# Modeling and Analysis of Rayleigh-Type Surface Waves in Elastic Solids with Double Porosity

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**Abstract.** This investigation examines Rayleigh-type surface waves in an isotropic, homogeneous elastic half-space possessing a dual-porosity structure. The surface is considered stress-free. From the general analysis, the frequency equations for elastic media with single porosity are recovered as a limiting case. Numerical solutions of the derived equations are obtained. Graphical representations for copper material illustrate the dependence of Rayleigh wave speed and attenuation coefficient on wave number.

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***Key words:*** dual porosity; elastic wave propagation; surface waves; frequency equation.

## Introduction

Rayleigh-type surface waves play a fundamental role in elastodynamics and seismology, as they propagate along free surfaces with amplitudes that decay exponentially with depth [2]. Due to their strong localization near the surface, Rayleigh waves are responsible for a significant portion of ground motion observed during seismic events and are therefore of central importance in earthquake engineering, near-surface geophysics, and nondestructive material characterization [2,3].

Classical Rayleigh wave theory, originally established for homogeneous, isotropic elastic half-spaces, has been extensively extended to account for additional physical effects such as thermal fields, diffusion processes, electromagnetic coupling, rotation, and material anisotropy. In particular, thermoelastic and diffusion-related extensions have demonstrated that coupled fields can significantly modify dispersion and attenuation characteristics of surface waves, as shown in a series of studies by Kumar and co-authors for thermoelastic diffusion and microstructured media [4,5].

Similarly, Abd-Alla and collaborators investigated Rayleigh wave propagation under magnetic, thermal, rotational, and viscoelastic effects, revealing substantial deviations from classical elastic behavior in granular and orthotropic media [8,9].

In parallel with these developments, the modeling of porous and fractured materials has emerged as a critical challenge in solid and geophysical mechanics. Classical single-porosity theories, while effective for relatively uniform pore distributions, are often inadequate for materials characterized by complex multiscale pore networks, such as fractured rocks, reservoir formations, and certain engineered composites. To address this limitation, dual-porosity theories were introduced, accounting for two interacting pore systems-typically representing matrix porosity and fissures or micro-macro pore structures. Early foundational contributions in this area include the seepage and consolidation models of Barenblatt and Biot, which laid the groundwork for modern poroelastic formulations [10,11].

More rigorous continuum-mechanical frameworks for materials with double porosity were subsequently developed and analyzed. In particular, the works of Iesan and Quintanilla established thermodynamically consistent field equations for elastic and thermoelastic solids with double porosity, while Svanadze provided

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fundamental solutions, uniqueness theorems, and stability analyses for elastic, viscoelastic, and thermoelastic media possessing dual pore structures [12,13].

These studies demonstrated that the presence of interacting pore systems introduces additional internal variables and coupling mechanisms, which can substantially alter wave propagation behavior [14].

Despite these advances, the investigation of Rayleigh-type surface waves in elastic solids with double porosity remains comparatively limited. Surface waves in such media are of particular interest because they are highly sensitive to near-surface microstructural features, including pore connectivity, fissure density, and fluid-solid interactions. Recent studies on surface wave propagation in elastic and poroelastic half-spaces, including seismic excitation models and harmonic wave formulations, have highlighted the importance of refined theoretical models for accurately capturing near-surface wave phenomena [15,16].

Moreover, modern applications such as synthetic accelerogram generation, baseline correction in seismic modeling, and stability analysis of dynamic half-space problems further emphasize the need for advanced wave models that incorporate complex material structures [17,18].

Motivated by these considerations, the present study investigates Rayleigh-type surface wave propagation in a homogeneous, isotropic elastic half-space endowed with a dual-porosity structure. Based on the continuum model of materials with double porosity, the governing field equations are formulated and reduced to a coupled system describing mechanical displacements and porosity fields. By imposing stress-free boundary conditions at the surface, an explicit frequency (secular) equation for Rayleigh waves is derived. Numerical solutions of this equation are then obtained to analyze the dispersion and attenuation characteristics of surface waves, with particular emphasis on the influence of wave number and phase velocity [19,20].

