Boundary Control Problem for the Heat Equation with a Nonlocal Boundary Condition

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Abstract. In this paper, we study a boundary control problem for the one-dimensional heat equation with a nonlocal boundary condition. The control is applied at one end of the domain, while the temperature at the opposite boundary is linked to the temperature at a fixed interior point. The control objective is formulated as an integral condition prescribing the average temperature of the rod. Using the method of separation of variables, the problem is reduced to a Volterra integral equation of the first kind. The existence of an admissible control function is proved by means of the Laplace transform method.

MSC (2020): 35K05; 35K15.

Key words: heat equation, initial-boundary problem, spectral problem, admissible control, rod.

Introduction

Control problems for the heat equation play a fundamental role in the theory of parabolic partial differential equations and have numerous applications in thermal processes. The foundations of optimal control theory for parabolic-type equations were developed by Fattorini and Friedman [1,2]. Later, Egorov [3] investigated control problems in infinite-dimensional spaces, extending Pontryagin's maximum principle to certain classes of equations in Banach spaces and establishing a bang-bang principle under appropriate conditions.

Boundary control problems for parabolic equations in multidimensional domains with piecewise smooth boundaries were studied in [4,5], where estimates for the minimum time required to reach a prescribed average temperature were obtained. Mathematical models of thermocontrol processes for parabolic equations were considered in [6], while control problems for the heat equation in three-dimensional domains were analyzed in [7].

Control problems in bounded one- and two-dimensional domains were investigated in [8–11]. In these works, estimates for the minimum time needed to achieve a given average temperature were derived, and the existence of a control function was established by means of the Laplace transform method.

Comprehensive treatments of optimal control theory for parabolic equations can be found in the monographs by Lions and Fursikov [12,13]. Numerical optimization methods and optimal control for second-order parabolic equations were studied in [14], and practical applications were presented in [15].

The aim of this work is to study the boundary control problem for the heat equation under fixed average temperature conditions. By applying the separation of variables method, the problem is reduced to a Volterra integral equation of the first kind. The solution of this integral equation is analyzed using the Laplace transform method, which allows us to precisely construct the control function and determine its existence.

Received: 11.02.2026, **Accepted:** 12.03.226
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Statement of the Problem

In this paper, we consider the following heat equation in the domain $\Omega_T := (0,1)\times(0,\infty)$:

$$u_t(x,t) = u_{xx}(x,t), \quad (x,t)\in\Omega_T, \tag{1}$$

with a nonlocal boundary condition of Bitsadze-Samarskii type

$$u(0,t)=\nu(t), \quad u(1,t)=u(x_0,t), \quad t\ge 0, \quad x_0\in(0,1), \tag{2}$$

and the initial condition

$$u(x,0)=0, \quad 0\le x\le 1. \tag{3}$$

Here $\nu(t)$ is a boundary control function applied at the left end of the rod, while $x_0\in(0,1)$ is a fixed interior point.

From the physical point of view, equation (1) describes the heat conduction process in a thin homogeneous rod of unit length. The function $u(x,t)$ represents the temperature at position x and time t. The initial condition (3) means that the rod is initially at zero temperature. The boundary condition $u(0,t)=\nu(t)$ corresponds to an external heating or cooling device located at the left end of the rod. The function $\nu(t)$ plays the role of a time-dependent boundary heat source and is regarded as the control input.

The nonlocal condition $u(1,t)=u(x_0,t)$ indicates a thermal coupling between the right end of the rod and an interior point $x=x_0$. Physically, this may model a situation in which the temperature at the right boundary is required to match the temperature at some prescribed internal location, for example, due to a feedback mechanism, a thermal sensor placed at $x=x_0$, or a design constraint that equalises the temperature at two distinct points. Such nonlocal constraints appear in various engineering applications, including thermal control systems, thermal imaging, and in problems with internal monitoring or symmetry requirements. The presence of this nonlocal condition significantly influences the heat distribution along the rod and introduces mathematical challenges that differ from those of classical boundary-value problems.

