Surface theory in four-dimensional Galilean space

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Abstract. This paper develops several fundamental aspects of the theory of surfaces in four-dimensional Galilean space. The first and second fundamental forms of a surface are introduced and used to define the normal curvature, principal curvatures, mean curvatures, and total curvature. The principal curvatures are characterized as extremal values of the normal curvature. Derivative formulas for surfaces are established, leading to relations that express the coefficients of the second fundamental form in terms of the coefficients of the first fundamental form and their partial derivatives. As a consequence, the mean and total curvatures are represented without explicit use of the coefficients of the second fundamental form.

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Introduction

Galilean geometry is a type of geometry in spaces with a special metric, and it is widely used in various fields of mathematics, mechanics, and theoretical physics. With the development of non-Euclidean geometry, interest in Galilean geometry has also grown. The first scientific works devoted to Galilean space appeared in the 1950s. The scientists who founded this field were R.G. Bukharaev and E.E. Khatipova, who laid the fundamental foundations of the geometry of spaces with broken metrics. The general theory of projective metric spaces, both metric and non-metric, is systematically presented in B. A. Rozenfeld's classic monograph "Non-Euclidean Spaces" [1]. Among the first to address the problems of "full" geometry in Galilean space were A. Artikbayev and Pankina [2],[3]. The theoretical foundations of three-dimensional Galilean space are described in detail in Roschel's monograph [4]. The subsequent development of "full" geometry is reflected in the monograph by A. Artikbayev and D.D. Sokolov [5]. The axiomatic construction of the geometry of the Galilean plane and space was studied by A.I. Dolgorev [6], while the solution of differential geometry problems and classical problems in this space was investigated by I.A. Dolgorev [7]. The above-mentioned studies are primarily from the late 20th century.

The theory of surfaces in Galilean space has been studied by authors such as Aydin, Dede, and Yoon, and the theory of surfaces in four-dimensional Euclidean and non-Euclidean spaces has been investigated by Y. Aminov and Dolgorev [8],[9],[10]. Ismoilov investigated application of isotropic geometry methods to solving Monge–Ampere equation in differential geometry problems [11]. Sharipov and Keunimjaev studied existence and uniqueness problems of polyhedra determined by conditional curvature values [12]. In three-dimensional Galilean space, A. Artikbayev and B. Sultonov conducted research on parabolic surface theory [13]. Dolgarev primarily worked on two-dimensional surface theory in four-dimensional space. The fundamental concepts of four-dimensional Galilean space are presented in the work [14] by A. Artikbayev and A. Nurbayev.

Preliminaries

In G_4 space, we take the surface equation in the following form for convenience:

$$\vec{r}(t,u,v) = t\vec{e} + x(t,u,v)\vec{i} + y(t,u,v)\vec{j} + z(t,u,v)\vec{k} \tag{1}$$

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In Euclidean geometry, the first quadratic form I is expressed as follows:

$$I = Adt^2 + 2Bdtdu + 2Cdtdv + Edu^2 + 2Fdudv + Gdv^2 \tag{2}$$

However, in Galilean geometry, its form is somewhat simplified with respect to the scalar product. For the surface (1) in G_4 space, the first quadratic form of the surface is determined by the formula [15]:

$$I = \begin{cases} dt^2 & \text{if } t \neq \text{const}, \\ Edu^2 + 2Fdudv + Gdv^2 & \text{if } t = \text{const} \end{cases} \tag{3}$$

Here

$$\begin{aligned} &E = \overrightarrow{r_u}^2 = {x'_u}^2 + {y'_u}^2 + {z'_u}^2, \quad F = \overrightarrow{r_u}\,\overrightarrow{r_v} = x'_u x'_v + y'_u y'_v + z'_u z'_v, \\ &G = \overrightarrow{r_v}^2 = {x'_v}^2 + {y'_v}^2 + {z'_v}^2 \end{aligned} \tag{4}$$

It is important to note that in determining some coefficients of the first quadratic form, there are components that are not involved in the product. We denote them as follows, and they are determined by the following formulas:

$$\begin{aligned} A &= \overrightarrow{r'_t}^2 = {x'_t}^2 + {y'_t}^2 + {z'_t}^2, \\ B &= \overrightarrow{r'_t}\,\overrightarrow{r'_u} = x'_t x'_u + y'_t y'_u + z'_t z'_u, \\ C &= \overrightarrow{r'_t}\,\overrightarrow{r'_v} = x'_t x'_v + y'_t y'_v + z'_t z'_v \end{aligned} \tag{5}$$

