Finiteness of Eigenvalues for Operator Matrices Arising in Quantum Mechanics

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Abstract. In this paper, an operator matrix of order three corresponding to a lattice system with a non-conserved number of particles not exceeding three, arising in quantum mechanics is considered as a linear, bounded, and self-adjoint operator acting in a Hilbert space. The essential spectrum of the considered third-order operator matrix is investigated. The two-particle and three-particle branches of the essential spectrum are identified. It is proved that, for an arbitrary value of the spectral parameter, the discrete spectrum of the operator matrix is finite.

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Introduction and statement of problem

Block operator matrices are matrices where the entries are linear operators between Banach or Hilbert spaces [1]. An important subclass of such matrices is formed by Hamiltonians describing lattice systems in which the number of quasi-particles is not conserved. Depending on the model, the number of quasi-particles may be infinite, as in spin-boson models, or finite, as in so-called truncated spin-boson models. Block operator matrices of this type naturally occur in various areas of theoretical physics, including solid-state physics [2], quantum field theory [3], statistical physics [4], and quantum mechanics.

Currently, extensive research is being carried out on the essential spectrum of operator matrices and on the problem of determining the number of their eigenvalues. In particular, special attention is paid to the analysis of threshold phenomena for families of generalized Friedrichs models, to the description of the structure of the essential spectrum of families of third-order operator matrices [5], and to the identification of conditions under which the number of eigenvalues is finite or infinite [6, 7].

In this paper, a third-order operator matrix associated with a lattice system with a non-conserved number of particles not exceeding three is studied for a linear, bounded, and self-adjoint operator acting in a Hilbert space. It is proved that the discrete spectrum of this operator matrix is finite.

Let us state the problem.

Let $\mathbb{T}$ be the one-dimensional torus, $\mathcal{H}_0 := \mathbb{C}$ be the field of complex numbers, $\mathcal{H}_1 := L_2(\mathbb{T})$ be the Hilbert space of square integrable (complex) functions defined on $\mathbb{T}$ and $\mathcal{H}_2 := L_2^{\rm s}(\mathbb{T}^2)$ be the Hilbert space of square-integrable symmetric (complex) functions defined on $\mathbb{T}^2$. We denote by $\mathcal{H}$ the direct sum of $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$, that is, $\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2$. The spaces $\mathcal{H}_0$, $\mathcal{H}_1$ and $\mathcal{H}_2$ are called the zero-particle, one-particle and two-particle subspaces of the bosonic Fock space $\mathcal{F}_{\rm b}(L_2(\mathbb{T}))$ over $L_2(\mathbb{T})$, respectively, where

$$\mathcal{F}_{\rm b}(L_2(\mathbb{T})) := \mathbb{C} \oplus L_2(\mathbb{T}) \oplus L_2^{\rm s}(\mathbb{T}^2) \oplus L_2^{\rm s}(\mathbb{T}^3) \oplus \cdots .$$

Elements of the Hilbert space $\mathcal{H}$ are identified with ordered triples (f_0, f_1, f_2). The inner product of the elements $f = (f_0, f_1, f_2)$ and $g = (g_0, g_1, g_2)$ of $\mathcal{H}$ is given by

$$(f, g) = f_0 \cdot \overline{g_0} + \int_{\mathbb{T}} f_1(x)\overline{g_1(x)}dx + \int_{\mathbb{T}^2} f_2(x, y)\overline{g_2(x, y)}dxdy.$$

In the Hilbert space $\mathcal{H}$, we introduce an operator matrix $\mathcal{A}_\mu$ of the form

$$\mathcal{A}_\mu := \begin{pmatrix} A_{00} & \mu A_{01} & 0 \\ \mu A_{01}^* & A_{11} & \mu A_{12} \\ 0 & \mu A_{12}^* & A_{22} \end{pmatrix}, \qquad \mu > 0.$$