The results demonstrate that dual porosity induces pronounced oscillatory behavior in both phase velocity and attenuation, distinguishing the response of double-porosity media from that of classical single-porosity or purely elastic solids. These findings provide new insight into surface wave behavior in fractured and multiscale porous materials and may contribute to improved interpretation of seismic data, nondestructive evaluation techniques, and wave-based characterization of complex elastic media [21,22].

## Governing Relations

Based on the model by Iesan and Quintanilla [14], the constitutive and field equations for a dual-porosity elastic solid, in the absence of body forces, are:

**Stress-Porosity Relations**

$$\begin{aligned} t_{ij} &= \lambda e_{rr}\delta_{ij} + 2\mu e_{ij} + b\,\delta_{ij}\phi + d\,\delta_{ij}\psi, \\ \sigma_i &= \alpha\,\phi_{,i} + b_1\,\psi_{,i}, \\ \tau_i &= b_1\,\phi_{,i} + \gamma\,\psi_{,i}. \end{aligned} \tag{1}$$

**Momentum Balance**

$$\mu\nabla^2 u_i + (\lambda+\mu)u_{j,ji} + b\phi_{,i} + d\psi_{,i} = \rho\ddot{u}_i, \tag{2}$$

**Porosity Evolution Equations**

$$\begin{aligned} \alpha\nabla^2\phi + b_1\nabla^2\psi - bu_{r,r} - \alpha_1\phi - \alpha_3\psi = \kappa_1\ddot{\phi}, \\ b_1\nabla^2\phi + \gamma\nabla^2\psi - du_{r,r} - \alpha_3\phi - \alpha_2\psi = \kappa_2\ddot{\psi}, \end{aligned} \tag{3}$$

Where: $\lambda, \mu-$ Lam constants; $\rho-$ density; $u_i-$ displacements; $t_{ij}-$ stress; $\kappa_1, \kappa_2-$ equilibrated inertia; $\phi, \psi-$ porosity fields (pores and fissures); $\sigma_i, \tau_i-$ associated equilibrated stresses; $b, d, b_1, \gamma, \alpha_i-$ constitutive constants; $\delta_{ij}-$Kronecker delta; dot indicates time derivative. The operators are:

$$\nabla = \hat{i}\partial_{x_1} + \hat{j}\partial_{x_2} + \hat{k}\partial_{x_3},\ \ \nabla^2 = \partial^2_{x_1} + \partial^2_{x_2} + \partial^2_{x_3}.$$

**Problem Statement**

Consider a half-space $x_3 \geq 0$. The $x_1$ -axis is the propagation direction, $x_3$ points inward. All fields are independent of $x_2$.

**Wave Solution**

Consider a half-space $x_3 \geq 0$. The $x_1$ -axis is the propagation direction, $x_3$ points inward. All fields are independent of $x_2$. Introduce dimensionless variables (primed):

$$\begin{array}{l} x'_i = \frac{\omega_1}{c_1} x_i, \ \ u'_i = \frac{\omega_1}{c_1} u_i, \ \ t'_{ij} = \frac{t_{ij}}{\lambda}, \\ \phi' = \frac{\kappa_1 \omega_1^2}{\alpha_1} \phi, \ \ \psi' = \frac{\kappa_1 \omega_1^2}{\alpha_1} \psi, \ \ t' = \omega_1 t, \\ \sigma'_1 = \frac{c_1}{\alpha \omega_1} \sigma_i, \ \ \tau'_1 = \frac{c_1}{\alpha \omega_1} \tau_1, \end{array} \tag{4}$$

with $c_1^2 = (\lambda + 2\mu)/\rho$ and $\omega_1 = \lambda/\kappa_1$.

Applying (4) to (2)-(3) and dropping primes yields:

$$\begin{array}{l} \left(\frac{\lambda+\mu}{\rho c_1^2}\right) \frac{\partial e}{\partial x_1} + \frac{\mu}{\rho c_1^2} \nabla^2 u_1 + a_1 \frac{\partial \phi}{\partial x_1} + a_2 \frac{\partial \psi}{\partial x_1} = \frac{\partial^2 u_1}{\partial t^2}, \\ \left(\frac{\lambda+\mu}{\rho c_1^2}\right) \frac{\partial e}{\partial x_3} + \frac{\mu}{\rho c_1^2} \nabla^2 u_3 + a_1 \frac{\partial \phi}{\partial x_3} + a_2 \frac{\partial \psi}{\partial x_3} = \frac{\partial^2 u_3}{\partial t^2}, \\ a_3 \nabla^2 \phi + a_4 \nabla^2 \psi - a_5 e - a_6 \phi - a_7 \psi = \frac{\partial^2 \phi}{\partial t^2}, \\ a_8 \nabla^2 \phi + a_9 \nabla^2 \psi - a_{10} e - a_{11} \phi - a_{12} \psi = \frac{\partial^2 \psi}{\partial t^2}, \end{array} \tag{5}$$