For a given surface, we introduce the unit normal vector in the following form:

$$\vec{n} = \frac{\overrightarrow{r'_u} \times \overrightarrow{r'_v}}{\left\| \overrightarrow{r'_u} \times \overrightarrow{r'_v} \right\|} = \frac{\begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix} \overrightarrow{e_2} - \begin{vmatrix} x'_u & z'_u \\ x'_v & z'_v \end{vmatrix} \overrightarrow{e_3} + \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix} \overrightarrow{e_4}}{\sqrt{EG - F^2}} = \frac{n_2 \overrightarrow{e_2} + n_3 \overrightarrow{e_3} + n_4 \overrightarrow{e_4}}{W} \tag{6}$$

Here

$$n_2 = \begin{vmatrix} y'_u & z'_u \\ y'_v & z'_v \end{vmatrix}, \quad n_3 = -\begin{vmatrix} x'_u & z'_u \\ x'_v & z'_v \end{vmatrix}, \quad n_4 = \begin{vmatrix} x'_u & y'_u \\ x'_v & y'_v \end{vmatrix}, \quad W = \sqrt{EG - F^2} \quad [13]$$

The second quadratic form of the surface is defined by the introduced normal vector. The general form of the second quadratic form is determined by the following equality [16]:

$$\mathrm{II} = \left(\vec{n}\, d^2 r\right) = Pdt^2 + Ldu^2 + Ndv^2 + 2Qdtdu + 2Rdtdv + 2Mdudv \tag{7}$$

Its coefficients are determined as follows:

$$\begin{cases} P = \overrightarrow{r''_{tt}} \cdot \vec{n} = \dfrac{x''_{tt} n_2 + y''_{tt} n_3 + z''_{tt} n_4}{W} \\ Q = \overrightarrow{r''_{tu}} \cdot \vec{n} = \dfrac{x''_{tu} n_2 + y''_{tu} n_3 + z''_{tu} n_4}{W} \\ R = \overrightarrow{r''_{tv}} \cdot \vec{n} = \dfrac{x''_{tv} n_2 + y''_{tv} n_3 + z''_{tv} n_4}{W} \\ L = \overrightarrow{r''_{uu}} \cdot \vec{n} = \dfrac{x''_{uu} n_2 + y''_{uu} n_3 + z''_{uu} n_4}{W} \\ M = \overrightarrow{r''_{uv}} \cdot \vec{n} = \dfrac{x''_{uv} n_2 + y''_{uv} n_3 + z''_{uv} n_4}{W} \\ N = \overrightarrow{r''_{vv}} \cdot \vec{n} = \dfrac{x''_{vv} n_2 + y''_{vv} n_3 + z''_{vv} n_4}{W} \end{cases} \tag{8}$$

Curvature of a curve on a surface

We determine the principal curvatures of the surface through the extremal values of the normal curvatures at a given point on the surface. It is well known that the normal curvature is defined as follows:

$$k_n = \frac{\mathrm{II}}{\mathrm{I}}.$$

In Galilean geometry, based on the first quadratic form presented above, we determine the normal curvature by dividing it into two parts.

First, for the case where $t \neq \text{const}$, we have:

$$k_n = \frac{\mathrm{II}}{\mathrm{I}} = \frac{Pdt^2 + Ldu^2 + Ndv^2 + 2Qdtdu + 2Rdtdv + 2Mdudv}{dt^2}$$

We define the principal curvatures of the surface as the extremal values of the normal curvature k_n. To achieve this, by taking derivatives with respect to the directions dt, du, dv, setting them to zero, and performing certain transformations, we arrive at the following system:

$$\begin{cases} Pdt + Qdu + Rdv = kdt \\ Qdt + Ldu + Mdv = 0 \\ Rdt + Mdu + Ndv = 0 \end{cases} \Rightarrow \begin{cases} (P-k)dt + Qdu + Rdv = 0 \\ Qdt + Ldu + Mdv = 0 \\ Rdt + Mdu + Ndv = 0 \end{cases} \tag{9}$$