If the control function $\nu(t)\in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\nu(0)=0$ and $|\nu(t)|\le 1$ on the half-line $t\ge 0$, then we call it an *admissible control*.

We note that the regularity condition $\nu\in W_2^1(\mathbb{R}_+)$ will be justified later in Section 4 as a consequence of the solvability of the control problem.

We now formulate the main control problem considered in this paper.

Control Problem. *For the given function $\phi(t)$, find an admissible control $\nu(t)$ such that the solution $u(x,t)$ of the initial-boundary value problem (1)-(3) exists and satisfies the integral condition*

$$\int_0^1 u(x,t)\,dx = \phi(t), \quad t\ge 0. \tag{4}$$

Condition (4) has a clear physical interpretation. The integral $\int_0^1 u(x,t)\,dx$ represents the total (or average) thermal energy of the rod at time t. Therefore, the control objective is to regulate the boundary temperature $\nu(t)$ in such a way that the overall heat content of the rod follows a prescribed evolution $\phi(t)$.

This type of control problem naturally arises in thermal engineering, where it is often impossible or unnecessary to control the temperature at every point of the domain. Instead, one aims to achieve a desired average temperature profile by acting on the system through boundary inputs. The presence of the nonlocal constraint $u(1,t)=u(x_0,t)$ reflects additional physical coupling between the boundary and an interior point, which may arise from design specifications, sensor-based feedback, or symmetry requirements in thermal systems. This coupling introduces new mathematical challenges in the analysis and synthesis of the control process, distinguishing it from classical boundary control problems.

For any constant $M>0$, we denote by $W(M)$ the set of functions $\phi\in W_2^2(-\infty,+\infty)$, $\phi(t)=0$ for all $t\le 0$ which satisfy the condition

$$\|\phi\|_{W_2^2(R_+)}\le M.$$

We present the following main theorem.

Theorem 1. *There exists $M>0$ such that for any function $\phi\in W(M)$ the solution $\nu(t)$ of the equation (4) exists and it satisfies condition $|\nu(t)|\le 1$.*

Integral equation for control function

In this section, we show how the considered control problem can be reduced to a Volterra integral equation of the first kind. For this purpose, we first eliminate the nonhomogeneous boundary conditions and then solve the corresponding auxiliary problem by the Fourier method.

We consider the spectral problem

$$X''(x) + \lambda X(x) = 0, \quad 0 < x < 1,$$

with the nonlocal boundary conditions

$$X(0) = 0, \quad X(1) = X(x_0), \quad x_0 \in (0,1).$$

It is easy to see that for $\lambda \leq 0$ this problem admits only the trivial solution. Therefore, we restrict ourselves to the case $\lambda > 0$. In this case, the spectrum consists of two countable sequences of eigenvalues given by

$$\lambda_{k,1} = \left(\frac{(2k-1)\pi}{1+x_0}\right)^2, \qquad \lambda_{k,2} = \left(\frac{2k\pi}{1-x_0}\right)^2, \quad k \in \mathbb{N}, \quad x_0 \in (0,1).$$

The corresponding eigenfunctions have the form

$$X_{k,1}(x) = \sin\bigl(\sqrt{\lambda_{k,1}}\, x\bigr), \qquad X_{k,2}(x) = \sin\bigl(\sqrt{\lambda_{k,2}}\, x\bigr), \quad k \in \mathbb{N}.$$

The system of eigenfunctions $\bigl\{X_{k,1}(x),\, X_{k,2}(x)\bigr\}_{k=1}^{\infty}$ is orthogonal and complete in the space $L_2(0,1)$. This system will be used to construct the solution of the auxiliary problem by the Fourier method (see [16]).

Let B be an arbitrary Banach space and let $T > 0$. By $C([0,T] \to B)$ we denote the Banach space of all continuous mappings $u : [0,T] \to B$ equipped with the norm

$$\|u\| = \max_{0 \leq t \leq T} \|u(t)\|_B.$$

By $\widetilde{W}_2^1(\Omega)$ we denote the subspace of the Sobolev space $W_2^1(\Omega)$ consisting of functions whose trace on $\partial\Omega$ is equal to zero. Since $\widetilde{W}_2^1(\Omega)$ is closed in $W_2^1(\Omega)$, the sum of a series of functions from $\widetilde{W}_2^1(\Omega)$ converging in the $W_2^1(\Omega)$-norm also belongs to $\widetilde{W}_2^1(\Omega)$, where $\Omega = (0,1)$.