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Throughout this paper we suppose that the matrix entries $\mathcal{A}_{ij} : \mathcal{H}_j \to \mathcal{H}_i$, $i \le j$, $i,j = 0,1,2$ are given by

$$A_{00}f_0 = af_0, \quad A_{01}f_1 = \int_{\mathbb{T}} v(t)f_1(t)dt;$$
$$(A_{11}f_1)(x) = f_1(x), \quad (A_{12}f_2)(x) = \int_{\mathbb{T}} f_2(x,t)dt;$$
$$(A_{22}f_2)(x,y) = w(x,y)f_2(x,y), \quad f_i \in \mathcal{H}_i, \quad i = 0,1,2.$$

where $a \in \mathbb{R}$, the function $v(\cdot)$ is a real-valued continuous function on $\mathbb{T}$ and $w(\cdot,\cdot)$ is defined by

$$w(x,y) := \varepsilon(x) + \varepsilon(x+y) + \varepsilon(y), \quad \varepsilon(x) := 1 - \cos x$$

We remark that the operators A_{01}, A_{12} and A_{01}^*, A_{12}^* are called annihilation and creation operators respectively [8]. A trivial verification shows

$$A_{01}^* : \mathcal{H}_0 \to \mathcal{H}_1, \quad (A_{01}^* f_0)(x) = v(x)f_0, \quad f_0 \in \mathcal{H}_0;$$
$$A_{12}^* : \mathcal{H}_1 \to \mathcal{H}_2, \quad (A_{12}^* f_1)(x,y) = \frac{f_1(x) + f_1(y)}{2}, \quad f_1 \in \mathcal{H}_1.$$

Under these assumptions, the operator matrix $\mathcal{A}_\mu$ is bounded and self-adjoint.

Essential spectrum of the operator matrix $\mathcal{A}_\mu$

In this section, we introduce the basic concepts necessary to state the main results.

For any fixed $\mu > 0$ and $x \in \mathbb{T}$, we define the function $\Delta_\mu(x;\cdot)$ in $\mathbb{R} \setminus [E_1(x); E_2(x)]$ by the following condition

$$\Delta_\mu(x\,;z) := \begin{cases} 1 - z - I_\mu(x\,;z), & z < E_1(x), \\ 1 - z + I_\mu(x\,;z), & z > E_2(x) \end{cases}$$

$$I_\mu(x\,;z) := \frac{\pi\mu^2}{\sqrt{(3 - \cos x - z)^2 - 4\cos^2\frac{x}{2}}},$$

where the numbers $E_1(x)$ and $E_2(x)$ are determined by

$$E_1(x) := \min_{y\in\mathbb{T}} w(x,y) = 3 - \cos x - \sqrt{2 + 2\cos x};$$
$$E_2(x) := \max_{y\in\mathbb{T}} w(x,y) = 3 - \cos x + \sqrt{2 + 2\cos x}.$$

Suppose that σ_μ is the set of points $z \in \mathbb{R}$ for which the equation $\Delta_\mu(x;\, z) = 0$ has a solution for at least one point $x \in \mathbb{T}$. Let us introduce the notation

$$\Sigma_\mu := \sigma_\mu \cup [0; \tfrac{9}{2}].$$

$\sigma(\cdot)$, $\sigma_{\text{ess}}(\cdot)$ and $\sigma_{\text{disc}}(\cdot)$ are denoted by the spectrum, the essential spectrum, and the discrete spectrum of a bounded self-adjoint operator respectively.

The following theorem is about the elements of essential spectrum of the operator matrix $\mathcal{A}_\mu$.

Theorem 1. *The essential spectrum $\sigma_{\text{ess}}(\mathcal{A}_\mu)$ of the operator matrix $\mathcal{A}_\mu$ coincides with the set Σ_μ, i.e., $\sigma_{\text{ess}}(\mathcal{A}_\mu) = \Sigma_\mu$.*

Theorem 1 is proved using the Weyl criterion [9], the characteristic property of the Faddeev equation, and the analytic Fredholm theorem.

The sets σ_μ and $[0; \frac{9}{2}]$ are called two-particle and three-particle branches of the essential spectrum of the operator matrix $\mathcal{A}_\mu$.

