A NON-LOCAL PROBLEM FOR THE BARENBLATT-ZHELTOV-KOCHINA TYPE FRACTIONAL EQUATIONS

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Abstract. This paper investigates a non-local problem associated with the fractional-order Barenblatt–Zheltov–Kochina equation involving the Caputo fractional derivative. The non-local problem under consideration is reduced to two auxiliary problems, and the solution of the corresponding Cauchy problem is employed in the analysis. The existence and uniqueness theorems are established for both the initial-boundary value problem. The results obtained extend the theoretical framework of fractional differential equations and provide a foundation for further theoretical developments as well as potential practical applications.

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Introduction

Consider a separable Hilbert space H. Let $A : H \to H$ be a self-adjoint, positive, unbounded operator defined in the domain $D(A)$. We assume that A has a compact inverse A^{-1}. Denote by $\{v_k\}$ a complete orthonormal set of eigenfunctions and by $\{\lambda_k\}$ the associated set of positive eigenvalues. The eigenvalues can be ordered in a non-decreasing sequence, which means $0 < \lambda_1 \le \lambda_2 \cdots \to +\infty$. The Caputo fractional derivative of order $\rho \in (0,\, 1)$ is defined as follows (see, for example [1])

$$D_t^\rho h(t) = \frac{1}{\Gamma(1-\rho)} \int\limits_0^t \frac{h'(\xi)}{(t-\xi)^\rho} d\xi, \quad t > 0,$$

provided that the right-hand side exists. Let $\rho \in (0,\, 1)$ be a given constant and $C((a,\, b);\, H)$ be a set of continuous functions $u(t)$ of $t \in (a,\, b)$ with values in H.

Let us consider the following *non-local problem*:

$$\begin{cases} D_t^\rho u(t) + A(1+\gamma D_t^\rho)u(t) = f, \quad \gamma > 0, \quad 0 < t \le T; \\ u(\tau_0) = \alpha u(0) + \varphi, \quad 0 < \tau_0 \le T, \end{cases} \tag{1}$$

here, $f, \varphi \in H$ are given functions. The parameters $\gamma > 0$, α is a constant, and τ_0 is a fixed point. When $\rho \in (0,\, 1)$ the first equation in (1) is called the Barenblatt–Zheltov–Kochina type fractional differential equation.

Definition 1. An absolutely continuous function $u(t) \in C([0,\, T];\, H)$ that has properties $Au(t)$, $D_t^\rho u(t)$, $A(D_t^\rho u(t)), \in C((0,\, T];\, H)$ and satisfies the conditions in (1) is called the solution of the non-local problem (1).

Let us provide an overview of the research conducted to date.

The Barenblatt–Zheltov–Kochina equation arises from the theory of fluid filtration in fissured porous media and other diffusion-type processes. The fractional version of this equation allows us to take into account memory effects and anomalous diffusion. Therefore, the study of boundary value problems for this equation is of both theoretical and practical interest.

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Barenblatt, Zheltov, and Kochina (see also [2]) developed the theory of unsteady filtration in fractured-porous media, which was later expanded and refined by numerous researchers ([3, 4, 5, 6, 7, 8, 9, 10, 11, 12]), who further elaborated its fundamental principles and governing equations.

The existence and uniqueness of a solution to the Cauchy problem for the Caputo derivative of the Barenblatt–Zheltov–Kochina type when $\gamma = 1$ was studied in [13].

In work [14], the Cauchy problem was studied in the case where $\gamma \geq 0$, the source term f is independent of time. The direct and inverse problems are considered for the cases $\gamma = 0$ and $\gamma > 0$. Later, the integral condition for a positive parameter was studied in [15].

If the operator D_t^ρ with $\rho \in (0,1)$ is replaced by the first-order time derivative $\frac{\partial}{\partial t}$, the equation reduces to the classical Rayleigh–Stokes equation.

In [16], a non-local problem for the Rayleigh–Stokes equation with the Riemann–Liouville fractional derivative was investigated. The non-local condition is given by

$$u(T) = \alpha u(0) + \varphi,$$

where $\alpha \in \{0, 1\}$ and $\gamma > 0$ is a constant. In particular, the case $\alpha = 1$ corresponds to a non-local problem. It was shown that when $\alpha = 0$, the associated backward problem is ill-posed. For certain values of α, the problem is well-posed; otherwise, additional restrictions on the function φ are imposed to guaranty well-posedness.

In [17], a non-local problem was studied for the Rayleigh–Stokes equation involving the Riemann–Liouville fractional derivative. In this work, the non-local condition is formulated as in (1). This paper studied a time non-local problem, instead of the initial condition, they considered the non-local condition. The results obtained were valid for the equation with the Laplace operator under the Dirichlet condition.

