

METALLIC RICCI SOLITON ON THE Sol^3 MANIFOLD

ERGASHEV MUXAMMADALI ABDUQAHHOR UGLI

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M. ULUGBEK, TASHKENT, UZBEKISTAN
turonsitiyim@gmail.com

ABSTRACT. In this paper, we study the manifold Sol^3 endowed with its standard left-invariant Riemannian metric together with a polynomial structure. First, we recall the Lie group structure of Sol^3 and compute the main geometric objects associated with the standard metric, including an orthonormal frame, Lie brackets, the Levi-Civita connection, and the Ricci tensor. Then we introduce a polynomial structure satisfying a quadratic relation and investigate its interaction with the Ricci tensor. In particular, we prove that Sol^3 is not an Einstein manifold and that the Ricci operator commutes with the polynomial structure. Finally, we study a metallic Ricci soliton type equation associated with the polynomial structure and obtain an explicit family of solutions for the corresponding vector field. These results provide new examples of curvature structures compatible with polynomial tensor fields on solvable Lie groups.

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Key words: Sol^3 manifold, metallic structure, Einstein manifold, Ricci tensor, Ricci soliton, metallic Ricci soliton.

Introduction

The geometry Sol^3 is one of Thurston's eight model geometries [1]. It can be realized as a three-dimensional solvable and non-nilpotent Lie group equipped with a left-invariant Riemannian metric. Among homogeneous spaces, Sol^3 occupies a distinguished position because its geometric behavior differs essentially from that of spaces of constant sectional curvature.

A broad account of the geometries of 3-manifolds was given by P. Scott [2]. A projective interpretation of the eight three-dimensional homogeneous geometries, including Sol geometry, can be found in [3]. These foundational results show that Sol^3 provides a natural model for the study of homogeneous Riemannian manifolds with anisotropic geometric properties.

Geometric aspects of curves and surfaces in the homogeneous space Sol^3 are treated in [4]. Further properties of Thurston-type geometries were considered by Erjavec [5]. Together, these works indicate that Sol^3 is a rich geometric setting in which curvature phenomena differ substantially from the Euclidean, spherical, and hyperbolic cases.

Ricci solitons play a central role in differential geometry because they arise as self-similar solutions of the Ricci flow introduced by Hamilton [6]. They also appear as a natural generalization of Einstein metrics, and a broad survey of their geometric significance is available in [7]. In dimension three, homogeneous Ricci solitons were analyzed in detail in [8].

The case of the Lie group Sol^3 was examined in [9], where Ricci soliton structures were investigated. This confirms that Sol^3 is an appropriate framework for the study of Ricci-type equations and related curvature properties on solvable Lie groups.

Another active line of research concerns polynomial structures on Riemannian manifolds. Metallic structures and their basic geometric properties were studied in [10], while related symmetry properties for diagonal metrics were discussed in [11]. For Lie groups endowed with left-invariant metrics, curvature computations are closely connected with the classical formulas established by Milnor [12].

Recent works of A.S. Sharipov and collaborators also deal with curvature-related questions in differential geometry, including isometric properties of surfaces and existence problems for geometric objects with prescribed

curvature conditions [13], [14]. These contributions further illustrate the continuing interest in curvature and their geometric applications.

Motivated by the above developments, in this paper we study the manifold Sol^3 endowed with its standard left-invariant metric together with a diagonal polynomial structure. We determine the corresponding orthonormal frame, Lie bracket relations, Levi-Civita connection, and Ricci tensor. Next, we examine the interaction between the Ricci operator and the polynomial structure, prove that Sol^3 is not an Einstein manifold, and obtain a decomposition of the Ricci tensor in terms of the tensors g and $g(JX, Y)$. Finally, we derive an explicit family of vector fields satisfying a metallic Ricci soliton type equation on Sol^3 .

Preliminaries

In this section, we recall the basic definitions and formulas that will be used throughout the paper.