Finiteness of the number of eigenvalues of $\mathcal{A}_\mu$

Following the classical definition introduced by Glazman [10], for any $\lambda \in \mathbb{R}$ and a bounded self-adjoint operator A acting in a Hilbert space H, we define the number $n(\lambda, A)$ as

$$n(\lambda, A) := \sup\Big\{\dim F \ : \ (Au, u) > \lambda,\ u \in F \subset H,\ \|u\| = 1\Big\}.$$

The quantity $n(\lambda, A)$ is infinite whenever $\lambda < \max \sigma_{\rm ess}(A)$.

If, on the other hand, $n(\lambda, A)$ is finite, then it coincides with the number of eigenvalues of the operator A that are greater than λ, counted according to their multiplicities.

Let $\tau_{\min}(\mathcal{A}_\mu)$ and $\tau_{\max}(\mathcal{A}_\mu)$ denote the lower and upper bounds of the essential spectrum $\sigma_{\rm ess}(\mathcal{A}_\mu)$ of the operator matrix $\mathcal{A}_\mu$, respectively, that is,

$$\tau_{\min}(\mathcal{A}_\mu) := \min \sigma_{\rm ess}(\mathcal{A}_\mu), \qquad \tau_{\max}(\mathcal{A}_\mu) := \max \sigma_{\rm ess}(\mathcal{A}_\mu).$$

By the definition of the quantity $N_{(a,b)}(\mathcal{A}_\mu)$, we have

$$N_{(-\infty,\, z)}(\mathcal{A}_\mu) = n\big(-z,\, -\mathcal{A}_\mu\big), \qquad z < \tau_{\min}(\mathcal{A}_\mu),$$

and

$$N_{(z,\, +\infty)}(\mathcal{A}_\mu) = n\big(z,\, \mathcal{A}_\mu\big), \qquad z > \tau_{\max}(\mathcal{A}_\mu).$$

Note that for any $x \in \mathbb{T}$ and $z < \tau_{\min}(\mathcal{A}_\mu)$ (resp. $z > \tau_{\max}(\mathcal{A}_\mu)$), the function $\Delta_\mu(x; z)$ (resp. $-\Delta_\mu(x; z)$) is positive and, therefore, admits a positive square root.

In the study of the discrete spectrum of the operator matrix $\mathcal{A}_\mu$, a fundamental role is played by the compact operator $\widehat{T}_\mu(z)$, $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$, acting in the space $\mathcal{H}_0 \oplus \mathcal{H}_1$ and given by

$$\widehat{T}_\mu(z) := \begin{pmatrix} \widehat{T}_{00}(\mu, z) & \widehat{T}_{01}(\mu, z) \\ \widehat{T}_{01}^*(\mu, z) & \widehat{T}_{11}(\mu, z) \end{pmatrix}.$$

where the matrix entries $\widehat{T}_{ij}(\mu, z) : \mathcal{H}_j \to \mathcal{H}_i$, $i \le j$, $i, j = 0, 1$, are defined as follows.

For $z < \tau_{\min}(\mathcal{A}_\mu)$,

$$\widehat{T}_{00}(\mu, z)\varphi_0 = (1 + z - a)\varphi_0, \qquad \widehat{T}_{01}(\mu, z)\varphi_1 = -\mu \int_{\mathbb{T}} \frac{v(t)\varphi_1(t)\, dt}{\sqrt{\Delta_\mu(t, z)}};$$

$$(\widehat{T}_{01}^*(\mu, z)\varphi_0)(x) = -\frac{\mu v(x)\varphi_0}{\sqrt{\Delta_\mu(x, z)}};$$

$$(\widehat{T}_{11}(\mu, z)\varphi_1)(x) = \frac{\mu^2}{2\sqrt{\Delta_\mu(x, z)}} \int_{\mathbb{T}} \frac{\varphi_1(t)\, dt}{\sqrt{\Delta_\mu(t, z)}\big(w(x, t) - z\big)}, \qquad \varphi_i \in \mathcal{H}_i,\ i = 0, 1.$$

