was later generalized to the case of multiple breaks by Khmelev [9], who used renormalizations of circle homeomorphisms with rational rotation numbers and two break points.

The main contribution of this paper is to extend these results to a broader class of circle maps. More precisely, we study the asymptotic behavior of renormalizations of circle homeomorphisms with rational rotation

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Received: 14.03.2026, **Accepted:** 30.03.2026

numbers and two break points under a weaker smoothness assumption, namely a Zygmund-type condition depending on a parameter γ, rather than the $C^{2+\varepsilon}$-smoothness condition considered in [8] and [9].

Preliminaries and Notations

Define a class of circle maps satisfying a Zygmund condition. Consider the function $\mathcal{Z}_\gamma : [0,1) \to (0,+\infty)$, defined as

$$\mathcal{Z}_\gamma(x) = |\log x|^{-\gamma}, \quad \text{for} \quad x \in (0,1)$$

and $\mathcal{Z}_\gamma(0) = 0$, where $\gamma > 0$.

Let $J = [a,\, b]$ be a finite interval and consider a differentiable function $K : J \mapsto K(J)$. Denote by $\Delta^2 K(\xi, \tau)$ the *second symmetric difference* of K on J, that is,

$$\Delta^2 K(\xi,\tau) = K'(\xi+\tau) + K'(\xi-\tau) - 2K'(\xi)$$

where $\xi \in J$ and $\tau \in [0,\, |J|/2]$ such that $\xi - \tau,\, \xi + \tau \in J$.

Suppose that there exists a constant $C > 0$ such that the following inequality holds:

$$\|\Delta^2 K(\cdot, \tau)\|_{L^\infty(J)} \le C\tau \mathcal{Z}_\gamma(\tau). \tag{1}$$

Note that the class of real valued functions satisfying (1) with $\mathcal{Z}_\gamma(\tau) \equiv 1$ is called the Zygmund class and denoted by Λ_* . The class Λ_* plays a key role to investigate trigonometric series. The class Λ_* was applied to the theory of circle homeomorphisms for the first time by Jun Hu and Sullivan. They extended the classical Denjoy's theorem to the class Λ_*.

We recall two important facts concerning the smoothness of functions satisfying inequality (1).

Proposition 1([2]). *Let $K : \mathbb{R} \to \mathbb{R}$ be a 1-periodic and continuous function that satisfying (1) for some $\gamma \in (1/2,\, 1]$. Then K is absolutely continuous and $K' \in L_p([0,\, 1])$ for every $p > 1$.*

Proposition 2([2]). *Let $K : \mathbb{R} \to \mathbb{R}$ be 1-periodic continuous function that satisfies (1) for some $\gamma > 1$. Then K is of class C^1.*

In this work we study the circle maps with rational rotation numbers and satisfying (1). Consider one parameter family of the orientation preserving circle homeomorphisms

$$f_t(x) = f_0(x) + t \,(\text{mod}\, 1), \;\; x \in S^1, \;\; t \in [0,1). \tag{2}$$

where the initial lift f_0 satisfies the following conditions:

(a) f_0 is a continuous and strictly increasing on $\mathbb{R}^1$;

(b) $f_0(0) = 0, \;\; f_0(x+1) = f_0(x) + 1, \;\; x \in \mathbb{R}^1$;

(c) there are two points $x_b^{(i)}$, $i = 1,2$ such that one-sided derivatives $f_0'(x_b^i \pm 0) > 0$, $i = 1,2$ exist and $f_0'(x_b^i - 0) \neq f_0(x_b^i + 0)$, $i = 1,2$.

(d) $f_0 \in C^{1+Z_\gamma}\left(S^1 \setminus \{x_b^{(1)},\, x_b^{(2)}\}\right), \;\; \gamma > 0$.

The points $x_b^{(i)}$, $i = 1,2$ are called break points and the ratio $c_i = \sqrt{\frac{f_0'(x_b^i - 0)}{f_0'(x_b^i + 0)}}$, $i = 1,2$ is called the jump ratio of f_t at $x_b^{(i)}$, $i = 1,2$. Denote by ρ_t the rotation number of f_t, i.e. $\rho_t = \lim\limits_{n\to\infty} \frac{f_t^n(x)}{n}$ (mod1), $x \in R^1$. Here and later, g^n denotes the *nth* iteration of g.

Denoted by $\mathbb{H}^{1+Z_\gamma}$ the class of circle homeomorphisms f whose derivative f' has bounded variation and satisfying conditions (a)-(d) and the inequality (1). Note also that the class $\mathbb{H}^{1+\mathcal{Z}_\gamma}$ is bigger than $\mathbb{H}^{2+\varepsilon}$ for any positive γ and ε.