In [18], nonlocal boundary value problems with $\gamma = 0$ and condition (1) were studied for equations involving fractional derivative Caputo and Riemann–Liouville. The authors established the existence and uniqueness results and analyzed the solvability of the problem depending on the parameter α. It was shown that the type of fractional derivative affects the corresponding coercivity-type inequalities. In addition, inverse problems related to the determination of φ were considered.

Preliminaries

In this section, we introduce the order and domain of definition of the operator A, some properties of the Mittag–Leffler function, and orthogonality conditions. The tools that will be used throughout the paper.

Let the vector function (or the simple function) $h(t)$ be defined in the interval $[0, +\infty)$ with values in the Hilbert space H. Let ν be an arbitrary real number. We define the power of the operator A, acting in the Hilbert space H, according to the following rule

$$A^\nu h = \sum_{k=1}^{\infty} \lambda_k^\nu h_k v_k,$$

here h_k are the Fourier coefficients of a function $h \in H$ defined by $h_k = (h, v_k)$. The domain of this operator is structured as follows

$$D(A^\nu) = \{h \in H : \sum_{k=1}^{\infty} \lambda_k^{2\nu} |h_k|^2 < \infty\}.$$

For elements of $D(A^\nu)$ we introduce the norm

$$\|h\|_\nu^2 = \sum_{k=1}^{\infty} \lambda_k^{2\nu} |h_k|^2 = \|A^\nu h\|^2,$$

and together with this norm $D(A^\nu)$ turn into a Hilbert space.

Now we note some properties of the Mittag–Leffler function. Let $\rho > 0$ and σ be an arbitrary complex number. Denote by $E_{\rho,\sigma}(z)$ the two parametric Mittag–Leffler functions:

$$E_{\rho,\sigma}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\rho n + \sigma)}.$$

If $\sigma = 1$, then we have one parametric Mittag–Leffler function:

$$E_{\rho,\,1}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\rho k + 1)} = E_{\rho}(z).$$

Lemma 1. If $t > 0$ then there is a constant $C > 0$ such that for any $\sigma \in \mathbb{C}$ one has (see, e.g., [19], p. 136)

$$|E_{\rho,\sigma}(-t)| \le \frac{C}{1+t}. \tag{2}$$

Lemma 2. The Mittag–Leffler function of the negative argument $E_{\rho}(-t)$ is a monotonically decreasing function for all $0 < \rho < 1$ and

$$0 < E_{\rho}(-t) < 1.$$

If $\alpha \in (0,\,1)$, there is $\lambda_0 > 0$ such that $E_{\rho}(-\lambda_0 \tau_0^{\rho}) = \alpha$. We suppose $\lambda_k = \lambda_0$ and let it be a multiple of p_0. We denote by

$$K_0 = \{k_0, k_0+1, k_0+2, \ldots, k_0+p_0-1\},$$

the set of numbers k by $\lambda_k = \lambda_0$.

Main part

Let us state the main theorem of the section below.

Theorem 1. Let $\varphi \in D(A)$ and $f \in H$.

If $\alpha \in (0,\,1)$ or $\alpha \notin [0,\,1)$, but $\lambda_k \ne \lambda_0$ for all $k \ge 1$, then the problem (1) has a unique solution and this solution has the form

$$u(t) = \sum_{k=1}^{\infty} \left[\frac{\varphi_k - V_k(\tau_0)}{E_{\rho}(-\mu_k \tau_0^{\rho}) - \alpha} E_{\rho}(-\mu_k t^{\rho}) + V_k(t) \right] v_k, \tag{3}$$

where

$$V_k(t) = \frac{f_k}{1+\gamma\lambda_k} t^{\rho} E_{\rho,\rho+1}(-\mu_k t^{\rho}).$$

If $\alpha \in (0,\,1)$ and $\lambda_k = \lambda_0$, $k \in K_0$, we assume that the orthogonality conditions

$$(\varphi, v_k) = 0, (f, v_k) = 0, \text{ for all } t > 0,\ k \in K_0,\ K_0 = \{k_0, k_0+1, \ldots, k_0+p_0-1\}, \tag{4}$$

are satisfied. The solution of problem (1) has the form

$$u(t) = \sum_{k \notin K_0}^{\infty} \left[\frac{\varphi_k - V_k(\tau_0)}{E_{\rho}(-\mu_k \tau_0^{\rho}) - \alpha} E_{\rho}(-\mu_k t^{\rho}) + V_k(t) \right] v_k + \sum_{k \in K_0} b_k E_{\rho}(-\mu_k t^{\rho})\, v_k. \tag{5}$$

with arbitrary coefficients b_k, $k \in K_0$.

Proof. Now, to prove the main theorem, problem (1) is divided into two auxiliary problems. The corresponding theorems and their proofs are presented.