For $z > \tau_{\max}(\mathcal{A}_\mu)$,

$$\widehat{T}_{00}(\mu, z)\varphi_0 = (1 + z - a)\varphi_0, \qquad (\widehat{T}_{01}(\mu, z)\varphi_1)(x) = -\mu \int_{\mathbb{T}} \frac{v(t)\varphi_1(t)\, dt}{\sqrt{-\Delta_\mu(t, z)}};$$

$$(\widehat{T}_{01}^*(\mu, z)\varphi_0)(x) = -\frac{\mu v(x)\varphi_0}{\sqrt{-\Delta_\mu(x, z)}};$$

$$(\widehat{T}_{11}(\mu, z)\varphi_1)(x) = -\frac{\mu^2}{2\sqrt{-\Delta_\mu(x, z)}} \int_{\mathbb{T}} \frac{\varphi_1(t)\, dt}{\sqrt{-\Delta_\mu(t, z)}\big(w(x, t) - z\big)}, \qquad \varphi_i \in \mathcal{H}_i,\ i = 0, 1.$$

The following lemma presents a modification of the well-known Birman-Schwinger principle for the operator matrix $\mathcal{A}_\mu$ (see. [11, 12]).

Lemma 1. *For all $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$, the operator $\widehat{T}_\mu(z)$ is compact and continuous with respect to z, and the following equalities hold:*

$$N_{(-\infty,\, z)}(\mathcal{A}_\mu) = n\big(1,\, \widehat{T}_\mu(z)\big), \quad \textit{for} \quad z < \tau_{\min}(\mathcal{A}_\mu),$$

$$N_{(z,+\infty)}(\mathcal{A}_\mu) = n\big(1, \widehat{T}_\mu(z)\big), \quad \textit{for} \quad z > \tau_{\max}(\mathcal{A}_\mu).$$

Since $\tau_{\min}(\mathcal{A}_\mu) \in \sigma_\mu$, $(\tau_{\max}(\mathcal{A}_\mu) \in \sigma_\mu)$, there exists a point $x_1 \in \mathbb{T}$ $(x_1' \in \mathbb{T})$ such that

$$\Delta_\mu\big(x_1; \tau_{\min}(\mathcal{A}_\mu)\big) = 0 \quad \big(\Delta_\mu\big(x_1'; \tau_{\max}(\mathcal{A}_\mu)\big) = 0\big).$$

Since $\tau_{\min}(\mathcal{A}_\mu) < 0$, the function $\Delta_\mu(\cdot\,; \tau_{\min}(\mathcal{A}_\mu))$ is regular on $\mathbb{T}$. Therefore, the number of zeros of this function is finite.

Let

$$\{x \in \mathbb{T} : \Delta_\mu(x; \tau_{\min}(\mathcal{A}_\mu)) = 0\} = \{x_1, \dots, x_n\},$$

and let k_j denote the multiplicity of the zero x_j for $j \in \{1, \dots, n\}$.

Since $\tau_{\max}(\mathcal{A}_\mu) > \frac{9}{2}$, the function $\Delta_\mu(\cdot\,; \tau_{\max}(\mathcal{A}_\mu))$ is regular on $\mathbb{T}$. Hence, the number of zeros of this function is finite.

Let

$$\{x \in \mathbb{T} : \Delta(x; \tau_{\max}(\mathcal{A}_\mu)) = 0\} = \{x_1', \dots, x_m'\},$$

and let $k_{j'}'$ denote the multiplicity of the zero $x_{j'}'$ for $j' \in \{1, \dots, m\}$.

Since $\tau_{\min}(\mathcal{A}_\mu) < 0$, it follows that the difference

$$w(x, y) - z$$

is positive for all $x, y \in \mathbb{T}$ and $z \le \tau_{\min}(\mathcal{A}_\mu)$. Hence, for all $z \le \tau_{\min}(\mathcal{A}_\mu)$ the function

$$\big(w(\cdot,\cdot) - z\big)^{-1}$$

is analytic on $\mathbb{T}^2$. Therefore, there exists a number $\delta > 0$ such that for any $i, j \in \{1, \dots, n\}$ and $z \le \tau_{\min}(\mathcal{A}_\mu)$ the following representations hold.

