LOCAL DERIVATIONS ON ALGEBRA OF MEASURABLE OPERATORS AFFILIATED WITH COMMUTATIVE REAL VON NEUMANN ALGEBRAS

RAKHIMOV ABDUGAFUR ABDUMADJIDOVICH*
NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN
rakhimov@ktu.edu.tr
KARIMOV ULUGBEK SHOKIROVICH
NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN
karimovulugbe07@gmail.com

ABSTRACT. Local derivations on algebra of measurable operators affiliated with commutative real von Neumann algebras are considered. A necessary and sufficient condition for the existence of a non-trivial (non-inner) derivation (local derivation) on an algebra of measurable operators affiliated with commutative real von Neumann algebra has been found. It turns out that such a condition is the not-atomicity of the lattice of projectors of the algebra.

MSC (2020): 46L10; 17A30; 17B30; 17B40; 17B56.

Key words: von Neumann algebras, derivations, local derivations.

Introduction

As is well known, the theory of derivations and local derivations on bounded and unbounded operator algebras, in particular, the algebras of all measurable (local and τ- measurable) operators on a Hilbert space with respect to von Neumann algebras, has been fairly well studied. Key achievements in this area are associated with names such as Sh.A.Ayupov, K.K.Kudaybergenov, V.I.Chilin, A.F.Ber, F.A.Sukochev and others. However, the real analogue of some results has not yet been described, i.e., local derivations on the algebras of all measurable (local and τ- measurable) operators with respect to real von Neumann algebras, have not been considered. In this article we will look at commutative real von Neumann algebra with a faithful normal semifinite trace.

Using the result of V.I.Chilin, A.F.Ber and F.A.Sukochev obtained for regular algebras, we obtained a real analogue of the results Sh.A.Ayupov, K.K.Kudaybergenov obtained for commutative (complex) von Neumann algebra. Namely, it is proved that in commutative real von Neumann algebra A with a faithful normal semifinite trace τ, the following conditions are equivalent:

- the lattice $P(\mathcal{A})$ of projections in $\mathcal{A}$ is not atomic;
- the algebra $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) admits a non-inner derivation;
- the algebra $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) admits a non-zero local derivation;
- the algebra $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) admits a local derivation which is not a derivation,

where $S(\mathcal{A})$ (resp. $S(\mathcal{A},\tau)$) is the algebra of all measurable (resp. τ-measurable) operators with respect to A.

Preliminaries

Let A be an algebra. A linear operator $D : A \to A$ is called a *derivation* if it satisfies the identity $D(xy) = D(x)y + xD(y)$, for all $x, y \in A$. Each element $a \in A$ defines a derivation D_a on A given as $D_a(x) = ax - xa$, $x \in A$. Such derivations D_a are said to be *inner derivations.* A linear map $\delta : A \to A$ is called a *local derivation*, if for every $x \in A$, there exists a derivation $\delta_x : A \to A$ such that $\delta(x) = \delta_x(x)$.

Let $B(H)$ be the *-algebra of all bounded linear operators on a Hilbert space H and let $\mathbb{1}$ be the identity operator on H. A W*-algebra* is a weakly closed *-subalgebra of $B(H)$, containing $\mathbb{1}$. A *real* W*-algebra* is

Received: 25.02.2026, **Accepted:** 15.03.2026
*Corresponding autor

a weakly closed real *-subalgebra $R \subset B(H)$ with the identity which satisfies the equality $R \cap iR = \{0\}$. W*-algebras are also known as *von Neumann algebras*.

Let M' is a set which consists of all bounded linear operators on the Hilbert space H, that commute with every element of M. That is $M' = \{y \in B(H) : xy = yx, \forall x \in M\}$. M' is called the commutant of M. The set of all projections in M we denote by $P(M)$. If $u(\mathcal{D}) \subseteq \mathcal{D}$ for any unitary $u \in M'$ then a linear subspace $\mathcal{D}$ in H is said to be affiliated with M (denoted as $\mathcal{D}\eta M$). A linear operator $x : \mathcal{D}(x) \to H$ is said to be affiliated with M (denoted as $x\eta M$) if $\mathcal{D}(x)\eta M$ and $u(x(\xi)) = x(u(\xi))$ for all $\xi \in \mathcal{D}(x)$ and for every unitary $u \in M'$, which domain is $\mathcal{D}(x)$ of x is a linear subspace of H. If $\mathcal{D}\eta M$ and there exists a sequence of projections $\{p_n\}_{n=1}^{\infty}$ in $P(M)$ such that $p_n \uparrow \mathbb{1}$, $p_n(H) \subseteq \mathcal{D}$ and $p_n^* = \mathbb{1} - p_n$ is finite in M for all $n \in \mathbb{N}$, then a linear subspace $\mathcal{D}$ in H is said to be strongly dense in H with respect to M. If $x\eta M$ and $\mathcal{D}(x)$ is strongly dense in H, then a closed linear operator x in the Hilbert space H is said to be measurable with respect to M. Let us denote by $S(M)$ all linear operators measurable with respect to M on H.