Definition 1. *By the solution of the problem (1)-(3) we mean a function $u(x,t)$, represented in the form*

$$u(x,t) = \nu(t) - w(x,t), \tag{5}$$

where the function $w(x,t)$ is a generalized solution from the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ of the following problem:

$$w_t(x,t) - w_{xx}(x,t) \;=\; \nu'(t),$$

with homogeneous initial and boundary conditions

$$w(0,t) = 0, \quad w(1,t) = w(x_0,t), \quad w(x,0) = 0.$$

We set

$$p_j := \|X_{k,j}\|_{L_2(0,1)} = \sqrt{\int_0^1 X_{k,j}^2 dx} > 0, \quad j = 1,2, \quad k \in \mathbb{N}.$$

Thus, we obtain (see [17])

$$w(x,t) \;=\; \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds\right) \sin\bigl(\sqrt{\lambda_{k,j}}\, x\bigr), \tag{6}$$

where

$$a_{k,j} = \frac{1 - \cos\sqrt{\lambda_{k,j}}}{p_j\, \sqrt{\lambda_{k,j}}}, \quad j = 1,2, \quad k \in \mathbb{N}.$$

Note that the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ is a subset of the class $W_2^{1,0}(\Omega_T)$, which was considered in monograph [18] for defining a solution to the problem homogeneous boundary conditions (see the corresponding uniqueness theorem in Ch. III, Theorem 3.2, pp. 173-176). Therefore, the above introduced generalized solution is also a generalized solution in the sense of [18]. However, unlike a solution from the class $W_2^{1,0}(\Omega_T)$, which is guaranteed to have a trace for almost everywhere $t \in [0,T]$, a solution from a class $C([0,T] \to \widetilde{W}_2^1(\Omega))$ continuously depends on $t \in [0,T]$ in the metric $L_2(\Omega)$.

Lemma 1. *Let $\nu \in W_2^1(\mathbb{R}_+)$ and $\nu(0) = 0$. Then the function*

$$u(x,t) = \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j}\, \lambda_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu(s)\, ds \right) \sin\big(\sqrt{\lambda_{k,j}}\, x\big), \tag{7}$$

is the solution of the initial-boundary value problem (1)-(3).

Proof. Using representations (5) and (6), we rewrite the solution of problem (1)-(3) in the form

$$u(x,t) = \nu(t) - \sum_{j=1}^{2} \frac{1}{p_i} \sum_{k=1}^{\infty} a_{k,j} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right) \sin\big(\sqrt{\lambda_{k,j}}\, x\big).$$

We prove that the function $w(x,t)$ defined by series (6) belongs to the class $C([0,T] \to \widetilde{W}_2^1(\Omega))$. It is sufficient to show that the spatial derivative $w_x(\cdot,t)$ belongs to $L_2(0,1)$ for every $t \in [0,T]$ and depends continuously on t with respect to the $L_2(0,1)$-norm.

Differentiating series (6) with respect to x, we obtain

$$w_x(x,t) = \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j} \sqrt{\lambda_{k,j}} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right) \cos\big(\sqrt{\lambda_{k,j}}\, x\big).$$

By Parseval's equality, the $L_2(0,1)$-norm of $w_x(\cdot,t)$ is given by

$$\|w_x(\cdot,t)\|^2_{L_2(0,1)} = \sum_{j=1}^{2} \sum_{k=1}^{\infty} \frac{a_{k,j}^2\, \lambda_{k,j}}{p_j^2} \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right)^2.$$