Note that for each rational number a the set $I(a) = \{\theta : \rho_\theta = a\}$ is nontrivial closed interval and $I(a)$ consists of only one point if a is irrational.

Let $\frac{p}{q} \in [0,1)$ be an arbitrary rational number of *rank* n, i.e. $\frac{p}{q} = [k_1,\, k_2,\, ...,\, k_n],\ k_n > 1$. Since the rank of $\frac{p}{q}$ equals n we write $p_n := p$ and $q_n := q$. Let us fix some $t \in I(\frac{p}{q})$ and denote $f = f_t$. Let $O_f(x_p; q_n) = \{f^j(x_p),\ j = 0,\, 1,\, ...,(q_n - 1)\}$ be a periodic orbit of f of period q_n. Denote by $\Delta_0 = [y_1; y_2]$ the closed interval formed by two consecutive points of orbit $O_f(x_p; q_n)$ that contains break point $x_b^{(1)}$. Also, denote $\Delta_i = f^i\Delta_0$. Then for some r we have $x_b^{(2)} \in \Delta_r$. Introduce the renormalized coordinate z on Δ_0 given by the formula $z = \frac{x - y_1}{y_2 - y_1}$. Denote new coordinates of $x_b^{(1)},\ x_b^{(2)}$ and $f^{-r}(x_b^{(2)})$ on intervals $\Delta_0,\ \Delta_r$ and Δ_0 by

$$d_1 = \frac{x_b^{(1)} - y_1}{y_2 - y_1} = \frac{x_b^{(1)} - y_1}{|\Delta_0|},\ \ d_2 = \frac{x_b^{(2)} - f^r y_1}{f^r y_2 - f^r y_1} = \frac{x_b^{(2)} - f^r y_1}{|\Delta_r|},\ \ \widetilde{d}_2 = \frac{f^{-r}x_b^{(2)} - y_1}{y_2 - y_1} = \frac{f^{-r}x_b^{(2)} - y_1}{|\Delta_0|}.$$

Now, we define the function $\mathbf{f}_n$ corresponding to $f^{q_n} : [y_1,\, y_2] \to [y_1,\, y_2]$ in this new coordinates by

$$\mathbf{f}_n(z) = \frac{f^{q_n}(y_1 + z(y_2 - y_1)) - y_1}{y_2 - y_1},\ \ z \in [0,\, 1].$$

The map $\mathbf{f}_n(z)$ is called *nth* renormalization of f on the interval $[y_1; y_2]$.

We need the following notations:

$$f^l_{c,\,d}(z) = \frac{c^2 z}{1 + d(c^2 - 1)},\ \ f^r_{c,\,d}(z) = \frac{z + d(c^2 - 1)}{1 + d(c^2 - 1)},\ \ f^{l,r}_{c,\,d}(z) = \begin{cases} f^l_{c,\,d}(z), & z \in [0,\, d], \\ f^r_{c,\,d}(z), & z \in (d,\, 1]. \end{cases}$$

$$F_M(z) = \frac{z}{M(1 - z) + z},\ \ M_1 = M(1; r) = \exp\left(\sum_{i=1}^{r-1} \frac{f'(b_i) - f'(a_i)}{f'(b_i)} dy\right),$$

$$M_2 = M(r + 1; q_n) = \exp\left(\sum_{i=r+1}^{q_n - 1} \frac{f'(b_i) - f'(a_i)}{f'(b_i)} dy\right),$$

where $a_i = f^i(y_1),\ b_i = f^i(y_2),\ 0 \le i \le q_n - 1$.

Define the function $G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}(z) : [0,\, 1] \to [0,\, 1]$ as follows: if $0 \le d_1 \le \widetilde{d}_2$, then

$$G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}(z) = \begin{cases} F_{\frac{c_1 c_2}{M_1}} \circ f^l_{c_2,\, d_2} \circ F_{M_1} \circ f^l_{c_1,\, d_1}(z) & \text{for} \ \ 0 \le z < d_1, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^l_{c_2,\, d_2} \circ F_{M_1} \circ f^r_{c_1,\, d_1}(z) & \text{for} \ \ d_1 \le z < \widetilde{d}_2, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^r_{c_2,\, d_2} \circ F_{M_1} \circ f^r_{c_1,\, d_1}(z) & \text{for} \ \ \widetilde{d}_2 \le z \le 1. \end{cases}$$