For this system to have a non-trivial (non-zero) solution, its principal determinant must be equal to zero. In this work, we consider the case where $LN - M^2 \neq 0$ for the given surface.

$$\begin{vmatrix} P-k & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix} = 0 \quad \Rightarrow \quad k_1 = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN - M^2} \tag{10}$$

If $t = \text{const}$, then according to equality (4), the normal curvature becomes:

$$k_n = \frac{II}{I} = \frac{Ldu^2 + 2Mdudv + Ndv^2}{Edu^2 + 2Fdudv + Gdv^2}$$

Since $dt = 0$ if $t = \text{const}$, the expression simplifies accordingly. To find the extremal values of k_n, by taking derivatives with respect to the directions du, dv, setting them to zero, and performing certain transformations, we arrive at the following system:

$$\begin{cases} (L-kE)du + (M-kF)dv = 0 \\ (M-kF)du + (N-kG)dv = 0 \end{cases} \tag{11}$$

For system (11) to have a non-trivial solution, the principal determinant must be equal to zero:

$$\begin{vmatrix} L-kE & M-kF \\ M-kF & N-kG \end{vmatrix} = 0$$

From this, the following relations are derived:

$$k_2 + k_3 = \frac{LG - 2MF + NE}{EG - F^2}, \qquad k_2 \cdot k_3 = \frac{LN - M^2}{EG - F^2} \tag{12}$$

Based on the definitions above, the first and second mean curvatures, as well as the total (Gaussian) curvature of the surface, can be determined as follows:

Mean curvature

$$3H_1 = k_1 + k_2 + k_3$$

Second-order curvature

$$H_2 = k_1k_2 + k_1k_3 + k_2k_3$$

Total curvature

$$K = k_1 k_2 k_3$$

Using relations (10) and (12), the following formulas for H_1, H_2, and K can be derived:

$$3H_1 = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN - M^2} + \frac{LG - 2MF + NE}{EG - F^2} \tag{13}$$

$$H_2 = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN - M^2} \cdot \frac{LG - 2MF + NE}{EG - F^2} + \frac{LN - M^2}{EG - F^2} \tag{14}$$

$$K = \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{EG - F^2} \tag{15}$$

Main part

Derivative formulas for the surface in G_4 space.

To study the surface (1) defined by the vector-valued function $\vec{r} = \vec{r}(t, u, v)$, we establish the derivation formulas. The second-order partial derivatives of the function are expressed as a linear combination of the vectors $\{\mathbf{r}_u, \mathbf{r}_v, \mathbf{n}\}$. To simplify the notation, we introduce the following conventions:

$$\vec{r_t} = r_t, \quad \vec{r_u} = r_u, \quad \vec{r_v} = r_v, \quad \vec{n} = n.$$

The derivation formulas are then given by the following systems of equations:

$$\begin{cases} r_{tt} = \Gamma_{11}^2 \mathbf{r}_u + \Gamma_{11}^3 \mathbf{r}_v + b_{11}\mathbf{n} \\ r_{tu} = \Gamma_{12}^2 \mathbf{r}_u + \Gamma_{12}^3 \mathbf{r}_v + b_{12}\mathbf{n} \\ r_{tv} = \Gamma_{13}^2 \mathbf{r}_u + \Gamma_{13}^3 \mathbf{r}_v + b_{13}\mathbf{n} \\ r_{uu} = \Gamma_{22}^2 \mathbf{r}_u + \Gamma_{22}^3 \mathbf{r}_v + b_{22}\mathbf{n} \\ r_{uv} = \Gamma_{23}^2 \mathbf{r}_u + \Gamma_{23}^3 \mathbf{r}_v + b_{23}\mathbf{n} \\ r_{vv} = \Gamma_{33}^2 \mathbf{r}_u + \Gamma_{33}^3 \mathbf{r}_v + b_{33}\mathbf{n} \end{cases} \tag{16}$$

$$\begin{cases} n_t = \Gamma_{14}^2 \mathbf{r}_u + \Gamma_{14}^3 \mathbf{r}_v \\ n_u = \Gamma_{24}^2 \mathbf{r}_u + \Gamma_{24}^3 \mathbf{r}_v \\ n_v = \Gamma_{34}^2 \mathbf{r}_u + \Gamma_{34}^3 \mathbf{r}_v \end{cases} \tag{17}$$

