Existence conditions for periodic solutions for differential equations with piecewise constant arguments of mixed type

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Abstract. This paper presents a way for locating n-periodic solutions to first-order differential equations with piecewise constant arguments of mixed type. The conditions for the existence of n-periodic solutions are thoroughly described, and an explicit formula for these solutions is derived. Additionally, an example is provided, illustrating a case where the problem admits infinitely many solutions.

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Introduction

Differential equations with piecewise constant arguments (DEPCAs) arise from extending the theory of functional differential equations with continuous arguments to those with discontinuous arguments. This extension holds significant applied interest since DEPCA, as particular cases, include impulsive and loaded equations used in control theory, as well as models similar to those in biomedical research. In [1], Cooke and Wiener introduced a novel differential equation of alternately retarded and advanced types, demonstrating that all equations with piecewise constant delays share characteristics with those studied in [2]. These equations are closely related to impulsive and loaded equations, especially discrete argument difference equations, and exhibit structural similarities to certain “sequential-continuous“ disease dynamic models [3].

The DEPCAs are often classified as hybrid systems and can be used to model certain harmonic oscillators with almost periodic forcing [4], [5]. For a comprehensive survey on ordinary and partial differential equations with piecewise constant arguments, we refer the reader to [6], [7]. Functional differential equations with deviated arguments serve as mathematical models for systems where changes in state depend on either past history or future states. DEPCA also emerge when certain terms in a differential equation are replaced by their piecewise constant approximations. This approach finds applications in impulsive or loaded differential equations in control theory, as well as in the stabilization of systems with discrete (sampled) control [7], [8].

In [10], the delay differential equation

$$x'(t)+p_0(t)x(t)+\sum_{i=1}^{m}p_i(t)x(t-\tau_i)=0, \quad t\ge 0,$$

was considered, establishing approximations of delay differential equation solutions via those of delay differential equations with piecewise constant arguments.

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The study of DEPCA of mixed type was initiated by S. M. Shah and J. Wiener [11] in 1983, followed by G. Ladas [12] in 1988. They observed that changes in the sign of the argument deviation not only introduced interesting periodic properties but also added complexity to the asymptotic and oscillatory behavior of solutions. Naturally, this led to efforts to study the oscillatory and stability properties of DEPCA of mixed type with general deviation arguments.

Criteria for the existence of oscillatory solutions of DEPCA have been explored by numerous authors [2], [3], [4], [9], [13], [14], [15], [16]. It is also important to understand the additional conditions required for the stability of oscillatory solutions. These issues have been addressed in the broader field of differential equations. For example, A. R. Aftabizadeh and J. Wiener examined the oscillatory properties of solutions to first-order linear DEPCA of retarded and advanced types in [14].

$$x'(t) + a(t)x(t) + p(t)x([t]) = 0,$$

$$x'(t) + a(t)x(t) + q(t)x([t+1]) = 0,$$

where $a(t)$, $p(t)$ and $q(t)$ are continuous on $\mathbb{R}_+ = [0,\infty)$, and $[\cdot]$ is the greatest-integer function.

Authors in [17], [18], [19], [20] have streamlined the problem of n-periodic solvability to a set of n linear equations. Leveraging established properties of linear systems in algebra, they systematically delineated all conditions necessary for the existence of n-periodic solutions and furnished explicit formulas for solving these equations.

In this paper, we investigate the existence conditions for periodic solutions of a linear differential equation with piecewise constant arguments of mixed type of the form:

$$T'(t) = a(t)T(t) + b(t)T([t]) + c(t)T([t+1]), \quad t > 0, \tag{1}$$

subject to the initial condition:

$$T(0) = T_0, \tag{2}$$

where the functions $a(t)$, $b(t)$, and $c(t)$ are nonzero, n-periodic, and continuous on $\mathbb{R}_+ = [0,\infty)$, with n being a positive integer. We identify the conditions that either admit a unique periodic solution or an unlimited number of periodic solutions to the initial value issue (1)–(2).

Existence condition of a periodic solution

Let us define the concept of a solution for the DEPCA of mixed type as given by equations (1)–(2).