where the dimensionless coefficients $a_1$ to $a_{12}$ are combinations of material constants, and $e = u_{1,1} + u_{3,3}$.

Displacements are expressed via potentials:

$$u_1 = \frac{\partial \phi_1}{\partial x_1} - \frac{\partial \psi_1}{\partial x_3}, \quad u_3 = \frac{\partial \phi_1}{\partial x_3} + \frac{\partial \psi_1}{\partial x_1}, \tag{6}$$

Substituting (6) into (5) gives:

$$\begin{array}{l} \left(\nabla^2 - \partial_t^2\right) \phi_1 + a_1 \phi + a_2 \psi = 0 \\ -a_5 \nabla^2 \phi_1 + \left(a_3 \nabla^2 - a_6 - \partial_t^2\right) \phi + \left(a_4 \nabla^2 - a_7\right) \psi = 0 \\ -a_{10} \nabla^2 \phi_1 + \left(a_8 \nabla^2 - a_{11}\right) \phi + \left(a_9 \nabla^2 - a_{12} - \partial_t^2\right) \psi = 0 \end{array} \tag{7}$$

and an equation for $\psi_1$ :

$$a_{12} \nabla^2 - \partial_t^2) \psi_1 = 0. \tag{8}$$

**Wave Solution**

Assume harmonic propagation:

$$[\phi_1, \phi, \psi, \psi_1] = [\phi_1^*, \phi^*, \psi^*, \psi_1^*] \exp\left[i\xi(x_1 - ct)\right]. \tag{9}$$

Here $\xi$ is wave number, $c$ is phase velocity, and $\omega = \xi c$. Inserting (9) into (7) leads to a system whose solvability condition provides the characteristic equation:

$$E_1 \frac{d^6}{dz^6} + E_2 \frac{d^4}{dz^4} + E_3 \frac{d^2}{dz^2} + E_4 = 0 \tag{10}$$

with coefficients $E_i$ depending on $a_i$, $\xi$, and $c$. For $\psi_1^*$:

$$\left(\frac{d^2}{dx_3^2} - \zeta_4^2\right) \psi_1^* = 0, \ \ \zeta_4^2 = \xi^2 \left(1 + \frac{\phi^2}{a_{12}}\right). \tag{11}$$

For surface waves decaying with depth $(x_3 \to \infty)$ :

$$(\phi_1, \phi, \psi) = \sum_{i=1}^{3} (1, r_i, s_i) B_i \exp(-m_i x_3 + i\xi(x_1 - ct)), \tag{12}$$

$$\psi_1 = B_4 \exp(-m_4 x_3 + i\xi(x_1 - ct)) \tag{13}$$

where $m_4 = \zeta_4$, $B_i$ are amplitudes, and $r_i, s_i$ are coupling constants derived from the system.

### Surface Conditions and Frequency Equation

At the free surface $x_3 = 0$ :

$$t_{33} = 0, \; t_{31} = 0, \;\; \sigma_3 = 0, \;\; \tau_3 = 0. \tag{14}$$

In dimensionless form:

$$\begin{aligned} t_{33} &= p_1 u_{3,3} + u_{1,1} + p_2\phi + p_3\psi, \\ t_{31} &= p_4\left(u_{3,1} + u_{1,3}\right), \\ \sigma_3 &= p_5\phi_{,3} + \bar{p}_6\psi_{,3}, \\ \tau_3 &= p_6\phi_{,3} + \bar{p}_7\psi_{,3}, \end{aligned} \tag{15}$$

with constants $p_1$ to $p_7$ defined from material parameters. Applying solutions (12)-(13) to conditions (14) yields a $4 \times 4$ linear system for $B_j$ :

$$\sum_{j=1}^{4} Q_{ij} B_j = 0, \quad i = 1, ..., 4. \tag{16}$$

The matrix elements $Q_{ij}$ involve $m_j, \xi, p_k, r_j, s_j$. Non-trivial solutions require:

$$\det[Q_{ij}] = 0, \tag{17}$$

which is the frequency (secular) equation for Rayleigh waves in the dual-porosity medium.