By taking the scalar product of each equation in the system with the vector $\mathbf{n}$, the coefficients can be easily determined as follows:

$$b_{11} = P, \quad b_{12} = Q, \quad b_{13} = R, \quad b_{22} = L, \quad b_{23} = M, \quad b_{33} = N$$

To determine the remaining Γ_{ij}^k coefficients $(k = 2, 3;\ i, j = 1, 2, 3)$, we take the scalar product of each equality in system (16) with the basis vectors $\{\mathbf{r}_u, \mathbf{r}_v\}$. For instance, considering the first equation:

$$\begin{cases} r_{tt} \cdot r_u = \Gamma_{11}^2 E + \Gamma_{11}^3 F = B_t - \frac{1}{2} A_u \\ r_{tt} \cdot r_v = \Gamma_{11}^2 F + \Gamma_{11}^3 G = C_t - \frac{1}{2} A_v \end{cases}$$

From this system, the solutions for Γ_{11}^2 and Γ_{11}^3 are found as:

$$\Gamma_{11}^2 = \frac{\begin{vmatrix} B_t - \frac{1}{2}A_u & F \\ C_t - \frac{1}{2}A_v & G \end{vmatrix}}{W^2}, \qquad \Gamma_{11}^3 = \frac{\begin{vmatrix} E & B_t - \frac{1}{2}A_u \\ F & C_t - \frac{1}{2}A_v \end{vmatrix}}{W^2}.$$

The remaining coefficients are determined in a similar manner:

$$\Gamma_{12}^2 = \frac{\begin{vmatrix} E_t & F \\ C_u - B_v + F_t & G \end{vmatrix}}{2W^2}, \quad \Gamma_{12}^3 = \frac{\begin{vmatrix} E & E_t \\ F & C_u - B_v + F_t \end{vmatrix}}{2W^2},$$

$$\Gamma_{13}^2 = \frac{\begin{vmatrix} B_v - C_u + F_t & F \\ G_t & G \end{vmatrix}}{2W^2}, \quad \Gamma_{13}^3 = \frac{\begin{vmatrix} E & B_v - C_u + F_t \\ F & G_t \end{vmatrix}}{2W^2},$$

$$\Gamma_{22}^2 = \frac{\begin{vmatrix} E_u & F \\ 2F_u - E_v & G \end{vmatrix}}{2W^2}, \quad \Gamma_{22}^3 = \frac{\begin{vmatrix} E & E_u \\ F & 2F_u - E_v \end{vmatrix}}{2W^2},$$

$$\Gamma_{23}^2 = \frac{\begin{vmatrix} E_v & F \\ G_u & G \end{vmatrix}}{2W^2}, \quad \Gamma_{23}^3 = \frac{\begin{vmatrix} E & E_v \\ F & G_u \end{vmatrix}}{2W^2},$$

$$\Gamma_{33}^2 = \frac{\begin{vmatrix} 2F_v - G_u & F \\ G_v & G \end{vmatrix}}{2W^2}, \quad \Gamma_{33}^3 = \frac{\begin{vmatrix} E & 2F_v - G_u \\ F & G_v \end{vmatrix}}{2W^2}.$$

The Γ_{ij}^k coefficients determined above are expressed solely through the coefficients A, B, C, E, F, G and their derivatives; notably, the coefficients of the second quadratic form are not involved.

We perform similar operations for system (17). By considering the following identities:

$$n_t \cdot r_t = -P, \quad n_t \cdot r_u = n_u \cdot r_t = -Q, \quad n_t \cdot r_v = n_v \cdot r_t = -R,$$
$$n_u \cdot r_u = -L, \quad n_u \cdot r_v = n_v \cdot r_u = -M, \quad n_v \cdot r_v = -N,$$

we take the scalar product of each equation in system (17) with the vectors $\mathbf{r}_u$ and $\mathbf{r}_v$. From the resulting equalities, the coefficients Γ_{i4}^k of system (17) are determined. From the first equation, the following system is obtained:

$$\begin{cases} n_t \cdot r_u = \Gamma_{14}^2 E + \Gamma_{14}^3 F = -Q \\ n_t \cdot r_v = \Gamma_{14}^2 F + \Gamma_{14}^3 G = -R \end{cases}$$