Using the Cauchy-Schwarz inequality, we estimate the integral term as

$$\left(\int_0^t e^{-\lambda_{k,j}(t-s)}\, \nu'(s)\, ds \right)^2 \le \int_0^t e^{-2\lambda_{k,j}(t-s)}\, ds \int_0^t |\nu'(s)|^2\, ds \le \frac{1}{2\lambda_{k,j}} \|\nu'\|^2_{L_2(0,T)}.$$

Substituting this estimate into the previous expression, we obtain

$$\|w_x(\cdot,t)\|^2_{L_2(0,1)} \le \frac{1}{2} \|\nu'\|^2_{L_2(0,T)} \sum_{j=1}^{2} \sum_{k=1}^{\infty} \frac{a_{k,j}^2}{p_j^2}.$$

Recalling that

$$a_{k,j} = \frac{1 - \cos\sqrt{\lambda_{k,j}}}{p_j \sqrt{\lambda_{k,j}}},$$

and taking into account the asymptotic behavior $\lambda_{k,j} \sim k^2$ as $k \to \infty$, we conclude that the series

$$\sum_{j=1}^{2} \sum_{k=1}^{\infty} \frac{a_{k,j}^2}{p_j^2}$$

is convergent. Hence, there exists a constant $C > 0$ such that

$$\|w_x(\cdot,t)\|^2_{L_2(0,1)} \le C\, \|\nu'\|^2_{L_2(0,T)}, \quad t \in [0,T].$$

Therefore, $w_x(\cdot,t) \in L_2(0,1)$ for all $t \in [0,T]$, and the mapping $t \mapsto w_x(\cdot,t)$ is continuous in the $L_2(0,1)$-norm. This implies that

$$w \in C([0,T] \to \widetilde{W}_2^1(\Omega)).$$

Finally, the fact that $w(x,t)$ is a generalized solution in the sense of the integral identity (3.5) of monograph [18] follows directly from Parseval's equality. The lemma is proved. □

Using the integral condition (4) and the solution (7), we can write

$$\phi(t) = \int\limits_0^1 u(x,t)\,dx = \sum_{j=1}^{2} \frac{1}{p_j} \sum_{k=1}^{\infty} a_{k,j}\,\sqrt{\lambda_{k,j}}\,(1-\cos\sqrt{\lambda_{k,j}}) \left(\int_0^t e^{-\lambda_{k,j}(t-s)}\,\nu(s)\,ds\right).$$

Set

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \Psi_{k,j}\, e^{-\lambda_{k,j} t}, \quad t>0, \tag{8}$$

where

$$\Psi_{k,j} = \frac{a_{k,j}}{p_j}\sqrt{\lambda_{k,j}}\,\big(1-\cos\sqrt{\lambda_{k,j}}\big), \quad j=1,2, \quad k \in \mathbb{N}.$$

Thus, we have the following Volterra integral equation of the first kind

$$\int\limits_0^t K(t-s)\,\nu(s)ds = \phi(t), \quad t>0. \tag{9}$$

Equation (9) is a Volterra integral equation of the first kind with a kernel $K(t)$ defined by (8). In order to investigate the solvability of this equation and the regularity properties of the control function $\nu(t)$, we first study the behaviour of the kernel $K(t)$ for $t>0$.

Lemma 2. *For the kernel function $K(t)$ defined by (8), the following estimate holds:*

$$K(t) \ \leq \ \frac{C}{\sqrt{t}}, \quad 0<t\leq 1,$$

where $C>0$ is a constant independent of t.

By definition, the kernel $K(t)$ has the representation

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{a_{k,j}}{p_j}\,\sqrt{\lambda_{k,j}}\,\big(1-\cos\sqrt{\lambda_{k,j}}\big)\,e^{-\lambda_{k,j}t}.$$

Recall that the coefficients $a_{k,i}$ are given by

$$a_{k,j} = \frac{1-\cos\sqrt{\lambda_{k,j}}}{p_j\sqrt{\lambda_{k,j}}}, \quad j=1,2, \quad k\in\mathbb{N}.$$

Substituting this expression into the series for $K(t)$, we obtain

$$K(t) = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{(1-\cos\sqrt{\lambda_{k,j}})^2}{p_j^2}\,e^{-\lambda_{k,j}t}.$$