If $\widetilde{d}_2 < d_1 \le 1$, then

$$G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}(z) = \begin{cases} F_{\frac{c_1 c_2}{M_1}} \circ f^l_{c_2,\, d_2} \circ F_{M_1} \circ f^l_{c_1,\, d_1}(z) & \text{for} \ \ 0 \le z < \widetilde{d}_2, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^r_{c_2,\, d_2} \circ F_{M_1} \circ f^l_{c_1,\, d_1}(z) & \text{for} \ \ \widetilde{d}_2 \le z < d_1, \\ F_{\frac{c_1 c_2}{M_1}} \circ f^r_{c_2,\, d_2} \circ F_{M_1} \circ f^r_{c_1,\, d_1}(z) & \text{for} \ \ d_1 \le z \le 1. \end{cases}$$

In the sequel, to simplify the notation, we write G for $G_{c_1,\, d_1,\, c_2,\, \widetilde{d}_2,\, M_1,\, d_2}$. The following is the main result of this article.

Theorem 1. *Let $f \in \mathbb{H}^{1+Z_\gamma},\ \gamma > 0,\ \ t \in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t = p/q$ of rank n. Then, there are constant $C > 0$ and natural number n_0 such that, for all $n \ge n_0$ the following inequalities hold:*

$$\|\mathbf{f}_n - G\|_{C^1([0,1]\setminus\{d_1,\, \widetilde{d}_2\})} \le \frac{C}{n^\gamma},\ \ \ \text{when} \ \ \gamma > 1/2,$$

$$\|\mathbf{f}_n - G\|_{C^1([0,1]\setminus\{d_1,\, \widetilde{d}_2\})} \le \frac{C}{n^\gamma},\ \ \|\mathbf{f}''_n - G''\|_{C^0([0,1]\setminus\{d_1,\, \widetilde{d}_2\})} \le \frac{C}{n^{\gamma - 1}},\ \ \ \text{when} \ \ \gamma > 1.$$

Note that similar result is obtained by D. Khmelev [9] for the $C^{2+\varepsilon}$-smooth circle homeomorphisms with two break points.

Previous results and proof of the main theorem

Consider a circle homeomorphism f with two break points. Let $A = \left(\frac{p_1}{q_1},\, \frac{p_2}{q_2}\right)$ be Farey interval such that $\rho(f) \in A$. See [8] for the definition and the main properties of Farey intervals. Take an arbitrary point $x_0 \in \mathbb{S}^1$ and consider orbit $\{x_i = f^i(x_0),\ 0 \le i \le q_1 + q_2\}$. We denote the intervals $[x_0,\, x_{q_1}]$ and $[x_{q_2},\, x_0]$ by $\Delta_0^{(1)}$ and $\Delta_0^{(2)}$. Also denote the images on these intervals under the action of f by $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$:

$$\Delta_i^{(1)} = f^i \Delta_0^{(1)}, \quad \Delta_j^{(2)} = f^j \Delta_0^{(2)}.$$

The following proposition was proved in [9].

Proposition 3.([9]) Suppose $\rho(f) \in \left(\frac{p_1}{q_1},\, \frac{p_2}{q_2}\right)$. The trajectory $\{x_i = f^i(x_0),\ 0 \le i \le q_1 + q_2\}$ forms a partition of the circle consisting of intervals

$$\Delta_i^{(1)},\ 0 \le i \le q_2; \quad \Delta_j^{(2)},\ 0 \le j \le q_1.$$

Denote $v = |\log \sigma_1^2| + |\log \sigma_2^2| + Var_{S^1} \log f',$ and $\lambda = (1 + e^{-v})^{-1/2} < 1$. The following lemma shows that the lengths of the intervals $\Delta_i^{(1)}$ and $\Delta_j^{(2)}$ are exponentially small.

Lemma 1.([9]) Assume that $\rho(f) \in \left[\frac{p_1}{q_1},\, \frac{p_2}{q_2}\right]$. Suppose the expression of $\frac{p_1}{q_1}$ to the continued fraction has length n: $\frac{p_1}{q_1} = [k_1,\, k_2,\, ...,k_n],\ k_n \ge 2$. Then

$$|\Delta_i^{(1)}|,\, |\Delta_j^{(2)}| \le Const\lambda^n,$$

for all $0 \le i \le q_2$ and $0 \le j \le q_1$.