For any $x \in \mathbb{T}$ and $y \in U_\delta(x_i)$,

$$\frac{\mu^2}{2\big(w(x,y) - z\big)} = \sum_{k=0}^{\lfloor k_i/2 \rfloor} a_{ik}^{(1)}(z; x)\,(y - x_i)^k + (y - x_i)^{\lfloor k_i/2 \rfloor + 1} b_i^{(1)}(z; x, y). \tag{1}$$

For any $y \in \mathbb{T}$ and $x \in U_\delta(x_i)$,

$$\frac{\mu^2}{2\big(w(x,y) - z\big)} = \sum_{k=0}^{\lfloor k_i/2 \rfloor} a_{ik}^{(2)}(z; y)\,(x - x_i)^k + (x - x_i)^{\lfloor k_i/2 \rfloor + 1} b_i^{(2)}(z; x, y). \tag{2}$$

For any $(x, y) \in U_\delta(x_i) \times U_\delta(x_j)$,

$$\begin{aligned} \frac{\mu^2}{2\big(w(x,y) - z\big)} &= \sum_{k=0}^{\lfloor k_i/2 \rfloor} \sum_{r=0}^{\lfloor k_j/2 \rfloor} d_{ij}^{kr}(z)(x - x_i)^k (y - x_j)^r \\ &+ \sum_{k=0}^{\lfloor k_i/2 \rfloor} \sum_{r=\lfloor k_j/2 \rfloor + 1}^{\infty} d_{ij}^{(kr)}(z)(x - x_i)^k (y - x_j)^r \\ &+ \sum_{k=\lfloor k_i/2 \rfloor + 1}^{\infty} \sum_{r=0}^{\lfloor k_j/2 \rfloor} d_{ij}^{(kr)}(z)(x - x_i)^k (y - x_j)^r \\ &+ (x - x_i)^{\lfloor k_i/2 \rfloor + 1} (y - x_j)^{\lfloor k_j/2 \rfloor + 1} q_{ij}(z; x, y), \end{aligned} \tag{3}$$

where $\lfloor m \rfloor$ denotes the integer part of m. For any $z \le \tau_{\min}(\mathcal{A}_\mu)$, the numbers $d_{ij}^{(kr)}(z)$ are some real coefficients, the functions $a_{ik}^{(\alpha)}(z; \cdot)$, $\alpha = 1, 2$, $b_i^{(1)}(z; \cdot, \cdot)$, $b_i^{(2)}(z; \cdot, \cdot)$, and $q_{ij}(z; \cdot, \cdot)$ are analytic functions on $\mathbb{T}$, $\mathbb{T} \times U_\delta(x_i)$, $U_\delta(x_i) \times \mathbb{T}$, and $U_\delta(x_i) \times U_\delta(x_j)$, respectively.

The case $z > \tau_{\max}(\mathcal{A}_\mu)$ can be treated analogously.

Lemma 2. *Let $j \in \{1, \dots, n\}$ and $j' \in \{1, \dots, m\}$. Then there exist constants $C_1, C_2 > 0$ and $\delta > 0$ such that for all $x \in U_\delta(x_j)$ and $x \in U_\delta(x'_{j'})$ the following estimates hold:*

$$\frac{|x - x_j|^{\lfloor k_j/2 \rfloor + 1}}{\sqrt{\Delta_\mu(x; z)}} \le C_1, \quad z \le \tau_{\min}(\mathcal{A}_\mu), \tag{4}$$

$$\frac{|x - x'_{j'}|^{\lfloor k'_{j'}/2 \rfloor + 1}}{\sqrt{-\Delta_\mu(x; z)}} \le C_2, \quad z \ge \tau_{\max}(\mathcal{A}_\mu). \tag{5}$$

The proof of this lemma is analogous to the proofs of the corresponding lemmas in [7, 13].