The solutions to this system are given by:

$$\Gamma_{14}^2 = \frac{\begin{vmatrix} F & Q \\ G & R \end{vmatrix}}{W^2}, \qquad \Gamma_{14}^3 = \frac{\begin{vmatrix} Q & E \\ R & F \end{vmatrix}}{W^2}.$$

The remaining coefficients are determined using the same method. Their values are given by:

$$\Gamma_{24}^2 = \frac{\begin{vmatrix} F & L \\ G & M \end{vmatrix}}{W^2}, \quad \Gamma_{24}^3 = \frac{\begin{vmatrix} L & E \\ M & F \end{vmatrix}}{W^2},$$

$$\Gamma_{34}^2 = \frac{\begin{vmatrix} F & M \\ G & N \end{vmatrix}}{W^2}, \quad \Gamma_{34}^3 = \frac{\begin{vmatrix} M & E \\ N & F \end{vmatrix}}{W^2}.$$

If the vector-valued function $\mathbf{r} = \mathbf{r}(t, u, v)$ is continuous in the domain under consideration and possesses mixed partial derivatives up to the third order, then according to Schwarz's theorem, the following identities must hold [17]:

$$\begin{aligned}
&(r_{tt})_u = (r_{tu})_t, \quad (r_{tt})_v = (r_{tv})_t,\\
&(r_{tu})_v = (r_{tv})_u, \quad (r_{uu})_t = (r_{tu})_u,\\
&(r_{vv})_t = (r_{tv})_v, \quad (r_{uv})_t = (r_{tu})_v,\\
&(r_{uu})_v = (r_{uv})_u, \quad (r_{uv})_v = (r_{vv})_u
\end{aligned} \tag{18}$$

From the first identity, $(\mathbf{r}_{tt})_u - (\mathbf{r}_{tu})_t = 0$, and by utilizing systems (16) and (17), the following expression can be derived:

$$\begin{aligned}
&\big[(\Gamma^2_{11})_u + \Gamma^2_{11}\Gamma^2_{22} + \Gamma^3_{11}\Gamma^2_{23} + P\Gamma^2_{24} - (\Gamma^2_{12})_t - \Gamma^2_{12}\Gamma^2_{12} - \Gamma^3_{12}\Gamma^2_{13} - Q\Gamma^2_{14}\big] r_u +\\
&+ \big[(\Gamma^3_{11})_u + \Gamma^2_{11}\Gamma^3_{22} + \Gamma^3_{11}\Gamma^3_{23} + P\Gamma^3_{24} - (\Gamma^3_{12})_t - \Gamma^2_{12}\Gamma^3_{12} - \Gamma^3_{12}\Gamma^3_{13} - Q\Gamma^3_{14}\big] r_v +\\
&+ \big[\Gamma^2_{11}L + \Gamma^3_{11}M + P_u - \Gamma^2_{12}Q - \Gamma^3_{12}R - Q_t\big] n = 0
\end{aligned}$$

Due to the linear independence of the vectors $\mathbf{r}_u, \mathbf{r}_v$, and $\mathbf{n}$ in the above equality, it follows that their respective coefficients must vanish. By setting these coefficients to zero and performing several transformations, we derive the following equations:

$$\begin{aligned}
&P\Gamma^2_{24} - Q\Gamma^2_{14} = (\Gamma^2_{12})_t + \Gamma^2_{12}\Gamma^2_{12} + \Gamma^3_{12}\Gamma^2_{13} - (\Gamma^2_{11})_u - \Gamma^2_{11}\Gamma^2_{22} - \Gamma^3_{11}\Gamma^2_{23} = \alpha_{11}\\
&P\Gamma^3_{24} - Q\Gamma^3_{14} = (\Gamma^3_{12})_t + \Gamma^2_{12}\Gamma^3_{12} + \Gamma^3_{12}\Gamma^3_{13} - (\Gamma^3_{11})_u - \Gamma^2_{11}\Gamma^3_{22} - \Gamma^3_{11}\Gamma^3_{23} = \alpha_{12}\\
&P_u - Q_t = \Gamma^2_{12}Q + \Gamma^3_{12}R - \Gamma^2_{11}L - \Gamma^3_{11}M = \beta_1
\end{aligned}$$

In a similar manner, the remaining 21 expressions for α_{lt} and β_i $(l = \overline{1,8};\ t = \overline{1,2})$ can be determined from the other seven identities. Consequently, a total of 24 equations are obtained from the eight identities.