In the case where the circle map f with rational rotation number $\frac{p_n}{q_n} := \frac{p}{q}$ of rank n has a single break point x_b, we also denote by $\Delta_0^{(n)} = [y_1,\, y_2]$ the interval of the periodic trajectory containing break point x_b. Then renormalization map $\mathbf{f}_n$ corresponding to the return map $f^{q_n} : \Delta_0^{(n)} \to \Delta_0^{(n)}$ is represented as a composition $\mathbf{f}_n = F_2 \circ F_1$ of two functions F_1 and F_2, corresponding to maps $f : \Delta_0^{(n)} \to \Delta_1^{(n)}$ and $f^{q_n - 1} : \Delta_1^{(n)} \to \Delta_{q_n}^{(n)} = \Delta_0^{(n)}$. In this case the intervals $\Delta_i^{(n)},\ 1 \le i \le q_n - 1$ do not cover the break point x_b and consequently define the following quantity

$$m_n = \exp\left\{\sum_{i=1}^{q_n - 1} \frac{f'(b_i) - f'(a_i)}{f'(b_i)}\right\}, \quad \text{where} \quad a_i = f^i(y_1),\ b_i = f^i(y_2),\ 0 \le i \le q_n - 1.$$

In [1], it was shown that the functions F_1 and F_2 are approximated by $f_{c,\,d}^{l,\,r}$ and F_{m_n}, respectively. More precisely, we have the following

Lemma 2.([1]) *Let $f \in \mathbb{H}^{1+Z_\gamma},\ \gamma > 0,\ \ t \in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t = p/q$ of rank n and with a break point x_b. Then, there are constant $C > 0$ and natural number n_0 such that, for all $n \ge n_0$ the following inequalities hold:*

$$\|F_2 - F_{m_n}\|_{C^1([0,1]\setminus\{d_1\})} \le \frac{C}{n^\gamma}, \quad \text{when} \quad \gamma > 1/2,$$

$$\|F_2 - F_{m_n}\|_{C^1([0,1]\setminus\{d_1\})} \le \frac{C}{n^\gamma}, \quad \|F_2'' - F_{m_n}''\|_{C^0([0,1]\setminus\{d_1\})} \le \frac{C}{n^{\gamma-1}}, \quad \text{when} \quad \gamma > 1.$$

Lemma 3.([1]) *Let $f \in \mathbb{H}^{1+Z_\gamma},\ \gamma > 0,\ \ t \in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t = p/q$ of rank n and with a break point x_b. Then, there are constant $C > 0$ and natural number n_0 such that, for all $n \ge n_0$ the following inequalities hold:*

$$\|F_1 - f^{l,\,r}\|_{C^1([0,1]\setminus\{d_1\})} \le C\lambda^n, \quad \text{when} \quad \gamma > 1/2,$$

$$\|F_1 - f^{l,\,r}\|_{C^2([0,1]\setminus\{d_1\})} \le C\lambda^n, \quad \text{when} \quad \gamma > 1.$$

We now return to the case where the circle map f from the family (2) has two break points and rational rotation number $\frac{p}{q}$ of rank n. Assume that the intervals $\Delta_i^{(n)}$, $i = r_1, ..., r_2 - 1$ do not cover break points $x_b^{(1)}$ and $x_b^{(2)}$. Define renormalized coordinates of the map $f^{r_2 - r_1} : \Delta_{r_1}^{(n)} \to \Delta_{r_2}^{(n)}$ in the new coordinates by

$$\mathcal{F}^{\,r_1,\,r_2}(z) = \frac{f^{r_2-r_1}((1-z)f^{r_1}y_1 + zf^{r_1}y_2) - f^{r_2}y_1}{|\Delta_{r_2}^{(n)}|}, \quad z \in [0,\,1].$$

Denote

$$M(r_1; r_2) = \exp\left(\sum_{i=r_1}^{r_2-1} \int_{\Delta_i} \frac{f'(b_i) - f'(a_i)}{f'(b_i)} dy\right), \quad \text{where} \quad a_i = f^i(y_1),\ b_i = f^i(y_2),\ 0 \le i \le q_n - 1.$$

The following lemma is proved similarly to Lemma 2.

