

ON THE EXISTENCE OF A SOLUTION TO THE INVERSE SOURCE PROBLEM FOR THE HOPF EQUATION

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ABSTRACT. In this paper, we consider a one-dimensional inverse source problem for the Hopf equation. The problem consists in determining the unknown solution and the time-dependent source term from the given initial condition and an additional overdetermination condition at a fixed spatial point. By reducing the original inverse problem to a loaded equation and applying the fixed-point method, we establish the existence of a solution in the class of functions of finite smoothness. The proof is based on a priori estimates, the Schauder fixed-point theorem for smooth data, and a limit transition argument using weak-* compactness for the general case. As a result, sufficient conditions for the solvability of the inverse source problem are obtained

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Introduction

The study of wave propagation in media for which the intrinsic pressure difference can be neglected, that is, in pressureless media, is of both theoretical and applied interest. A direct mathematical model of such media is given by the equations of gas dynamics in which the pressure term is formally set equal to zero. From the applied point of view, pressureless media arise in the description of various physical phenomena, including multiphase flows, the motion of dispersed media, dust particles, droplets, cumulative processes, and granular materials. Various examples of gas-dynamic problems involving pressureless media can be found in classical monographs and research papers [1-4].

From the mathematical point of view, pressureless media also generate a wide class of nontrivial problems [5-6]. In particular, the theory of classical solutions for the corresponding systems has been extensively developed up to the moment when singularities arise. It has been shown that singularities may occur on manifolds of different dimensions. At the same time, earlier studies demonstrated, at the physical level of rigor, that solutions for pressureless media may still preserve their meaning even after the formation of singularities. In such situations, new types of discontinuous solutions appear, in which strong density concentrations are formed on hypersurfaces of different codimensions. These phenomena are also closely related to the mathematical modeling of two-phase and multiphase flows [7-10].

In the present paper, we consider a one-dimensional inverse source problem for the Hopf equation. More precisely, we study the problem of determining the pair $u(t, x)$ and $(g(t))$ from the equation

$$u_t + uu_x = f(x)g(t),$$

subject to the initial condition

$$u|_{t=0} = u_0(x),$$

and the additional condition

$$u|_{x=0} = \varphi(t).$$

Here the function $f(x)$ is assumed to be known and satisfies the normalization condition $f(0) = 1$. The compatibility condition $\varphi(0) = u_0(0)$ is also imposed.

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To investigate this inverse problem, we first eliminate the unknown function $g(t)$ by means of the additional condition and reduce the original problem to a loaded equation for the derivative $v = u_x$. After that, the solvability of the resulting problem is established by the fixed-point method. For sufficiently smooth input data, the existence of a solution is obtained by applying the Schauder fixed-point theorem together with suitable a priori estimates. Then, by approximating the initial data with smooth functions and using weak-* compactness arguments based on the Banach–Alaoglu theorem, we pass to the limit and obtain a solution in the class of functions of finite smoothness.

Thus, the main result of the paper is an existence theorem for the one-dimensional inverse source problem for the Hopf equation in the class of finitely smooth functions.

Inverse source problem for the Hopf equation

Let us consider a one-dimensional dynamic inverse problem of determining the function $u(t, x)$, $g(t)$, if the inhomogeneous Hopf equation and the following conditions are satisfied:

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} = f(x) \cdot g(t), \quad t > 0, \quad x \in R, \tag{1}$$

$$u|_{t=0} = u_0(x), \quad x \in R \tag{2}$$

$$u|_{x=0} = \varphi(t), \quad t > 0 \tag{3}$$

The function $f(x)$ is given. Without loss of generality, we can assume that $f(0) = 1$. Let us assume that the agreement condition is satisfied

$$\varphi(0) = u_0(0) \tag{4}$$

Using (3), we eliminate the unknown function $g(t)$ from (1) and obtain the Cauchy problem for the loaded equation. For this purpose, we formally differentiate with respect to x both parts of (1) and introduce the notation $v = u_x$. Thus, we arrive at the following problem for the loaded equation [11]

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v^2 = F(t, x) + G(t, x) \cdot v(t, 0), \tag{5}$$

$$v(0, x) = u_0'(x), \tag{6}$$

$$\frac{\partial u}{\partial x} = v, \tag{7}$$

$$u(t, 0) = \varphi(t), \tag{8}$$

where $F(t, x) = f'(x) \cdot \varphi'(t)$ and $G(t, x) = f'(x) \cdot \varphi(t)$.