Definition 1. [2] The manifold Sol^3 is the Lie group on \mathbb{R}^3 with the group law

$$(x, y, z) \cdot (x', y', z') = (x + e^{-z}x', y + e^z y', z + z').$$

Its matrix representation is given by

$$\begin{pmatrix} e^{-z} & 0 & x \\ 0 & e^z & y \\ 0 & 0 & 1 \end{pmatrix}.$$

The standard left-invariant Riemannian metric on Sol^3 is given by

$$g = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2.$$

We consider the orthonormal frame

$$E_1 = e^{-z} \frac{\partial}{\partial x}, \quad E_2 = e^z \frac{\partial}{\partial y}, \quad E_3 = \frac{\partial}{\partial z}. \tag{1}$$

With respect to the frame in (1), the Lie bracket relations are

$$[E_1, E_2] = 0, \quad [E_1, E_3] = E_1, \quad [E_2, E_3] = -E_2. \tag{2}$$

Using the Koszul formula together with (2), we obtain the Levi-Civita connection

$$\nabla_{E_1} E_1 = -E_3, \quad \nabla_{E_1} E_3 = E_1, \quad \nabla_{E_2} E_2 = E_3, \quad \nabla_{E_2} E_3 = -E_2, \tag{3}$$

and all remaining components vanish, that is, $\nabla_{E_i} E_j = 0$.

From the above connection formulas, the Ricci tensor is diagonal in the orthonormal frame. Recall that the Ricci tensor is defined by

$$Ric(X, Y) = \sum_{k=1}^n g(R(E_k, X)Y, E_k).$$

where any smooth vector fields X, Y . Therefore, we have

$$Ric(E_1, E_1) = 0, \quad Ric(E_2, E_2) = 0, \quad Ric(E_3, E_3) = -2, \tag{4}$$

while

$$Ric(E_i, E_j) = 0, \quad i \neq j.$$

From (3), a direct curvature computation yields

$$K(E_1, E_2) = 1, \quad K(E_1, E_3) = -1, \quad K(E_2, E_3) = -1.$$

Hence, Sol^3 does not have constant sectional curvature.

Definition 2. [7] A Riemannian manifold (M, g) is called an Einstein manifold if there exists a constant $\lambda \in \mathbb{R}$ such that

$$Ric = \lambda g. \tag{5}$$

Definition 3. [6] A Riemannian manifold (M, g) is called a Ricci soliton if there exist a smooth vector field V and a constant $\lambda \in \mathbb{R}$ such that

$$\mathcal{L}_V g + 2\text{Ric} = 2\lambda g.$$

If $V = \nabla f$ for some smooth function f , then the Ricci soliton is called a gradient Ricci soliton.

Definition 4. [10] Let (M, g) be a Riemannian manifold. A $(1, 1)$ -tensor field J is called a metallic structure on the Riemannian manifold if

$$J^2 = pJ + qI. \tag{6}$$

Here p and q are positive integers, and I denotes the identity operator.

Let us denote the roots of the equation $x^2 = px + q$ by

$$\sigma_1 = \frac{p + \sqrt{p^2 + 4q}}{2}, \quad \sigma_2 = \frac{p - \sqrt{p^2 + 4q}}{2}. \tag{7}$$

In what follows, we consider the tensor field J defined by

$$J(E_1) = \sigma_1 E_1, \quad J(E_2) = \sigma_1 E_2, \quad J(E_3) = \sigma_2 E_3, \tag{8}$$

on the orthonormal frame of Sol^3 manifold.

For the metric, the relation

$$g(JX, Y) = g(X, JY)$$

holds for all vector fields X, Y .

Main Results

We now derive the principal results of the paper.

Proposition 1. The manifold Sol^3 endowed with the above metric is not an Einstein manifold.

Proof. we have (4) equalities. Now, assume that Sol^3 is an Einstein manifold. Then, by Definition 2, there exists a constant λ such that

$$\text{Ric}(E_i, E_j) = \lambda g(E_i, E_j), \quad i, j = 1, 2, 3. \tag{9}$$

Since $\{E_1, E_2, E_3\}$ is an orthonormal frame, from (9) we obtain

$$\text{Ric}(E_i, E_j) = \begin{cases} \lambda, & i = j, \\ 0, & i \neq j. \end{cases}$$

Comparing these equalities with (4), we obtain

$$\lambda = 0, \quad \lambda = -2,$$

which is impossible. Therefore, Sol^3 is not an Einstein manifold. The proof is complete.

Proposition 2. On the manifold Sol^3 , any diagonal left-invariant polynomial operator J of the form (8) commutes with the Ricci operator.