Using the elementary inequality

$$0 \leq 1-\cos y \leq 2, \quad y\in\mathbb{R},$$

we immediately get

$$(1-\cos\sqrt{\lambda_{k,j}})^2 \leq 4.$$

Hence,

$$K(t) \le \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{4}{p_j^2}\, e^{-\lambda_{k,j} t}.$$

Since the constants p_j do not depend on k, there exists a constant $C_1 > 0$ such that

$$K(t) \le C_1 \sum_{j=1}^{2}\sum_{k=1}^{\infty} e^{-\lambda_{k,j} t}.$$

Taking into account that the eigenvalues satisfy the asymptotic relation

$$\lambda_{k,j} \sim C_2 k^2, \quad k \to \infty,$$

we use the classical estimate

$$\sum_{k=1}^{\infty} e^{-ck^2 t} \le \frac{C_3}{\sqrt{t}}, \quad 0 < t \le 1.$$

Combining the above inequalities, we conclude that

$$K(t) \le \frac{C}{\sqrt{t}}, \quad 0 < t \le 1,$$

where $C > 0$ is a constant independent of t. The lemma is proved. □

Main result

We solve the Volterra integral equation (9) by means of the Laplace transform. Recall that the Laplace transform of the control function $\nu(t)$ is defined by

$$\widetilde{\nu}(p) = \int_0^{\infty} e^{-pt}\nu(t)\, dt, \quad \Re p > 0.$$

Applying the Laplace transform to both sides of equation (9) and using the convolution theorem, we obtain

$$\widetilde{\phi}(p) = \widetilde{K}(p)\, \widetilde{\nu}(p).$$

Hence,

$$\widetilde{\nu}(p) = \frac{\widetilde{\phi}(p)}{\widetilde{K}(p)}, \quad p = \xi + i\tau,\ \xi > 0,\ \tau \in \mathbb{R}.$$

By the inverse Laplace transform, the control function $\nu(t)$ can be written as

$$\nu(t) = \frac{1}{2\pi i}\int_{\xi - i\infty}^{\xi + i\infty} \frac{\widetilde{\phi}(p)}{\widetilde{K}(p)} e^{pt}\, dp = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{\widetilde{\phi}(\xi + i\tau)}{\widetilde{K}(\xi + i\tau)} e^{(\xi + i\tau)t}\, d\tau. \tag{10}$$

Lemma 3. *For $\xi > 0$ and $\tau \in \mathbb{R}$ the estimate*

$$|\widetilde{K}(\xi + i\tau)| \ge \frac{C_\xi}{\sqrt{1 + \tau^2}}$$

holds, where $C_\xi > 0$ is a constant depending only on ξ.

Proof. Using the definition of $K(t)$, we write

$$\widetilde{K}(p) = \int_0^{\infty} K(t) e^{-pt}\, dt = \sum_{j=1}^{2}\sum_{k=1}^{\infty} \frac{\Psi_{k,j}}{p + \lambda_{k,j}}.$$

For $p=\xi+i\tau$, we have

$$\widetilde{K}(\xi+i\tau)=\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+\tau^2}-i\tau\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+\tau^2}.$$

Therefore,

$$\Re\widetilde{K}(\xi+i\tau)=\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+\tau^2},$$

$$\Im\widetilde{K}(\xi+i\tau)=-\tau\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+\tau^2}.$$

Since

$$(\xi+\lambda_{k,j})^2+\tau^2\le((\xi+\lambda_{k,j})^2+1)(1+\tau^2),$$

we obtain

$$\frac{1}{(\xi+\lambda_{k,j})^2+\tau^2}\ge\frac{1}{1+\tau^2}\frac{1}{(\xi+\lambda_{k,j})^2+1}.$$