Definition. *A function $T(t)$ is called a solution of the DEPCA of mixed type (1)–(2) if the following conditions are satisfied:*

i) $T(t)$ is continuous on $\mathbb{R}_+$.

ii) The derivative $T'(t)$ exists at each point $[t] \in \mathbb{R}_+$, with the possible exception of the points $[t] \in \mathbb{R}_+$, where the one-side derivatives exist.

iii) Equation (1) is satisfied for T on each interval $(k, k+1)$, $k \in \mathbb{N}$, and it holds for the right derivative at the points k, $k \in \mathbb{Z}$.

The following assumption will be used throughout the paper. Let us denote:

$$M(i,t) = e^{\int_i^t a(m)dm}\left(1 + \int_i^t b(s)e^{-\int_i^s a(r)dr}ds\right), \quad t > i, \quad i = 0,1,2,...,$$

$$P(i,t) = e^{\int_i^t a(m)dm}\int_i^t c(s)e^{-\int_i^s a(r)dr}ds, \quad t > i, \quad i = 0,1,2,....$$

The following theorem provides a representation formula for the solutions of the DEPCA of mixed type (1)–(2) for $t > 0$.

Theorem 1. *Let $a(t), b(t),$ and $c(t)$ be continuous function on $[0, \infty)$, and suppose that*

$$P(i, i+1) \neq 1, \quad i = 0, 1, 2, \ldots.$$

Then, the solution of the DEPCA of mixed type (1) and (2) is well defined for all $t \geq 0$ and given by

$$T(t) = \frac{M(n,t)}{1 - P(n,t)} \left(\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1 - P(i,i+1)} \right) T_0, \quad t \in [n, n+1), \quad n = 0, 1, 2, \ldots. \tag{3}$$

Proof. Suppose that the function $T(t)$ is a solution of the DEPCA of mixed type (1) and (2) on the interval $k \leq t < k+1$, then we have

$$T'(t) = a(t)T(t) + b(t)T(k) + c(t)T(k+1). \tag{4}$$

The solution of(1) and (2) in $t \in [0, 1)$ has the form

$$T(t) = T_0 \frac{\exp\left[\int_0^t a(m)dm\right] (1 + \int_0^t b(s) e^{-\int_0^s a(r)dr} ds)}{1 - \exp\left[\int_0^t a(m)dm\right] \int_0^t c(s) \exp\left[-\int_0^s a(r)dr\right] ds} = T_0 \frac{M(0,t)}{1 - P(0,t)} \quad \text{for} \quad t \in [0, 1).$$

Then, for $t \to 1 - 0$, we have

$$T(1) = \lim_{t \to 1-0} T(t) = T_0 \frac{M(0,1)}{1 - P(0,1)}.$$

For $t \in [1, 2)$, the function $T(t)$ has the form

$$T(t) = T_1 \frac{\exp\left[\int_0^t a(m)dm\right] \left(1 - \int_1^t b(s) \exp\left[-\int_1^s a(r)dr\right] ds\right)}{1 - \exp\left[\int_1^t a(m)dm\right] \int_1^t c(s) \exp\left[-\int_1^s a(r)dr\right] ds}$$

or

$$T(t) = T_0 \frac{M(0,1)}{1 - P(0,1)} \frac{M(1,t)}{1 - P(1,t)} \quad \text{for} \quad t \in [1, 2).$$

Let the function

$$T(t) = T(0) \prod_{i=0}^{k-2} \Big(\frac{M(i,i+1)}{1 - P(i,i+1)} \Big) \frac{M(k-1,t)}{1 - P(k-1,t)} \quad \text{for} \quad t \in [k-1, k), \quad k = 3, 4, \ldots$$

be solution of (1) and (2) in $[k-1, k)$ and

$$T(k) = \lim_{t \to k-0} T(t) = T_0 \frac{M(k-1,k)}{1 - P(k-1,k)}.$$

Then, integrating (4), we get the function

$$T(t) = T(0) \prod_{i=0}^{k-1} \Big(\frac{M(i,i+1)}{1 - P(i,i+1)} \Big) \frac{M(k,t)}{1 - P(k,t)} \quad \text{for} \quad t \in [k, k+1), \quad k = 3, 4, \ldots.$$