Result. Since the terms α_{lt} in the derived equations are expressed linearly through the Christoffel symbols Γ^k_{ij} $(k = 2, 3;\ i, j = 1, 2, 3)$ and their partial derivatives, they can be determined without involving the coefficients of the second quadratic form. To define these terms, it is sufficient to obtain the coefficients A, B, C, E, F, G and their respective partial derivatives.

Lemma. *The ratios of the coefficients of the second quadratic form of the surface are fully expressed through the coefficients A, B, C, E, F, and G, as well as their partial derivatives.*

Proof. By substituting the values of Γ^k_{i4} into the equations derived from the identities, we isolate the six relations necessary for the proof of the Lemma:

$$\begin{cases}
P\Gamma^2_{24} - Q\Gamma^2_{14} = \dfrac{PFM - PLG - QFR + Q^2G}{W^2} = \alpha_{11}\\[2mm]
P\Gamma^3_{24} - Q\Gamma^3_{14} = \dfrac{PFL - PEM + QER - Q^2F}{W^2} = \alpha_{12}\\[2mm]
P\Gamma^2_{34} - R\Gamma^2_{14} = \dfrac{PFN - PGM + QGR - R^2F}{W^2} = \alpha_{21}\\[2mm]
L\Gamma^3_{14} - Q\Gamma^3_{24} = \dfrac{-E(LR - QM)}{W^2} = \alpha_{42}\\[2mm]
M\Gamma^3_{14} - Q\Gamma^3_{34} = \dfrac{-E(MR - QN)}{W^2} = \alpha_{62}\\[2mm]
L\Gamma^3_{34} - M\Gamma^3_{24} = \dfrac{-E(LN - M^2)}{W^2} = \alpha_{72}
\end{cases} \tag{19}$$

The system of equations in (19) can be rewritten in the following form:

$$\begin{cases} F(PM-QR)-G(PL-Q^2)=\alpha_{11}W^2 \\ -E(PM-QR)+F(PL-Q^2)=\alpha_{12}W^2 \\ F(PN-R^2)-G(PM-QR)=\alpha_{21}W^2 \\ QM-RL=\dfrac{\alpha_{42}W^2}{E} \\ QN-RM=\dfrac{\alpha_{62}W^2}{E} \\ LN-M^2=\dfrac{-\alpha_{72}W^2}{E} \end{cases}$$

From the given system, we determine the values of the expressions involving only the coefficients of the second quadratic form:

$$\begin{cases} PM-QR=\begin{vmatrix} -G & \alpha_{11} \\ F & \alpha_{12} \end{vmatrix}=a \\ PL-Q^2=\begin{vmatrix} \alpha_{11} & F \\ \alpha_{12} & -E \end{vmatrix}=b \\ PN-R^2=\begin{vmatrix} -G & \alpha_{21} \\ F & \alpha_{22} \end{vmatrix}=c \\ QN-RM=\dfrac{\alpha_{62}W^2}{E}=d \\ QM-RL=\dfrac{\alpha_{42}W^2}{E}=e \\ LN-M^2=\dfrac{-\alpha_{72}W^2}{E}=l \end{cases} \tag{20}$$

As seen from this system of equations, the terms $a,b,c,d,e,$ and l are all expressed in terms of the coefficients A,B,C,E,F,G and their partial derivatives. Using the established system (20), the following ratios can be constructed:

$$\begin{aligned} &\frac{Q}{L}=\frac{ae-bd}{e^2-bl}; \quad \frac{L}{M}=\frac{e^2-bl}{de-al}; \quad \frac{M}{N}=\frac{de-al}{d^2-cl}; \\ &\frac{N}{P}=\frac{d^2-cl}{a^2-bc}; \quad \frac{P}{R}=\frac{a^2-bc}{ce-ad}. \end{aligned} \tag{21}$$