Lemma 4. *Let $f \in \mathbb{H}^{1+Z_\gamma}$, $\gamma > 0$, $t \in I(p_n/q_n)$ be circle homeomorphisms from the family (2) with rational rotation number $\rho_t = p/q$ of rank n and with two break points $x_b^{(1)}$ and $x_b^{(2)}$. Then, there are constant $C > 0$ and natural number n_0 such that, for all $n \ge n_0$ the following inequalities hold:*

$$\|\mathcal{F}^{\,r_1,\,r_2} - F_{M(r_1;r_2)}\|_{C^1([0,1]\setminus\{d\})} \le \frac{C}{n^\gamma}, \quad \text{when} \quad \gamma > 0,$$

$$\|\mathcal{F}^{\,r_1,\,r_2} - F_{M(r_1;r_2)}\|_{C^1([0,1]\setminus\{d\})} \le \frac{C}{n^\gamma}, \quad \|(\mathcal{F}^{\,r_1,\,r_2})'' - F''_{M(r_1;r_2)}\|_{C^0([0,1]\setminus\{d\})} \le \frac{C}{n^{\gamma-1}}, \quad \text{when} \quad \gamma > 1.$$

Proof of Theorem 1. It is clear that the map $\mathbf{f}_n$ can be represented as

$$\mathbf{f}_n = \mathcal{F}^{\,r+1,\,q} \circ \mathcal{F}^{\,r,\,r+1} \circ \mathcal{F}^{\,1,\,r} \circ \mathcal{F}^{\,0,\,1}. \tag{2}$$

Then by Lemma 3 the map $\mathcal{F}^{\,0,\,1}$ is approximated by $f^{l,r}_{c_1,d_1}$ and $\mathcal{F}^{\,r,\,r+1}$ is approximated by $f^{l,r}_{c_2,d_2}$. On the other hand, Lemma 4 implies that $\mathcal{F}^{\,1,\,r}$ is close to F_{M_1} and $\mathcal{F}^{\,r+1,\,q}$ is close to F_{M_2}. To complete the proof of the theorem, we only need to compare M_1M_2 with c_1c_2.

By Propositions 1 and 2, the function f' is absolute continuous and $f'' \in L_p$ for every $p > 1$ in the case $\gamma \in (1/2,\,1]$ and it is differentiable in the case $\gamma > 1$. Consequently,

$$\frac{f'(b_i) - f'(a_i)}{f'(b_i)} dy = \int_{\Delta_i} \frac{f''(y)}{f'(y)} dy + \int_{\Delta_i} \frac{f''(y)}{f'(y)} \left(\int_{a_i}^{y} \frac{f''(t)}{f'(t)} dt\right) dy.$$

By Lemma 1, $|\Delta_i^{(n)}| \le C\lambda^n$ for $i = 0, 1, ..., (q_n - 1)$, and therefore (sse also [3])

$$|\int_{\Delta_i} \frac{f''(y)}{f'(y)} \left(\int_{a_i}^{y} \frac{f''(t)}{f'(b_i)} dt\right) dy| = O(\lambda_1^n \int_{\Delta_i} |\frac{f''(y)}{f'(y)} dy|), \ \ \lambda_1 = \lambda^{1-\frac{1}{p}}.$$

Now we estimate M_1M_2 as

$$\log M_1M_2 = \sum_{i=1}^{q-1} \int\limits_{\Delta_i^{(n)}} \frac{f''(y)}{2f'(y)} dy - \int\limits_{\Delta_0^{(n)}} \frac{f''(y)}{2f'(y)} dy - \int\limits_{\Delta_1^{(n)}} \frac{f''(y)}{2f'(y)} dy + O(\lambda_1^n) =$$

$$= \log c_1c_2 - \int\limits_{\Delta_0^{(n)}} \frac{f''(y)}{2f'(y)} dy - \int\limits_{\Delta_1^{(n)}} \frac{f''(y)}{2f'(y)} dy + O(\lambda_1^n).$$

Applying Lemma 1 again, we obtain

$$\left| \int\limits_{\Delta_0^{(n)}} \frac{f''(y)}{2f'(y)} dy + \int\limits_{\Delta_1^{(n)}} \frac{f''(y)}{2f'(y)} dy \right| = O(\lambda_1^n).$$

Hence, $M_1M_2 = c_1c_2 + O(\lambda_1^n)$. Moreover, M_1 and M_2 are bounded.

It can be easily verified that functions $f_{c,d}^l$, $f_{c,d}^r$ and F_M have the following useful properties:

$$F_M \circ F_N = F_{MN}, \;\; f_{c,d}^l(d) = f_{c,d}^r(d) = F_{1/c^2}(d),$$

for all M, N, c, d. These properties and relation (2), together with Lemmas 3 and 4 imply the assertions of the Theorem 1.

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Cite this article

Begmatov A. S. *Low Smoothness: Renormalizations of circle maps with rational rotation numbers again behave as Möbius functions*, **Acta NUUz**, 2026, No 2/1, pp. 18-23.