Theorem 2. *Let $a(t)$, $b(t)$ and $c(t)$ be n-periodic continuous functions, and assume $P(i, i+1) \neq 1, \quad i = 0, 1, 2, \ldots$. Then the solution for the DEPCA of mixed type (1)–(2) is n-periodic if and only if*

$$\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1 - P(i,i+1)} = 1.$$

Proof. Let $T(t)$ be a n-periodic solution for the DEPCA of mixed type (1)–(2). Then $T(n) = T_0$. From (3), when $t = n$ we obtain the condition: $\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1-P(i,i+1)} = 1$.

Conversely, suppose that $\prod_{i=0}^{n-1} \frac{M(i,i+1)}{1-P(i,i+1)} = 1$. We will show that $T(n+t) = T(t)$ for all $t \in \mathbb{R}_+$. Let $t \in [k, k+1)$, which implies that $t+n \in [n+k, n+k+1)$, where k is an integer number. Then, we have

$$T(t+n) = \frac{M(n+k,t+n)}{1-P(n+k,t+n)} \left(\prod_{i=0}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} \right) T_0 \quad \text{for} \quad t+n \in [n+k, n+k+1).$$

Changing the variables $r' = \rho + n$, $s = s' + n$ in the integral, we get:

$$M(n+k,t+n) = e^{-\int_{n+k}^{t+n} a(r')dr'} \left(1 + \int_{n+k}^{t+n} b(s) e^{\int_{n+k}^{s} a(r)dr} ds \right),$$

$$P(n+k,t+n) = e^{-\int_{n+k}^{t+n} a(r')dr'} \int_{n+k}^{t+n} c(s) e^{\int_{n+k}^{s} a(r)dr} ds.$$

After the change of variables, we obtain:

$$M(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} (1 + \int_{k}^{t} b(s') e^{\int_{n+k}^{s'+n} a(r)dr} ds'),$$

$$P(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \int_{k}^{t} c(s') e^{\int_{n+k}^{s'+n} a(r)dr} ds'.$$

Then, by changing the variable $r = r' + n$, we obtain:

$$M(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \left(1 + \int_{k}^{t} b(s') e^{\int_{k}^{s'} a(r')dr'} ds' \right),$$

$$P(n+k,t+n) = e^{-\int_{k}^{t} a(\rho)d\rho} \int_{k}^{t} c(s') e^{\int_{k}^{s'} a(r')dr'} ds',$$

i.e. $\frac{M(n+k,t+n)}{1-P(n+k,t+n)} = \frac{M(k,t)}{1-P(k,t)}$ for all $t+n \in [n+k, n+k+1)$. By applying this equation, we present the subsequent calculations.

$$\prod_{i=0}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} = \prod_{i=0}^{n-1} \frac{M(i,i+1)}{1-P(i,i+1)} \prod_{i=n}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} = \prod_{i=n}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)}$$
$$= \prod_{j=0}^{k-1} \frac{M(j+n,j+n+1)}{1-P(j+n,j+n+1)} = \prod_{j=0}^{k-1} \frac{M(j,j+1)}{1-P(j,j+1)}.$$

Therefore

$$T(t+n) = T(0) \prod_{i=0}^{n+k-1} \frac{M(i,i+1)}{1-P(i,i+1)} \frac{M(n+k,t+n)}{1-P(n+k,t+n)}$$
$$= T(0) \prod_{i=0}^{k-1} \frac{M(i,i+1)}{1-P(i,i+1)} \frac{M(k,t)}{1-P(k,t)} = T(t) \quad \text{for} \quad t \in [k, k+1).$$

Hence, $T(n+t) = T(t)$ for $t \in [k, k+1)$.

Theorem 3. *Assuming $M(i,i+1) = 0$ and $P(i,i+1) = 1$, then the Cauchy problem for the DEPCA of mixed type (1) and (2) admits an unlimited number of solutions for $i \in \mathbb{N} \cup \{0\}$. Specifically, the issue has an infinite number of n-periodic solutions and a unique one-periodic solution, $n = 2, 3, \ldots$.*

The demonstration of the theorem closely resembles the proof provided for Theorem 2 (1) in [20].