Let us take $u(t) = V(t) + W(t)$ and solve the following two auxiliary problems:

$$\begin{cases} D_t^{\rho} V(t) + A(1+\gamma D_t^{\rho}) V(t) = f, & 0 < t \le T; \\ V(0) = 0, \end{cases} \tag{6}$$

$$\begin{cases} D_t^{\rho} W(t) + A(1+\gamma D_t^{\rho}) W(t) = 0, & 0 < t \le T; \\ W(\tau_0) = \alpha W(0) + \psi, & 0 < \tau_0 \le T, \end{cases} \tag{7}$$

where a function $\psi \in H$ is given.

It is easy to check that the function $u(t) = W(t) + V(t)$ solves problem (7) if we set $\psi = \varphi - V(\tau)$, and $W(t)$ and $V(t)$ are solutions of the problems (6) and (7), respectively.

The problem (6) has been considered in [14]. Therefore, the corresponding theorem is presented without proof.

Theorem 2. Let $f \in H$. Then, there is a unique solution to the problem (6), which takes the following form:

$$V(t)=\sum_{k=1}^{\infty}\frac{f_k}{1+\gamma\lambda_k}t^{\rho}E_{\rho,\rho+1}(-\mu_k t^{\rho})v_k. \tag{8}$$

Let us present the following theorem for the problem (7).

Theorem 3. Let $\psi \in H$.

If $\alpha \notin [0,\,1)$ or $\alpha \in (0,\,1)$, but $\lambda_k \neq \lambda_0$ for all $k \geq 1$, the problem (6) exists a unique solution. This solution takes the form of

$$W(t)=\sum_{k=1}^{\infty}\frac{\psi_k}{E_{\rho}(-\mu_k\tau_0^{\rho})-\alpha}E_{\rho}(-\mu_k t^{\rho})\,v_k. \tag{9}$$

If $\alpha \in (0,\,1)$ and $\lambda_k=\lambda_0, k \in K_0$, that the orthogonality conditions

$$\psi_k=(\psi,v_k)=0,\quad k\in K_0,\quad K_0=\{k_0,k_0+1,k_0+2,\ldots,k_0+p_0-1\}, \tag{10}$$

are satisfied.

The equation

$$W(t)=\sum_{k\neq K_0}\frac{\psi_k}{E_{\rho}(-\mu_k\tau_0^{\rho})-\alpha}E_{\rho}(-\mu_k t^{\rho})\,v_k+\sum_{k\in K_0}b_k E_{\rho}(-\mu_k t^{\rho})\,v_k, \tag{11}$$

with arbitrary coefficients b_k, $k \in K_0$, is the solution to the problem (7).

Proof. The auxiliary problem (7) is now solved. We search in the following form for a series:

$$W(t)=\sum_{k=1}^{\infty}T_k(t)\,v_k.$$

After substituting this expression into (7), we have the following problem:

$$\begin{cases} D_t^{\rho}T_k(t)+\mu_k T_k(t)=0, & 0<t\leq T;\\ T_k(\tau_0)=\alpha T_k(0)+\psi_k, \end{cases} \tag{12}$$

where $\mu_k=\dfrac{\lambda_k}{1+\gamma\lambda_k}$ and ψ_k are the Fourier coefficients of the function $\psi \in H$.

Let $T_k(0)=b_k$ be denoted. Then, given this initial condition, the unique solution to the differential equation (12) takes the following form: $T_k(t)=b_k E_{\rho}(-\mu_k t^{\rho})$ (see, e.g., [20], p.174).

Using the non-local condition of (12), we can find the unknown numbers b_k using the following equation:

$$b_k E_{\rho}(-\mu_k\tau_0^{\rho})=\alpha b_k+\psi_k. \tag{13}$$

For every $\alpha \geq 1$ and $\alpha < 0$, by Lemma 2, the Mittag-Leffler function has $E_{\rho}(-\mu_k\tau_0^{\rho}) \neq \alpha$. Consequently, using (13), we have

$$b_k=\frac{\psi_k}{E_{\rho}(-\mu_k\tau_0^{\rho})-\alpha},\quad |b_k|\leq C_{\alpha}|\psi_k|,\quad k\geq 1\ \text{ and }\ \alpha\geq 1\ \text{ or }\ \alpha<0, \tag{14}$$

here and below, we will refer to a constant as C_{α}, which will rely on α, but not always on.

Assume that $0<\alpha<1$. Then, by Lemma 2, there exists a unique $\lambda_0>0$ such that $E_{\rho}(-\mu_0\tau^{\rho})=\alpha$. The estimate in (14) holds provided that $\lambda_k \neq \lambda_0$ for all $k \geq 1$, with some constant $C_{\alpha}>0$.