Existence theorem

Theorem 1. Let the functions $u_0(x), f(x) \in C^1[-1, 1]$, $\varphi(t) \in C^1[0, 1]$, and let $f(0) = 1$. Then a solution to problem (1)–(4) exists and

$$u(t, x), u_t(t, x), u_x(t, x) \in L_\infty(\Omega), \quad g(t) \in L_\infty(0, t_0).$$

Proof. First, we prove the solvability of problem (5)–(8). To do this, we use the fixed point method. First, we assume that the input data $u_0(x), f(x)$ and $\varphi(t)$ are sufficiently smooth functions.

Fixed point

Let be $z(t, x)$, $h(t)$ some functions. Instead of system (5), (6) we consider the system

$$\frac{\partial v}{\partial t} + z \frac{\partial v}{\partial x} + v^2 = F(t, x) + G(t, x) \cdot h(t), \tag{9}$$

$$v(0, x) = u_0'(x), \tag{10}$$

And then another problem

$$\frac{\partial u}{\partial x} = v, \tag{11}$$

$$u(t, 0) = \varphi(t). \tag{12}$$

The sets Z and Ω will depend on some positive parameters M, V, δ, τ , which will be specified later. Let's put it this way

$$M = 1 + 2 \left(\sup_{|x| < 1} |u_0(x)| + \sup_{0 < t < 1} |\varphi(t)| \right),$$

$$V = 1 + 2 \sup_{|x| < 1} |u'_0(x)|.$$

Next, for some quantities $0 < \delta < 1$ and $0 < \tau < \delta/M$ we assume

$$\Omega_{\delta, \tau} = \{ (t, x) \mid 0 < t < \tau, |x| < \delta - tM \}.$$

The domain $\Omega_{\delta, \tau}$ is a trapezoid with base $(-\delta, \delta)$ and height τ . Further, $Z \subset C(\bar{\Omega}_{\delta, \tau}) \times C[0, \tau]$ is the set of such pairs $(z(t, x), h(t))$ that

$$\|z(t, x)\|_{C(\bar{\Omega}_{\delta, \tau})} \leq M,$$

$$\|z_x(t, x)\|_{C(\bar{\Omega}_{\delta, \tau})} \leq V,$$

$$z(t, 0) = \varphi(t),$$

$$\|h(t)\|_{C[0, \tau]} \leq V.$$

Let us consider the system (9)-(12) with coefficients. Let us put

$$F_0 = \sup_{|x| < 1, 0 < t < 1} |F(t, x)|, \quad G_0 = \sup_{|x| < 1, 0 < t < 1} |G(t, x)|.$$

The problem for $v(t, x)$ is equivalent to a nonlinear system

$$\dot{v}(t, y) + v^2(t, y) = (F + Gh)(t, y), \tag{13}$$

$$\dot{y}(t, x) = z(t, y), \tag{14}$$

$$v(0, x) = u'_0(x), \tag{15}$$

$$y(0, x) = x. \tag{16}$$

Locally in time this system is solvable and for small t , the estimate holds

$$|\dot{v}| \leq v^2 + F_0 + G_0V.$$

Since $|u'_0(x)| < V/2$, then on some interval $(0, t_0)$ the estimate $|v(t, x)| < V$ holds, which means

$$|v(t, x)| \leq V/2 + \tau (V^2 + F_0 + G_0V).$$

We choose τ so small that

$$\tau (V^2 + F_0 + G_0V) < V/2.$$

Then $\tau \leq t_0$. This means that system (13)-(16) is solvable in the domain $(0, \tau) \times (-\delta, \delta)$ and

$$|v(t, x)| \leq V.$$

Further, the following estimate holds:

$$|y(t, x) - x| \leq Mt.$$

This means that

$$y(t, -\delta) \leq -\delta + Mt, y(t, \delta) \geq \delta - Mt.$$

By the theorem on the continuous dependence of the solution on the parameter, we conclude that the values $y(t, x)$ cover the interval $(-\delta + Mt, \delta - Mt)$ when x runs through the interval $(-\delta, \delta)$.

In addition, differentiability with respect to the parameter takes place, which means that this covering is univalent. Consequently, system (9)-(12) is solvable in the domain Ω . In addition, the estimate holds

$$|u(t, x)| \leq |\varphi(t)| + \delta V \leq M/2 + \delta V.$$

We choose the value δ so that

$$\delta V \leq M/2.$$

Then inside Ω

$$|u(t, x)| \leq M.$$

Thus, for the given choice of parameters δ, τ , system (9)-(12) is solvable in the domain Ω , the functions $v(t, x)$, $u(t, x)$, $u_x(t, x)$ are continuous and the inequalities are satisfied

$$\begin{aligned} \|u(t, x)\|_{C(\bar{\Omega}_{\delta, \tau})} &\leq M, \\ \|u_x(t, x)\|_{C(\bar{\Omega}_{\delta, \tau})} &\leq V, \\ u(t, 0) &= \varphi(t), \\ \|v(t, 0)\|_{C[0, \tau]} &\leq V. \end{aligned}$$

Therefore, the operator L is well defined and $L(Z) \subset Z$.