Illustrative examples

We will introduce appropriate example in this section. The example will show the usefulness of our theoretical results.

Example 1. Suppose $a(t) = a \in \mathbb{R}$, $b(t) = \beta \cos 2\pi t$, and $c(t) = \gamma \cos 2\pi t$, where $a \neq 0$, $\beta = \frac{(a^2+4\pi^2)e^a}{a(1-e^a)}$, and $\gamma = \frac{a^2+4\pi^2}{a(e^a-1)}$. Then, we have

$$M(i,t) = e^{a(t-i)} + \frac{\beta}{a^2+4\pi^2}\Big(2\pi \sin 2\pi t - a\cos 2\pi t - e^{a(t-i)}(2\pi \sin 2\pi i - a \cos 2\pi i)\Big) \quad \text{for} \quad t > i,$$

$$P(i,t) = \frac{\gamma}{a^2+4\pi^2}\Big(2\pi \sin 2\pi t - a\cos 2\pi t - e^{a(t-i)}(2\pi \sin 2\pi i - a \cos 2\pi i)\Big) \quad \text{for} \quad t > i.$$

It can be readily verified that the DEPCA of mixed type (1) and (2) satisfies the conditions outlined in Theorem 3. The function

$$T_2(t) = \begin{cases} M(0,t)T_0 + P(0,t)T_1, & t \in [0,1), \\ M(1,t)T_1 + P(1,t)T_0, & t \in [1,2), \end{cases}$$

is a 2-periodic solution of the DEPCA of mixed type (1) and (2), where T_1 is an arbitrary constant. As a result, the DEPCA of mixed type (1) and (2) admits infinitely many 2-periodic solutions. This result is further demonstrated by the simulations presented in Figs. 1 and 2.

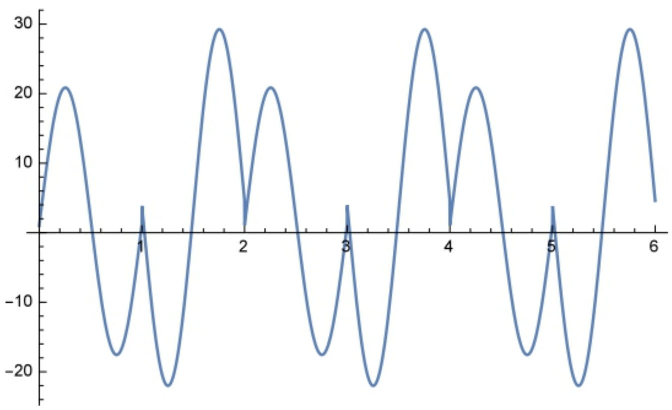

Рис. 4: The 2-periodic solution $T_2(t)$ for the DEPCA of mixed type with the parameters $T_1 = 4$ and $a = \frac{1}{6}$.

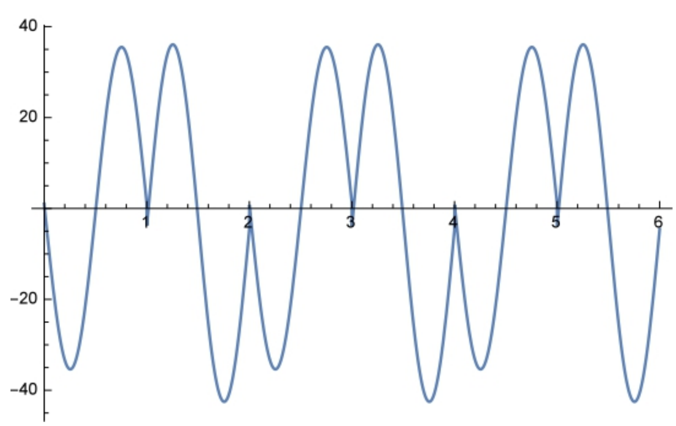

Рис. 5: The 2-periodic solution $T_2(t)$ for the DEPCA of mixed type with the parameters $T_1 = -4$ and $a = \frac{1}{6}$.

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