For a von Neumann algebra M, we typically define a trace on its set of positive elements, denoted as M_+. A *trace* on M is a function $\tau : M_+ \to [0, \infty]$ that satisfies the following three conditions for all $x, y \in M_+$ and scalar $\lambda \geq 0$: i) $\tau(x+y) = \tau(x)+\tau(y)$, ii) $\tau(\lambda x) = \lambda\tau(x)$ (with the convention $0\cdot\infty = 0$) and iii) $\tau(xx^*) = \tau(x^*x)$ for every $x \in M$. Traces are rarely "just"traces; they are usually categorized by how well-behaved they are:

- *faithful*: $\tau(x) = 0$ implies $x = 0$. This means the trace doesn't "miss"any non-zero operators.
- *normal*: If a directed increasing net of operators x_i converges to x, then $\tau(x_i)$ converges to $\tau(x)$. This is a continuity requirement essential for the von Neumann setting.
- *finite*: $\tau(\mathbb{1}) < \infty$. If $\tau(\mathbb{1}) = 1$, it is often called a *tracial state*.
- *semifinite*: For every non-zero $x \in M_+$, there exists a non-zero $y \leq x$ such that $\tau(y) < \infty$.

Let M be a von Neumann algebra with semifinite trace τ. If $x\eta M$ and $\mathcal{D}(x)$ is τ-dense in H, i.e. for any $\epsilon > 0$ there is a projection $p \in M$ such that $p(H) \subseteq \mathcal{D}(x)$ and $\tau(p^{\perp}) < \epsilon$, then a closed linear operator x is said to be τ-measurable with respect to M. Let $S(M, \tau)$ is a set of all τ-measurable operators on H with respect to M. It is worth mentioning that $S(M, \tau) = S(M)$ if the trace τ is finite. All the above concepts are similarly defined for real von Neumann algebras.

Main part.

Below we discuss the problem of existence of local derivations which are not derivations on the algebras $S(\mathcal{R})$ and $S(\mathcal{R}, \tau)$ in the case where the real von Neumann algebra $\mathcal{R}$ is commutative. In many places, we use the concepts and results of the work [3], which considers a regular algebra over an arbitrary field K, in particular $K = \mathbb{C}$ or $K = \mathbb{R}$.

Let A be a commutative algebra with the unit $\mathbb{1}$ over the field $\mathbb{C}$ or $\mathbb{R}$. We denote by ∇ the set

$$\nabla = \{e \in A : e^2 = e\}$$

of all idempotents in A. For $e, f \in \nabla$ we set $e \leq f$ if $ef = e$. With respect to this partial order, the lattice operations

$$e \vee f = e + f - ef, \qquad e \wedge f = ef,$$

and the complement $e^{\perp} = \mathbb{1} - e$, the set ∇ forms a Boolean algebra.

A non-zero element q from the Boolean algebra ∇ is called an *atom* if

$$0 \neq e \leq q, \ e \in \nabla \quad \Rightarrow \quad e = q.$$

If for any non-zero $e \in \nabla$ there exists an atom q such that $q \leq e$, then the Boolean algebra ∇ is said to be *atomic*.

An algebra A is called *regular* (in the sense of von Neumann) if for any $a \in A$ there exists $b \in A$ such that

$$a = aba.$$

Further, we shall always assume that A is a unital commutative regular algebra over $\mathbb{C}$ or $\mathbb{R}$, and that ∇ is the Boolean algebra of all its idempotents. In this case, given any element $a \in A$, there exists an idempotent

$e \in \nabla$ such that $ea = a$, and if $ga = a$, $g \in \nabla$, then $e \leq g$. This idempotent is called the *support* of a and denoted by $s(a)$.

Suppose that μ is a strictly positive countably additive finite measure on the Boolean algebra ∇ of idempotents from A and consider the metric

$$\rho(a, b) = \mu(s(a - b)), \qquad a, b \in A.$$

From now on we shall assume that (A, ρ) is a complete metric space (cf. [2,3]).