Now we will show that the operator L is completely continuous. To do this, we will show $|v_x(t, x)| \leq N$ that for some constant N . It is at this point that additional smoothness of the input data is required. We will prove this statement later. For now, we assume that such an estimate has already been obtained. Then

$$|v_t(t, x)| \leq MN + V^2 + F_0 + VG_0 = N_1.$$

Hence, $v(t, 0)$ is Lipschitz with constant N_1 . Further,

$$|u_{xx}(t, x)| = |v_x(t, x)| \leq N,$$

and

$$|u_t(t, x)| \leq |\varphi'(t)| + \delta N_1.$$

These estimates show that pairs $(u(t, x), v(t, 0))$ have some excess smoothness, and therefore the set of such pairs is compactly embedded in Z . Consequently, by the Schauder theorem [12] the operator has a fixed point. Thus, problem (5)-(8) is solvable, provided that the input functions are smooth. We will show that in this case the function $u(t, x)$ is a solution to problem (1)-(4). Indeed, from the construction of the solution it follows that the trace $v(t, 0)$ is defined. Then

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - f(x)g(t) \right) = 0,$$

$$g(t) = \varphi'(t) + \varphi(t)v(t, 0).$$

Hence,

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} - f(x)g(t) = H(t),$$

with some function $H(t)$. Putting in the last equality $x = 0$, we obtain $H(t) \equiv 0$. So, the function $u(t, x)$ is a solution to problem (1)-(4). Now we can get rid of the assumption about the increased smoothness of the input data. We only needed it to evaluate $v_x(t, x)$ and prove the COMPACTNESS of the operator L .

To do this, we note that the following estimates hold:

$$\begin{aligned} \|u(t, x)\|_{C(\bar{\Omega}_{\delta, \tau})} &\leq M, \\ \|u_x(t, x)\|_{C(\bar{\Omega}_{\delta, \tau})} &\leq V, \end{aligned}$$

$$\|u_x(t, 0)\|_{C[0, \tau]} \leq V.$$

In addition, from equation (1) we easily obtain

$$g(t) = \varphi'(t) + \varphi(t)v(t, 0),$$

$$|u_t| \leq MV + |f(x)| (|\varphi'(t)| + |\varphi(t)| V).$$

Moreover, the quantities M , V and the domain Ω depend only on the norms of the functions $u_0(x)$, $u'_0(x)$, $\varphi(t)$, $\varphi'(t)$, $f'(x)$.

Consider a sequence of smooth functions $u_0^\tau(x)$, $\varphi^\tau(t)$, $f^\tau(x)$. For them we solve problem (5)-(8) and obtain a solution to problem (1)-(4) $\{u^\tau\}$, $\{g^\tau\}$. In this case

$$u^\tau(t, x), u_t^\tau(t, x), u_x^\tau(t, x) \in L_\infty(\Omega), g^\tau(t) \in L_\infty(0, t_0).$$

Since $(L_1(\Omega))' = L_\infty(\Omega)$ by the Banach-Alaoglu theorem [13] the functions $u_t^\tau(t, x)$, $u_x^\tau(t, x)$, $g^\tau(t)$ * - weakly converge to the corresponding functions $u_t(t, x)$, $u_x(t, x)$, $g(t)$. And the functions $u^\tau(t, x)$ converge strongly in $C(\Omega)$ to $u(t, x)$.

After this, it only remains to pass to the limit in system (1)-(4).

Estimate $v_x(t, x)$.

So, let the input data be smooth. Let us differentiate equation (9) with respect to and denote by $w(t, x) = v_x(t, x)$

$$\frac{\partial w}{\partial t} + z \frac{\partial w}{\partial x} + z_x w + 2vw = F_x(t, x) + G_x(t, x)h(t), \quad (17)$$

There is an inequality on the characteristics

$$|\dot{w}| \leq 3V |w| + F_1 + G_1 V.$$

Hence (by Gronwall's lemma) when $t < \tau$

$$|w(t, x)| \leq ((F_1 + G_1 V) \tau + |w(0, x)|) e^{3V\tau} \leq \tilde{N}.$$

The value \tilde{N} depends on some additional derivatives of the input data, but this is not important. The main thing is that such a value exists. **Theorem 1 is proven.**

Corollary. Let the conditions of Theorem 1 be satisfied. Then the solution to problem (1)-(4) $u(t, x)$ is Lipschitz.

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