Proof. We shall prove that

$$\text{Ric} \circ J = J \circ \text{Ric}. \tag{10}$$

For the both Ric and J operators sufficient to verify equality (10) on the orthonormal basis $\{E_1, E_2, E_3\}$.

From (4), the Ricci operator is given by

$$\text{Ric}(E_1) = 0, \quad \text{Ric}(E_2) = 0, \quad \text{Ric}(E_3) = -2E_3.$$

On the other hand, using by (8), we compare the two compositions on each basis vector.

For E_1 , we obtain

$$\text{Ric}(J(E_1)) = \text{Ric}(\sigma_1 E_1) = \sigma_1 \text{Ric}(E_1) = 0.$$

Also,

$$J(\text{Ric}(E_1)) = J(0) = 0.$$

Hence

$$\text{Ric}(J(E_1)) = J(\text{Ric}(E_1)).$$

For E_2 , similarly,

$$\text{Ric}(J(E_2)) = \text{Ric}(\sigma_1 E_2) = \sigma_1 \text{Ric}(E_2) = 0,$$

and

$$J(\text{Ric}(E_2)) = J(0) = 0.$$

Therefore,

$$\text{Ric}(J(E_2)) = J(\text{Ric}(E_2)).$$

For E_3 , we have

$$\text{Ric}(J(E_3)) = \text{Ric}(\sigma_2 E_3) = \sigma_2 \text{Ric}(E_3) = -2\sigma_2 E_3,$$

while

$$J(\text{Ric}(E_3)) = J(-2E_3) = -2J(E_3) = -2\sigma_2 E_3.$$

Thus,

$$\text{Ric}(J(E_3)) = J(\text{Ric}(E_3)).$$

Since these equalities hold for each basis vector, relation (10) follows by linearity on the whole tangent space. After that $[\text{Ric}, J] = 0$ for it is symmetric. Hence, the Ricci operator commutes with the polynomial structure J . The proposition is proved.

Proposition 3. Although the Ricci operator commutes with the metallic structure J , the Ricci tensor is not, in general, J -invariant. In other words,

$$\text{Ric}(JX, JY) \neq \text{Ric}(X, Y) \tag{11}$$

for any vector fields X and Y on Sol^3 .

Proof. Indeed, let X and Y be arbitrary vector fields on Sol^3 . With respect to the orthonormal frame $\{E_1, E_2, E_3\}$, we may write

$$X = x_1 E_1 + x_2 E_2 + x_3 E_3, \quad Y = y_1 E_1 + y_2 E_2 + y_3 E_3,$$

where $x_1, x_2, x_3, y_1, y_2, y_3$ are smooth functions on the Sol^3 manifold.

From the form of the Ricci tensor, we know that

$$\text{Ric}(E_1, E_1) = 0, \quad \text{Ric}(E_2, E_2) = 0, \quad \text{Ric}(E_3, E_3) = -2,$$

and

$$\text{Ric}(E_i, E_j) = 0, \quad i \neq j.$$

Using the bilinearity of the Ricci tensor, we compute

$$\text{Ric}(X, Y) = \text{Ric}(x_1 E_1 + x_2 E_2 + x_3 E_3, y_1 E_1 + y_2 E_2 + y_3 E_3).$$

Expanding this expression term by term, we see that every term vanishes except the one involving the E_3 -components. Therefore,

$$\text{Ric}(X, Y) = -2x_3 y_3.$$

On the other hand, the metallic structure J on the Sol^3 is defined by (8). Hence,

$$JX = \sigma_1 x_1 E_1 + \sigma_1 x_2 E_2 + \sigma_2 x_3 E_3,$$

and similarly,

$$JY = \sigma_1 y_1 E_1 + \sigma_1 y_2 E_2 + \sigma_2 y_3 E_3.$$

Now we evaluate the Ricci tensor on the pair (JX, JY) :

$$\text{Ric}(JX, JY) = \text{Ric}(\sigma_1 x_1 E_1 + \sigma_1 x_2 E_2 + \sigma_2 x_3 E_3, \sigma_1 y_1 E_1 + \sigma_1 y_2 E_2 + \sigma_2 y_3 E_3).$$

Again, since the Ricci tensor is zero on all components except the E_3 -direction, all mixed terms disappear, and we obtain

$$\text{Ric}(JX, JY) = -2\sigma_2^2 x_3 y_3.$$

Now comparing the two expressions, namely

$$\text{Ric}(X, Y) = -2x_3 y_3, \quad \text{Ric}(JX, JY) = -2\sigma_2^2 x_3 y_3,$$

we conclude that

$$\text{Ric}(JX, JY) = \text{Ric}(X, Y)$$

if and only if

$$(\sigma_2^2 - 1)x_3 y_3 = 0.$$

Thus, this equality holds only in special cases, for example, when $\sigma_2^2 = 1$, i.e., when the structure is not metallic, or when at least one of the vectors X and Y has a vanishing E_3 -component. In general, however, this condition is not satisfied.