Example 3.1 The most important example of a complete commutative regular algebra (A, ρ) is the algebra $A = L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$ of all (classes of equivalence of) measurable complex (or real) functions on a measure space (Ω, Σ, μ), where μ is a finite countably additive measure on Σ, and

$$\rho(a, b) = \mu(s(a - b)) = \mu\big(\{\omega \in \Omega : \ a(\omega) \neq b(\omega)\}\big)$$

(see for details [1, Lemma] and [3, Example 2.5]).

Remark 3.2 If (Ω, Σ, μ) is a general localizable measure space, i.e. the (not finite in general) measure μ has the finite sum property, then the algebra $L^0(\Omega, \Sigma, \mu)$ is a unital regular algebra, but $\rho(a, b) = \mu(s(a - b))$ is not a metric in general. But one can represent Ω as a union of pairwise disjoint measurable sets with finite measures and thus this algebra is a direct sum of commutative regular complete metrizable algebras from the above example.

Following [3] we call an element $a \in A$ *finitely valued* (respectively, *countably valued*) if

$$a = \sum_{k=1}^{n} \alpha_k e_k,$$

where $\alpha_k \in \mathbb{C}$ or $\alpha_k \in \mathbb{R}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, \ldots, n$, $n \in \mathbb{N}$ (respectively,

$$a = \sum_{k=1}^{\omega} \alpha_k e_k,$$

where $\alpha_k \in \mathbb{C}$ or $\alpha_k \in \mathbb{R}$, $e_k \in \nabla$, $e_k e_j = 0$, $k \neq j$, $k, j = 1, \ldots, \omega$, and ω is a natural number or ∞; in the latter case the convergence of the series is understood with respect to the metric ρ).

We denote by $K(\nabla)$ (respectively, $K_c(\nabla)$) the set of all finitely valued (respectively, countably valued) elements in A. It is known that

$$\nabla \subset K(\nabla) \subset K_c(\nabla),$$

both $K(\nabla)$ and $K_c(\nabla)$ are regular subalgebras in A, and moreover the closure of $K(\nabla)$ in (A, ρ) coincides with $K_c(\nabla)$ (see [3], Proposition 2.8).

Now let D be a derivation on the given regular commutative algebra A. By [3], Proposition 2.3 we have that

$$s(D(a)) \leq s(a), \qquad \forall a \in A,$$

and $D|_\nabla = 0$. Therefore by the definition, each local derivation Δ on A satisfies the following two conditions:

$$s(\Delta(a)) \leq s(a), \qquad \forall a \in A, \quad (1)$$

$$\Delta|_\nabla \equiv 0. \quad (2)$$

This means that (1) and (2) are necessary conditions for a linear operator Δ to be a local derivation on the algebra A. We are going to show that these two conditions are in fact also sufficient.

First we recall some further notions from the paper [3]. Let B be a unital subalgebra in the algebra A. An element $a \in A$ is called:

• *algebraic with respect to* B, if there exists a polynomial $p \in B[x]$ (i.e. a polynomial in x with coefficients from B) such that $p(a) = 0$;

• *integral with respect to* B, if there exists a unitary polynomial $p \in B[x]$ (i.e. the coefficient of the largest degree of x in $p(x)$ is equal to $\mathbb{1} \in B$) such that $p(a) = 0$;

• *transcendental with respect to* B, if a is not algebraic with respect to B;

• *weakly transcendental with respect to* B, if $a \neq 0$ and for any non-zero idempotent $e \leq s(a)$ the element ea is not integral with respect to B.

In the paper [2] the following two lemmas are proved using the results of paper [3]. Note that the fields K do not play a role in the proofs of these lemmas.

Lemma 3.3 [2, Lemma 3.3] Given any element $a \in A$ there exists an idempotent $e \in \nabla$ such that

(i) ea is integral with respect to $K_c(\nabla)$, moreover in this case $ea \in K_c(\nabla)$;

(ii) $e^{\perp}a$ is weakly transcendental with respect to $K_c(\nabla)$, if $e \neq \mathbb{1}$.

The following lemma is the crucial step for the proof of the main results in this section.

Lemma 3.4 [2, Lemma 3.4] Each linear operator on the algebra A satisfying the conditions (1) and (2) is a local derivation on A.

The following is the main result concerning the existence of local derivations on commutative regular algebras.

Theorem 3.5 Let A be a unital commutative regular algebra over $\mathbb{C}$ or $\mathbb{R}$ and let μ be a finite strictly positive countably additive measure on the Boolean algebra ∇ of all idempotents of A. Suppose that A is complete in the metric

$$\rho(a, b) = \mu(s(a - b)), \qquad a, b \in A.$$

Then the following conditions are equivalent:

(i) $K_c(\nabla) \neq A$;

(ii) the algebra A admits a non-zero derivation;

(iii) the algebra A admits a non-zero local derivation;

(iv) the algebra A admits a local derivation which is not a derivation.