Lemma 3. *Then, for every $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$, the operator $\widehat{T}_\mu(z)$ admits the representation*

$$\widehat{T}_\mu(z) = \widehat{T}^{(0)}_\mu(z) + \widehat{T}^{(1)}_\mu(z),$$

where the operator-valued function $\widehat{T}^{(0)}_\mu(\cdot)$ is continuous in the operator norm on the intervals $(-\infty, \tau_{\min}(\mathcal{A}_\mu)]$ and $[\tau_{\max}(\mathcal{A}_\mu), +\infty)$, and for all $z \in \mathbb{R} \setminus [\tau_{\min}(\mathcal{A}_\mu), \tau_{\max}(\mathcal{A}_\mu)]$ the operator $\widehat{T}^{(1)}_\mu(z)$ is finite-dimensional and does not depend on z.

Proof. Assume that $z \le \tau_{\min}(\mathcal{A}_\mu)$. Since the operators $\widehat{T}_{00}(\mu, z)$, $\widehat{T}_{01}(\mu, z)$, and $\widehat{T}^*_{01}(\mu, z)$ are one-dimensional and independent of z, it suffices to analyze the operator $\widehat{T}_{11}(\mu, z)$.

Let $\widehat{T}_{11}(\mu, z)$ denote the kernel of the integral operator $\widehat{T}_{11}(\mu, z)$

$$\widehat{T}_{11}(\mu; z; x, y) := \frac{\mu^2}{\sqrt{\Delta_\mu(x; z)}\,(w(x, y) - z)\,\sqrt{\Delta_\mu(y; z)}}.$$

In this case, $\tau_{\min}(\mathcal{A}_\mu) < 0$, and by using representations (1)–(4) we obtain

$$\widehat{T}_{11}(\mu, z) = \widehat{T}^0_{11}(\mu, z) + \widehat{T}^1_{11}(\mu, z),$$

where the kernels $\widehat{T}^0_{11}(\mu; z; x, y)$ and $\widehat{T}^1_{11}(\mu; z; x, y)$ of the integral operators $\widehat{T}^0_{11}(\mu, z)$ and $\widehat{T}^1_{11}(\mu, z)$, respectively, are given by

$$\begin{aligned}
\widehat{T}^0_{11}(\mu; z; x, y) &:= (1 - \chi_{V_\delta}(x))(1 - \chi_{V_\delta}(y))\widehat{T}_{11}(\mu; z; x, y) \\
&+ \frac{1 - \chi_{V_\delta}(x)}{\sqrt{\Delta_\mu(x; z)}} \sum_{i=1}^{n} \frac{\chi_{V_\delta}(y)(y - x_i)^{\lfloor k_i/2 \rfloor + 1}}{\sqrt{\Delta_\mu(y; z)}} B^{(1)}_i(z; x, y) \\
&+ \frac{1 - \chi_{V_\delta}(y)}{\sqrt{\Delta_\mu(y; z)}} \sum_{i=1}^{n} \frac{\chi_{V_\delta}(x)(x - x_i)^{\lfloor k_i/2 \rfloor + 1}}{\sqrt{\Delta_\mu(x; z)}} B^{(2)}_i(z; x, y) \\
&+ \chi_{V_\delta}(x)\chi_{V_\delta}(y) \sum_{i,j=1}^{n} \frac{(x - x_i)^{\lfloor k_i/2 \rfloor + 1}(y - x_j)^{\lfloor k_j/2 \rfloor + 1}}{\sqrt{\Delta_\mu(x; z)}\,\sqrt{\Delta_\mu(y; z)}} Q_{ij}(z; x, y);
\end{aligned}$$

$$\widehat{T}^1_{11}(\mu;z;x,y) := \frac{(1-\chi_{V_\delta}(x))\chi_{V_\delta}(y)}{\sqrt{\Delta_\mu(x;z)}\sqrt{\Delta_\mu(y;z)}}\sum_{i=1}^{n}\sum_{k=0}^{\lfloor k_i/2\rfloor}(y-x_i)^k\,a^{(1)}_{ik}(z;x)$$