Therefore, for arbitrary vector fields X and Y , one has in general

$$\text{Ric}(JX, JY) \neq \text{Ric}(X, Y).$$

This shows that the Ricci tensor is not J -invariant on Sol^3 , even though the Ricci operator commutes with the metallic structure J .

Theorem 1. Sol^3 manifold is a metallic-Einstein type if and only if

$$\text{Ric}(X, Y) = \alpha g(X, Y) + \beta g(JX, Y), \tag{12}$$

where

$$\alpha = -\frac{p + \sqrt{p^2 + 4q}}{\sqrt{p^2 + 4q}}, \quad \beta = \frac{2}{\sqrt{p^2 + 4q}}.$$

Proof. We determine the constants α and β by comparing both sides of (12) on the orthonormal basis $\{E_1, E_2, E_3\}$. Since both sides of (12) are symmetric bilinear forms, it is sufficient to verify the equality on the basis vectors.

From the fact that $\{E_1, E_2, E_3\}$ is an orthonormal frame, we obtain

$$g(E_1, E_1) = g(E_2, E_2) = g(E_3, E_3) = 1,$$

and all remaining metrics vanish, that is, $g(E_i, E_j) = 0$. Substituting $X = Y = E_1 = E_2$ into (12), we obtain

$$\text{Ric}(E_1, E_1) = \alpha g(E_1, E_1) + \beta g(JE_1, E_1),$$

Using relations (4) and (8), we get

$$\alpha + \beta\sigma_1 = 0.$$

In addition, this relation also holds for all non-diagonal cases.

Now, taking $X = Y = E_3$ in (12), we get

$$\text{Ric}(E_3, E_3) = \alpha g(E_3, E_3) + \beta g(JE_3, E_3),$$

and therefore

$$\alpha + \beta\sigma_2 = -2.$$

Consequently, the unknown constants α and β satisfy the system

$$\begin{cases} \alpha + \beta\sigma_1 = 0, \\ \alpha + \beta\sigma_2 = -2. \end{cases}$$

Subtracting the first equation from the second one, we obtain

$$\beta(\sigma_2 - \sigma_1) = -2.$$

Using by (7),

$$\sigma_2 - \sigma_1 = -\sqrt{p^2 + 4q}.$$

Hence,

$$\beta = \frac{2}{\sqrt{p^2 + 4q}}.$$

Substituting this value into the relation $\alpha + \beta\sigma_1 = 0$, we obtain

$$\alpha = -\beta\sigma_1 = -\frac{2}{\sqrt{p^2 + 4q}} \cdot \frac{p + \sqrt{p^2 + 4q}}{2} = -\frac{p + \sqrt{p^2 + 4q}}{\sqrt{p^2 + 4q}}.$$

Thus, the Ricci tensor admits the decomposition (12). The proof is complete.

Now, we consider the metallic Ricci soliton. Recall that a metallic Ricci soliton is defined by the equation

$$\mathcal{L}_V g(X, Y) + \text{Ric}(X, Y) = \lambda g(X, Y) + \mu g(JX, Y), \tag{13}$$

where V is a smooth vector field and $\lambda, \mu \in \mathbb{R}$.

Theorem 2. The metallic Ricci soliton admits the family of vector fields

$$V = \left(\left(\frac{\lambda + \mu\sigma_1}{2} - c_1 \right) x + c_2 \right) \frac{\partial}{\partial x} + \left(\left(\frac{\lambda + \mu\sigma_1}{2} + c_1 \right) y + c_3 \right) \frac{\partial}{\partial y} + c_1 \frac{\partial}{\partial z}, \tag{14}$$

where $c_1, c_2, c_3 \in \mathbb{R}$, provided that $\lambda + \mu\sigma_2 + 2 = 0$ on the Sol^3 manifold.