Consequently,

$$|\Re\widetilde{K}(\xi+i\,\tau)|=\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}\,(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+\tau^2}\ge$$

$$\ge\frac{1}{1+\tau^2}\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}\,(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+1}=\frac{C_{1,\xi}}{1+\tau^2},$$

and

$$|\Im\widetilde{K}(\xi+i\,\tau)|=|\tau|\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+\tau^2}\ge$$

$$\ge\frac{|\tau|}{1+\tau^2}\sum_{j=1}^{2}\sum_{k=1}^{\infty}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+1}=\frac{C_{2,\xi}\,|\tau|}{1+\tau^2},$$

where

$$C_{1,\xi}=\sum_{j,k}\frac{\Psi_{k,j}(\xi+\lambda_{k,j})}{(\xi+\lambda_{k,j})^2+1},\qquad C_{2,\xi}=\sum_{j,k}\frac{\Psi_{k,j}}{(\xi+\lambda_{k,j})^2+1}.$$

Hence,

$$|\widetilde{K}(\xi+i\tau)|^2\ge\frac{\min(C_{1,\xi}^2,C_{2,\xi}^2)}{1+\tau^2},$$

which proves the lemma. □

Then, proceed to the limit as $\xi\to0$ from (10), we obtain the equality

$$\nu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)}\,e^{i\,\tau t}\,d\tau. \tag{11}$$

Lemma 4. ([11]) *Assume that $\phi\in W(M)$. Then for the imaginary part of the Laplace transform of function $\phi(t)$ the following inequality holds:*

$$\int_{-\infty}^{+\infty}|\widetilde{\phi}(i\,\tau)|\sqrt{1+\tau^2}\,d\tau\le C\,\|\phi\|_{W_2^2(R_+)},$$

where $C>0$ is a constant.

Now we prove Theorem 1.

Proof of Theorem 1. First of all, we prove that $\nu \in W_2^1(\mathbb{R}_+)$. Using lemmas 3 and 4, we get the estimate

$$\int_{-\infty}^{+\infty} |\widetilde{\nu}(\tau)|^2(1+|\tau|^2)\,d\tau \;=\; \int_{-\infty}^{+\infty} \left|\frac{\widetilde{\phi}(i\,\tau)}{\widetilde{K}(i\,\tau)}\right|^2 (1+|\tau|^2)\,d\tau \;\le$$

$$\le C_0 \int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)|^2(1+|\tau|^2)^2\,d\tau \;=\; C_0\|\phi\|^2_{W_2^2(\mathbb{R})},$$

where $C_0 = \min(C_{1,0}, C_{2,0})$.

Besides, we have

$$|\nu(t)-\nu(s)| \;=\; \left|\int_s^t \nu'(y)\,dy\right| \;\le\; \|\nu'\|_{L_2}\,(t-s)^{1/2}.$$

From (11), lemmas 3 and 4, we can write

$$|\nu(t)| \le \frac{1}{2\pi}\int_{-\infty}^{+\infty} \frac{|\widetilde{\phi}(i\,\tau)|}{|\widetilde{K}(i\,\tau)|}d\tau \le \frac{1}{2\pi C_0}\int_{-\infty}^{+\infty} |\widetilde{\phi}(i\,\tau)|\sqrt{1+\tau^2}d\tau \le$$

$$\le \frac{C}{2\pi C_0}\|\phi\|_{W_2^2(\mathbb{R}_+)} \le \frac{C\,M}{2\pi C_0} = 1,$$

where M is as follows:

$$M = \frac{2\pi C_0}{C}.$$

Conclusion

In this paper, a boundary control problem for the one-dimensional heat equation with a nonlocal boundary condition has been studied. The control objective was to steer the average temperature of the rod to a prescribed function. By applying the Fourier method, the problem was reduced to a Volterra integral equation of the first kind. Using the Laplace transform method, the existence of the admissible control function was proved and its regularity properties were established. The obtained results demonstrate that nonlocal boundary conditions lead to new analytical features that distinguish the problem from classical boundary control settings.

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Cite this article

Dekhkonov F.N., Usubjonov A.A., Nematjonova M.R. *Boundary control problem for the heat equation with a nonlocal boundary condition*, **Acta NUUz**, 2026, No 2/1, pp. 24-32.