$$+\frac{\chi_{V_\delta}(x)(1-\chi_{V_\delta}(y))}{\sqrt{\Delta_\mu(x;z)}\sqrt{\Delta_\mu(y;z)}}\sum_{i=1}^{n}\sum_{k=0}^{\lfloor k_i/2\rfloor}(x-x_i)^k\,a^{(2)}_{ik}(z;y)$$

$$+\frac{\chi_{V_\delta}(x)\chi_{V_\delta}(y)}{\sqrt{\Delta_\mu(x;z)}\sqrt{\Delta_\mu(y;z)}}\sum_{i,j=1}^{n}\Big(\sum_{k=0}^{\lfloor k_i/2\rfloor}\sum_{r=0}^{\lfloor k_j/2\rfloor}d^{kr}_{ij}(z)(x-x_i)^k\,(y-x_j)^r$$

$$+\sum_{k=0}^{\lfloor k_i/2\rfloor}\sum_{r=\lfloor k_j/2\rfloor+1}^{\infty}d^{kr}_{ij}(z)(x-x_i)^k\,(y-x_j)^r$$

$$+\sum_{k=\lfloor k_i/2\rfloor+1}^{\infty}\sum_{r=0}^{\lfloor k_j/2\rfloor}d^{kr}_{ij}(z)(x-x_i)^k\,(y-x_j)^r\Big),$$

where $V_\delta := \bigcup_{i=1}^n U_\delta(x_i)$, $\chi_\Omega(\cdot)$ is the characteristic function of the set $\Omega\subset\mathbb{T}$.

$$B^{(1)}_i(z;x,y) := \begin{cases} b^{(1)}_i(z;x,y), & (x,y)\in\mathbb{T}\times U_\delta(x_i),\\ 0, & (x,y)\notin\mathbb{T}\times U_\delta(x_i),\end{cases}$$

$$B^{(2)}_i(z;x,y) := \begin{cases} b^{(2)}_i(z;x,y), & (x,y)\in U_\delta(x_i)\times\mathbb{T},\\ 0, & (x,y)\notin U_\delta(x_i)\times\mathbb{T},\end{cases}$$

$$Q_{ij}(z;x,y) := \begin{cases} q_{ij}(z;x,y), & (x,y)\in U_\delta(x_i)\times U_\delta(x_j),\\ 0, & (x,y)\notin U_\delta(x_i)\times U_\delta(x_j).\end{cases}$$

Applying Lemma 2, we obtain that the function $\widehat{T}^0_{11}(\mu;z;\cdot,\cdot)$ is square-integrable on $\mathbb{T}^2$ for $z\le\tau_{\min}(\mathcal{A}_\mu)$ and converges almost everywhere to $\widehat{T}^0_{11}(\mu;\tau_{\min}(\mathcal{A}_\mu);\cdot,\cdot)$ as $z\to\tau_{\min}(\mathcal{A}_\mu)-0$. Then by the Lebesgue dominated convergence theorem the operator $\widehat{T}^0_{11}(\mu,z)$ converges in the operator-norm to $\widehat{T}^0_{11}(\mu,\tau_{\min}(\mathcal{A}_\mu))$ as $z\to\tau_{\min}(\mathcal{A}_\mu)-0$.

The finite-dimensionality of the operator $\widehat{T}^1_{11}(\mu,\tau_{\min}(\mathcal{A}_\mu))$ follows directly from the definition of the function $\widehat{T}^1_{11}(\mu;z;x,y)$.