**Single-Porosity Limit:** When $b_1, \gamma, \alpha_3, \alpha_2, d \to 0$, equation (17) reduces to the known result for a material with single porosity.

### Numerical Analysis and Graphs

Copper material constants:

$$\begin{aligned} &\lambda = 7.76 \times 10^{10} Nm^{-2}, \;\; \mu = 3.86 \times 10^{10} Nm^{-2}, \;\; \rho = 8.954 \times 10^{3} Kgm^{3}, \\ &\alpha = 1.3 \times 10^{-5} N, \quad \alpha_1 = 1.65 \times 10^{10} Nm^{-2}, \quad \alpha_2 = 1.96 \times 10^{10} Nm^{-2}, \\ &\alpha_3 = 1.86 \times 10^{10} Nm^{-2}, \;\; \gamma = 0.19 \times 10^{-5} N, \;\; b_1 = 0.12 \times 10^{-5} N, \\ &d = 0.49 \times 10^{10} Nm^{-2}, \quad \kappa_1 = 0.1456 \times 10^{-12} Nm^{-2}s^2, \;\; \kappa_2 = 0.1546 \times 10^{-12} Nm^{-2}s^2, \\ &b = 0.4 \times 10^{10} Nm^{-2}, \quad \omega_1 = 1 \times 10^{11} s^{-1}, \;\; t = 0.1s. \end{aligned}$$

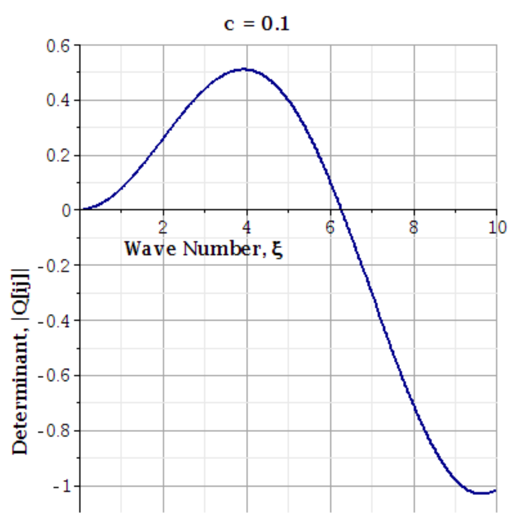


Figure 1. Determinant of the secular equation for Rayleigh waves corresponding to different values of the parameter $\xi$ at $c = 0.1$.

Figure 1 plots the determinant of (17) versus wave number $\xi$ for fixed $c = 0.1$. The response oscillates harmonically, with amplitude growing with $\xi$.

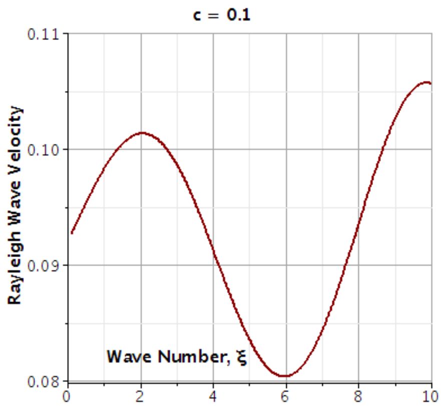


Figure 2. Rayleigh wave velocity for different values of the parameter $\xi$ at $c = 0.1$.

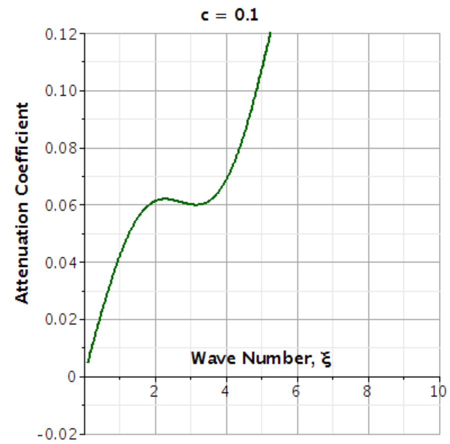


Figure 3. Attenuation coefficient for different values of the parameter $\xi$ at $c = 0.1$.