The implications $(i) \Leftrightarrow (ii)$ are proved in [3], Theorem 3.2. The implication $(ii) \Rightarrow (iii)$ is trivial because any derivation is a local derivation.

To prove $(iii) \Rightarrow (iv)$ we need the following lemma.

Lemma 3.6 Let A be a commutative regular algebra over the field K, where $K = \mathbb{R}$ or $K = \mathbb{C}$. If D is a derivation on A, then D^2 is a derivation if and only if $D = 0$.

Proof. For $K = \mathbb{C}$ the lemma is proved in [2, Lemma 3.6]. Let us prove it for $K = \mathbb{R}$. Let R be a commutative regular real algebra and $A = R + iR$. As is known, the regularity of the ring is preserved during its complexification. Therefore A is a commutative regular algebra. Let $D : R \to R$ be a derivation on R. Let us extend a derivation D on A as $\overline{D}(x + iy) = D(x) + iD(y)$. If $D = 0$ then $\overline{D} = 0$ and by [2, Lemma 3.6] we have $\overline{D}^2$ is a derivation. Since

$$\overline{D}^2(x + iy) = \overline{D}(D(x) + iD(y)) = D^2(x) + iD^2(y) = \overline{D^2}(x + iy)$$

then $\overline{D}^2 = \overline{D^2}$. Hence $\overline{D^2}$ is also a derivation. Therefore D^2 is a derivation.

Now let D^2 be a derivation. Then $\overline{D^2}$ is also a derivation, hence $\overline{D}^2$ is a derivation. By [2, Lemma 3.6] we have $\overline{D} = 0$, hence we obtain $D = 0$. Proof is complete.

Now $(iii) \Rightarrow (iv)$ follows. Let $a \in A \setminus K_c(\nabla)$. By Lemma 3.3 there exists an idempotent $e \in \nabla$ such that $ea \in K_c(\nabla)$ and $b = e^{\perp}a$ is weakly transcendental with respect to $K_c(\nabla)$. By [3, Proposition 3.7, Theorem 3.1] there exists a derivation D on A such that $D(b) = b$. Consider $\Delta = D^2$. Then Δ satisfies (1) and (2), hence by Lemma 3.4 Δ is a local derivation. Moreover,

$$\Delta(b) = D(D(b)) = D(b) = b \neq 0,$$

so $\Delta \neq 0$. By Lemma 3.6 Δ is not a derivation.

Finally, $(iv) \Rightarrow (i)$ follows from [3, Theorem 3.2].

Corollary 3.7 Let (Ω, Σ, μ) be a finite measure space and let $L^0(\Omega) = L^0(\Omega, \Sigma, \mu)$. The following conditions are equivalent:

(i) the Boolean algebra of all idempotents from $L^0(\Omega)$ is not atomic;

(ii) $L^0(\Omega)$ admits a non-zero derivation;

(iii) $L^0(\Omega)$ admits a non-zero local derivation;

(iv) $L^0(\Omega)$ admits a local derivation which is not a derivation.

Proof. This follows easily from Theorem 1.3, Example 3.1, [2, Corollary 3.7] and [3, Corollary 3.7].

Theorem 3.8 Let $\mathcal{A}$ be a commutative complex or real von Neumann algebra with a faithful normal semifinite trace τ. The following conditions are equivalent:

(i) the lattice $P(\mathcal{A})$ of projections in $\mathcal{A}$ is not atomic;

(ii) the algebra $S(\mathcal{A})$ (respectively $S(\mathcal{A}, \tau)$) admits a non-inner derivation;

(iii) the algebra $S(\mathcal{A})$ (respectively $S(\mathcal{A}, \tau)$) admits a non-zero local derivation;

(iv) the algebra $S(\mathcal{A})$ (respectively $S(\mathcal{A}, \tau)$) admits a local derivation which is not a derivation.

In the complex case, this theorem was proved in [2, Theorem 3.8]. In the real case, it is proved by similar arguments using the above results.

Remark 3.9 For general (non-commutative) complex or real von Neumann algebras the above conditions are not equivalent. Some implications remain valid, but others fail.

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Cite this article

Rakhimov A.A., Karimov U.Sh. *Local derivations on algebra of measurable operators affiliated with commutative real von Neumann algebras*, **Acta NUUz**, 2026, No 2/1, pp. 99-103.