Setting

$$\widehat{T}^{(0)}_\mu(z) := \begin{pmatrix}0 & 0\\ 0 & \widehat{T}^0_{11}(\mu,z)\end{pmatrix},\qquad \widehat{T}^{(1)}_\mu(z) := \begin{pmatrix}\widehat{T}_{00}(\mu,z) & \widehat{T}_{01}(\mu,z)\\ \widehat{T}^*_{01}(\mu,z) & \widehat{T}^1_{11}(\mu,z)\end{pmatrix}.$$

The case $z>\tau_{\max}(\mathcal{A}_\mu)$ can be proved analogously. □

Theorem 2. *For any $\mu>0$, the operator $\mathcal{A}_\mu$ has only a finite number of eigenvalues lying to the left of $\tau_{\min}(\mathcal{A}_\mu)$ and to the right of $\tau_{\max}(\mathcal{A}_\mu)$.*

Proof. Using Weyl's inequality, we obtain

$$\begin{aligned} n(1,\widehat{T}_\mu(z)) &\le n\Big(\tfrac{2}{3},\widehat{T}^{(0)}_\mu(z)\Big)+n\Big(\tfrac{1}{3},\widehat{T}^{(1)}_\mu(z)\Big)\\ &\le n\Big(\tfrac{1}{3},\,\widehat{T}^{(0)}_\mu(z)-\widehat{T}^{(0)}_\mu(\tau_{\min}(\mathcal{A}_\mu))\Big)+n\Big(\tfrac{1}{3},\,\widehat{T}^{(0)}_\mu(\tau_{\min}(\mathcal{A}_\mu))\Big)+n\Big(\tfrac{1}{3},\,\widehat{T}^{(1)}_\mu(z)\Big), \end{aligned}\tag{6}$$

for all $z<\tau_{\min}(\mathcal{A}_\mu)$.

According to Lemma 3, the operator $\widehat{T}^{(0)}_\mu(\tau_{\min}(\mathcal{A}_\mu))$ is compact and, consequently,

$$n\Big(\tfrac{1}{3},\widehat{T}^{(0)}_\mu(\tau_{\min}(\mathcal{A}_\mu))\Big)<\infty,$$

and moreover,

$$n\Big(\tfrac{1}{3}, \widehat{T}_\mu^{(0)}(z) - \widehat{T}_\mu^{(0)}(\tau_{\min}(\mathcal{A}_\mu)\Big) \to 0 \quad \text{as} \quad z \to \tau_{\min}(\mathcal{A}_\mu) - 0.$$

Since the operator $\widehat{T}_\mu^{(1)}(z)$ is finite-dimensional and the dimension of its range does not depend on z, $z < \tau_{\min}(\mathcal{A}_\mu)$, there exists a constant $C > 0$ such that for all $z < \tau_{\min}(\mathcal{A}_\mu)$ the inequality

$$n\Big(\tfrac{1}{3}, \widehat{T}_\mu^{(1)}(z)\Big) \le C < \infty$$

holds. Therefore, by inequality (6), we conclude that the number

$$n(1, \widehat{T}_\mu(z))$$

is finite for all $z < \tau_{\min}(\mathcal{A}_\mu)$.

Now, Lemma 1 implies that

$$N_{(-\infty,\, z)}(\mathcal{A}_\mu) = n(1, \widehat{T}_\mu(z)), \quad \text{for} \quad z < \tau_{\min}(\mathcal{A}_\mu),$$

and hence

$$\lim_{z \to \tau_{\min}(\mathcal{A}_\mu)} N_{(-\infty,\, z)}(\mathcal{A}_\mu) = N_{(-\infty,\, \tau_{\min}(\mathcal{A}_\mu))}(\mathcal{A}_\mu)$$
$$\le n\Big(\tfrac{1}{3}, \widehat{T}_\mu^{(0)}(\tau_{\min}(\mathcal{A}_\mu))\Big) + n\Big(\tfrac{1}{3}, \widehat{T}_\mu^{(1)}(\tau_{\min}(\mathcal{A}_\mu))\Big) < \infty.$$

This proves that the operator $\mathcal{A}_\mu$ has only a finite number of eigenvalues lying to the left of $\tau_{\min}(\mathcal{A}_\mu)$. The case $z > \tau_{\max}(\mathcal{A}_\mu)$ can be treated analogously. □

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