Proof. Let

$$V = v_1 E_1 + v_2 E_2 + v_3 E_3.$$

Substituting (4), (8), and the expression of $\mathcal{L}_V g$ into equation (13), we obtain the following system:

$$\begin{cases} 2(e^{-z} \frac{\partial v_1}{\partial x} + v_3) = \lambda + \mu\sigma_1, \\ 2(e^z \frac{\partial v_2}{\partial y} - v_3) = \lambda + \mu\sigma_1, \\ 2\frac{\partial v_3}{\partial z} - 2 = \lambda + \mu\sigma_2, \\ e^{-z} \frac{\partial v_2}{\partial x} + e^z \frac{\partial v_1}{\partial y} = 0, \\ e^z \frac{\partial v_3}{\partial y} + \frac{\partial v_2}{\partial z} + v_2 = 0, \\ e^{-z} \frac{\partial v_3}{\partial x} + \frac{\partial v_1}{\partial z} - v_1 = 0. \end{cases}$$

From the first, third and fifth equations we have

$$v_3 = c_1, \quad c_1 \in \mathbb{R}, \quad \lambda + \mu\sigma_2 + 2 = 0.$$

Next, the first two equations become

$$\frac{\partial v_1}{\partial x} = e^z \left(\frac{\lambda + \mu\sigma_1}{2} - c_1 \right), \quad \frac{\partial v_2}{\partial y} = e^{-z} \left(\frac{\lambda + \mu\sigma_1}{2} + c_1 \right).$$

Integrating with respect to x and y , respectively, we obtain

$$v_1 = e^z \left(\left(\frac{\lambda + \mu\sigma_1}{2} - c_1 \right) x + \phi_1(y, z) \right),$$

$$v_2 = e^{-z} \left(\left(\frac{\lambda + \mu\sigma_1}{2} + c_1 \right) y + \phi_2(x, z) \right).$$

Since v_3 is constant, the last two equations imply

$$\frac{\partial v_1}{\partial z} = 0, \quad \frac{\partial v_2}{\partial z} = 0.$$

Therefore, ϕ_1 and ϕ_2 do not depend on z , and hence

$$\phi_1 = \phi_1(y), \quad \phi_2 = \phi_2(x).$$

Now we use the mixed relation:

$$e^{-z} \frac{\partial v_2}{\partial x} + e^z \frac{\partial v_1}{\partial y} = 0.$$

Substituting the above expressions for v_1 and v_2 , we get

$$e^{-z} \phi_2'(x) + e^z \phi_1'(y) = 0.$$

Since this identity holds for every z , we must have

$$\phi_2'(x) = 0, \quad \phi_1'(y) = 0.$$

Hence,

$$\phi_1 = c_2, \quad \phi_2 = c_3, \quad c_2, c_3 \in \mathbb{R}.$$

Thus,

$$v_1 = e^z \left(\left(\frac{\lambda + \mu\sigma_1}{2} - c_1 \right) x + c_2 \right), \quad v_2 = e^{-z} \left(\left(\frac{\lambda + \mu\sigma_1}{2} + c_1 \right) y + c_3 \right), \quad v_3 = c.$$

Substituting these expressions into the equality for V , we arrive at (14). The theorem is proved.

Conclusion

In this paper, we studied the manifold Sol^3 endowed with its standard left-invariant metric together with a diagonal polynomial structure. First, we recalled the Lie group structure of Sol^3 , its standard metric, the associated orthonormal frame, the Lie bracket relations, and the Levi-Civita connection.

We then showed that the manifold Sol^3 is not Einstein and does not have constant sectional curvature. After introducing the polynomial structure J , we proved that the Ricci operator commutes with J . At the same time, we observed that the Ricci tensor is not J -invariant in general.

Next, we established that the Ricci tensor admits the decomposition

$$\text{Ric}(X, Y) = \alpha g(X, Y) + \beta g(JX, Y),$$

where α and β are explicitly expressed through the parameters of the polynomial structure. Finally, we derived an explicit family of vector fields satisfying a metallic Ricci soliton type equation associated with J .

These results show that Sol^3 provides a natural framework for the study of polynomial structures and Ricci-type equations on solvable Lie groups. They may also be useful in further investigations of generalized Einstein conditions, geometric flows, and higher-dimensional analogues.

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