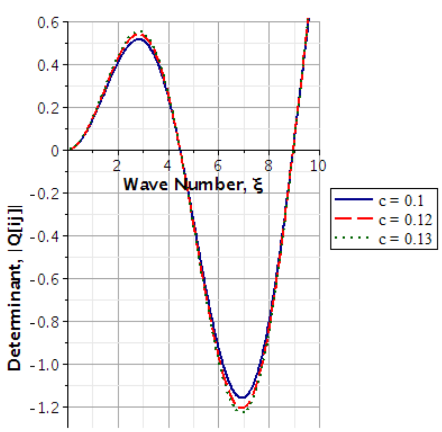


Figure 4. Determinant of the Rayleigh wave secular equation for different values of $c$ with respect to the parameter $\xi$.

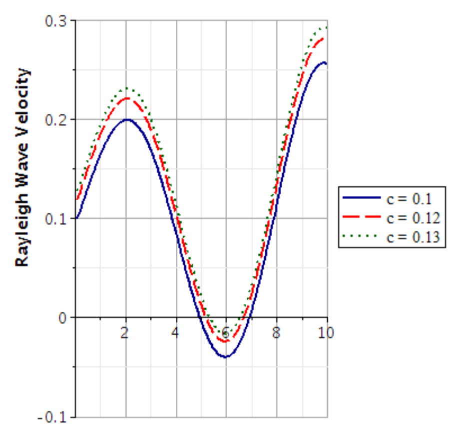


Figure 5. Rayleigh wave velocity for different values of $c$ with respect to the parameter $\xi$.

Figures 2 and 3 show phase velocity and attenuation coefficient versus $\xi$ for $c = 0.1$. Both exhibit oscillatory dependence, with increasing oscillation amplitude at higher $\xi$.

Figures 4-5 present comparisons for different phase velocities ($c = 0.1, 0.12, 0.13$). The determinant (Fig. 4) maintains harmonic variation with larger magnitude at higher $\xi$. Phase velocity (Fig. 5) and attenuation.

These results demonstrate that dual porosity introduces significant oscillatory dispersion and attenuation, strongly dependent on wave number and phase velocity.

## Conclusions

This study has presented a rigorous theoretical and numerical investigation of Rayleigh-type surface wave propagation in a homogeneous, isotropic elastic half-space characterized by a dual-porosity structure. By employing a continuum mechanics framework for materials with interacting pore systems, the governing coupled field equations were systematically derived and reduced to a tractable form suitable for surface wave analysis.

A closed-form frequency (secular) equation for Rayleigh waves was explicitly obtained under stress-free boundary conditions. Unlike classical elastic or single-porosity models, the derived dispersion relation incorporates additional coupling mechanisms associated with the interaction between matrix porosity and fissures. This formulation provides a clear mathematical foundation for assessing how multiscale pore structures influence surface wave behavior.

Numerical simulations revealed that dual porosity induces pronounced oscillatory dispersion and attenuation characteristics with respect to the wave number and phase velocity. In particular, both the Rayleigh wave speed and attenuation coefficient exhibit non-monotonic, wave-number-dependent behavior, with oscillation amplitudes increasing at higher phase velocities. These features are absent in conventional elastic models and demonstrate that internal pore interactions play a decisive role in controlling near-surface wave dynamics.

The results obtained in this work significantly advance the understanding of surface wave propagation in fractured and multiscale porous media. From a practical perspective, the proposed model offers a physically consistent tool for interpreting surface wave measurements in geophysics, seismic hazard assessment, and nondestructive evaluation of porous and composite materials. Moreover, the derived framework establishes a solid basis for further extensions to more complex settings, including anisotropic, thermo-poroelastic, and fluid-saturated layered media.

In summary, this study provides both a novel theoretical formulation and quantitative insight into Rayleigh wave propagation in dual-porosity elastic solids, thereby contributing a meaningful and original advancement to the broader field of wave propagation in complex continua.

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