In conclusion, the mutual ratios of the coefficients of the second quadratic form are expressed through the coefficients A,B,C,E,F,G and their partial derivatives. The Lemma is proved. □

Theorem 1. *The second-order curvature of the surface, defined by* $H_2=k_1k_2+k_1k_3+k_2k_3$*, is not expressible through the coefficients of the first quadratic form alone; however, there exists a representation of this curvature in which the coefficients of the second quadratic form do not explicitly appear.*

Proof. To prove the theorem, we must demonstrate that the second-order curvature of the surface, expressed as:

$$H_2=\frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN-M^2}\cdot\frac{LG-2MF+NE}{EG-F^2}+\frac{LN-M^2}{EG-F^2}$$

can be rewritten without the coefficients $P,Q,R,L,M,$ and N. We must show that while the coefficients $E,F,$ and G are insufficient to represent this equality, the additional coefficients $A,B,$ and C (which do not appear in the standard first quadratic form) are required.By expanding the determinant in the expression for H_2, we divide both the numerator and the denominator of the first fraction by P^2:

$$H_2=\frac{P(LN-M^2)-Q(QN-MR)+R(QM-LR)}{LN-M^2}\cdot\frac{LG-2MF+NE}{EG-F^2}+\frac{LN-M^2}{EG-F^2}$$

$$H_2 = \frac{(LN - M^2) - \frac{Q}{P}(QN - MR) + \frac{R}{P}(QM - LR)}{\frac{L}{P}\frac{N}{P} - \left(\frac{M}{P}\right)^2} \cdot \frac{\frac{L}{P}G - 2\frac{M}{P}F + \frac{N}{P}E}{EG - F^2} + \frac{LN - M^2}{EG - F^2} \tag{22}$$

According to the Lemma and the system of equations (20), H_2 can be expressed as follows:

$$\begin{aligned} H_2 &= \frac{l - \frac{ae-bd}{a^2-bc}d + \frac{ce-ad}{a^2-bc}e}{\frac{e^2-bl}{a^2-bc}\frac{d^2-cl}{a^2-bc} - \left(\frac{de-al}{a^2-bc}\right)^2} \cdot \frac{\frac{e^2-bl}{a^2-bc}G - 2\frac{de-al}{a^2-bc}F + \frac{d^2-cl}{a^2-bc}E}{EG - F^2} + \frac{l}{EG - F^2} \\ &= \frac{1}{EG - F^2} \cdot \left\{ \frac{\left[l(a^2 - bc) - d(ae - bd) + e(ce - ad)\right]\left[G(e^2 - bl) - 2F(de - al) + E(d^2 - cl)\right]}{(e^2 - bl)(d^2 - cl) - (de - al)^2} + l \right\} \\ &= \frac{l^2 - G(e^2 - bl) + 2F(de - al) - E(d^2 - cl)}{l(EG - F^2)} \end{aligned}$$

Thus, the expression for the second-order curvature takes the following form:

$$H_2 = \frac{l^2 - G(e^2 - bl) + 2F(de - al) - E(d^2 - cl)}{l(EG - F^2)}. \tag{23}$$

In the derived formula (23), the expressions a, b, c, d, e, and l consist solely of the coefficients E, F, G and A, B, C, along with their derivatives. The theorem is proved.The aforementioned Theorem 1 is of significant importance for surface theory. If, under certain conditions, the terms in H_2 containing A, B, and C vanish, it follows that the second-order curvature of the surface belongs to its intrinsic geometry.

Theorem 2. *The total curvature (Gaussian curvature) of the surface is determined by the following formula:*

$$K = \pm\frac{\sqrt{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}}{EG - F^2}$$