Therefore, if $\alpha \notin [0,\,1)$ or $\alpha \in (0,\,1)$, $\lambda_k \neq \lambda_0$ for $k \geq 1$, then the solution to the problem (7) has the form (9).

Lastly, consider $\alpha \in (0;1)$ and $\lambda_k=\lambda_0$ for $K_0=\{k_0,k_0+1,k_0+2,\ldots,k_0+p_0-1\}$, where the number p_0 is a multiple of the eigenvalue λ_{k_0}. If the function ψ satisfies the orthogonality condition (10), then the non-local problem (12) has a solution. So, if $k \in K_0$, then b_k arbitrary numbers are solutions to the equation (13). For all other $k \notin K_0$, b_k are defined by the following equality:

$$b_k = \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha}, \quad |b_k| \le C_\alpha |\psi_k|, \quad k \notin K_0. \tag{15}$$

Thus, in this case, the solution to the problem (7) has the form (11).

We shall assume that the orthogonality condition (10) is satisfied, whenever $\alpha \in (0;1)$ and $\lambda_k = \lambda_0$.

Let us prove the uniqueness of the solution to problem (7). The solution to problem (12) with the condition $\psi_k = 0$ has been defined subject to the condition $T_k(\tau_0) = \alpha T_k(0)$.

If $\alpha \notin [0, 1)$, then according to Lemma 2, this implies $b_k \equiv 0$, and consequently $T_k(t) \equiv 0$. By the completeness of the system $\{v_k\}$, it follows that $W(t) \equiv 0$.

Consider $\alpha \notin [0, 1)$ or $\alpha \in (0, 1)$, but for any $k \ge 1$, $\lambda_k \ne \lambda_0$. Then $T_k(t) = 0$ in this situation due to the completeness of the set of eigenfunctions $\{v_k\}$, we can deduce that $W(t) \equiv 0$. In this case, the problem (7) has a unique solution.

Assume that $\alpha \in (0, 1)$ and $k \in K_0$, with $\lambda_k = \lambda_0$. Therefore, in this case, the solution to the problem (7) is not unique.

The partial sum of the series (9) is denoted by $S_n(t)$. Next

$$AS_n(t) = \sum_{k=1}^{n} \lambda_k \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k.$$

The Parseval's equality allows us to write

$$\|AS_n(t)\|^2 = \sum_{k=1}^{n} \lambda_k^2 \left| \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) \right|^2.$$

Using estimates (2), (14) and (15)

$$\|AS_n(t)\|^2 \le C_\alpha \sum_{k=1}^{n} \lambda_k^2 \left| \frac{\psi_k}{1 + \mu_k t^\rho} \right|^2 \le C_\alpha t^{-2\rho} \sum_{k=1}^{n} \lambda_k^2 |\psi_k|^2.$$

Hence, if $\psi \in D(A)$, then $Au(t) \in C((0, T]; H)$.

Let us evaluate $D_t^\rho S_n(t)$. First, we find $D_t^\rho S_n(t)$

$$D_t^\rho S_n(t) = -(I + \gamma A)^{-1} AS_n(t) = -\sum_{k=1}^{n} \frac{\lambda_k}{1 + \gamma \lambda_k} \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k.$$

Applying Parseval's equality, estimates (2), (14) and (15)

$$\|D_t^\rho S_n(t)\|^2 = \sum_{k=1}^{n} \left| \frac{\mu_k \psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) \right|^2 \le C_\alpha \sum_{k=1}^{n} \frac{|\psi_k|^2}{(1 + \mu_k t^\rho)^2} \le C_\alpha t^{-2\rho} \sum_{k=1}^{n} |\psi_k|^2.$$

Let us evaluate $AD_t^\rho S_n(t)$. First, we find $AD_t^\rho S_n(t)$

$$A(D_t^\rho S_n(t)) = -A((I + \gamma A)^{-1} AS_n(t)) = -\sum_{k=1}^{n} \lambda_k \mu_k \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho)\, v_k.$$

Applying Parseval's equality, estimates (2), (14), and (15)

$$A(D_t^\rho S_n(t)) = \left| \sum_{k=1}^{n} \lambda_k \mu_k \frac{\psi_k}{E_\rho(-\mu_k \tau_0^\rho) - \alpha} E_\rho(-\mu_k t^\rho) \right|^2$$

$$\le C_\alpha \sum_{k=1}^{n} \lambda_k^2 \frac{|\psi_k|^2}{(1 + \mu_k t^\rho)^2} \le C_\alpha t^{-2\rho} \sum_{k=1}^{n} \lambda_k^2 |\psi_k|^2.$$

Hence, if $\psi \in D(A)$, then $A(D_t^\rho u(t)) \in C((0, T]; H)$.

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