Proof. For a given surface, the total curvatureвЂ”defined as the product of the principal curvaturesвЂ”is determined by equation (15). Certain coefficients of the second quadratic form can be expressed as follows:

$$\begin{aligned} P^2 &= \frac{P^2(LN - M^2)}{LN - M^2} = \frac{LN - M^2}{\frac{L}{P}\frac{N}{P} - \left(\frac{M}{P}\right)^2} = \frac{l}{\frac{e^2-bl}{a^2-bc}\frac{d^2-cl}{a^2-bc} - \left(\frac{de-al}{a^2-bc}\right)^2} \\ &= \frac{(a^2 - bc)^2}{2ade - e^2c - bd^2 + bcl - a^2l} = \frac{\begin{vmatrix} a & b \\ c & a \end{vmatrix}^2}{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}} \end{aligned}$$

$$Q^2 = \frac{Q^2(LN - M^2)}{LN - M^2} = \frac{(ae - bd)^2}{2ade - e^2c - bd^2 + bcl - a^2l} = \frac{\begin{vmatrix} a & b \\ d & e \end{vmatrix}^2}{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}$$

$$R^2 = \frac{R^2(LN - M^2)}{LN - M^2} = \frac{(ce - ad)^2}{2ade - e^2c - bd^2 + bcl - a^2l} = \frac{\begin{vmatrix} c & a \\ d & e \end{vmatrix}^2}{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}$$

By utilizing these relations, the formula for the total curvature can be derived as follows:

$$\begin{aligned} K &= \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{EG-F^2} = \frac{P(LN-M^2)-Q(QN-MR)+R(QM-LR)}{EG-F^2} \\ &= \frac{\pm\dfrac{\begin{vmatrix} a & b \\ c & a \end{vmatrix} l - \begin{vmatrix} a & b \\ d & e \end{vmatrix} d + \begin{vmatrix} c & a \\ d & e \end{vmatrix} e}{\sqrt{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}}}{EG-F^2} = \pm\frac{\sqrt{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}}{EG-F^2} \end{aligned}$$

The theorem is proved. □

In surface theory, several fundamental equations can be derived from the system of equations (18). In particular, using system (20), the following relation can be established:

$$\frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{EG-F^2} = \pm\frac{\sqrt{\begin{vmatrix} a & b & e \\ d & e & l \\ c & a & d \end{vmatrix}}}{EG-F^2} \tag{24}$$

The right-hand side of this equation is not expressed solely through the coefficients of the first quadratic form. However, it is important to note that the coefficients of the second quadratic form do not appear in this expression either. Instead, the right-hand side is fully defined by the first quadratic form, its derivatives, and the additional coefficients A, B, and C, along with their derivatives. This equation serves as the analogue of the Gauss equation in the G_4 space.

Theorem 3. *The mean curvature of a surface in Galilean space is determined by the following identity:*

$$3H_1 = \frac{G(e^2-bl)-2F(de-al)+E(d^2-cl)}{(a^2-bc)(EG-F^2)} - \frac{1}{l} \tag{25}$$

Proof. The mean curvature of the given surface is defined as shown in (13). Based on the previously established Lemma and Theorems, the following derivation can be constructed:

$$\begin{aligned} 3H_1 &= \frac{\begin{vmatrix} P & Q & R \\ Q & L & M \\ R & M & N \end{vmatrix}}{LN-M^2} + \frac{LG-2MF+NE}{EG-F^2} = \frac{-1}{l} + \frac{\frac{e^2-bl}{a^2-bc}G - 2\frac{de-al}{a^2-bc}F + \frac{d^2-cl}{a^2-bc}E}{EG-F^2} \\ &= \frac{G(e^2-bl)-2F(de-al)+E(d^2-cl)}{(a^2-bc)(EG-F^2)} - \frac{1}{l} \end{aligned}$$

The theorem is proved. □

Conclusion

In this study, several fundamental aspects of surface theory in four-dimensional Galilean space were investigated. The first and second quadratic forms of a surface were introduced, and their roles in defining the normal curvature and the principal curvatures were analyzed. Based on these concepts, formulas for the principal curvatures, mean curvature, second-order curvature, and total curvature of the surface were derived.

A system of derivative formulas describing the behavior of the surface was also established. By analyzing the relationships obtained from these formulas and applying the identities derived from the Schwarz theorem, it was shown that the ratios of the coefficients of the second quadratic form can be expressed through the coefficients of the first quadratic form and their partial derivatives. As a consequence, new representations for the mean curvature, second-order curvature, and total curvature were obtained in which the coefficients of the second quadratic form do not explicitly appear.

The obtained results contribute to the development of the differential geometry of surfaces in Galilean spaces and may be useful for further investigations of geometric properties of multidimensional non-Euclidean spaces.

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