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BOUNDARY VALUE PROBLEM FOR THE GELLERSTEDT EQUATION IN AN UNBOUNDED DOMAIN

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ABSTRACT. The Tricomi boundary value problem for the Gellerstedt equation is investigated in a domain where the elliptic part is the first quadrant of the plane. The unique solvability of the problem under consideration is established using the method of integral equations. The problem is equivalently reduced to solving a singular integral equation with a Cauchy kernel. By regularizing this equation using the Carleman-Vekua method, an explicit solution is obtained.

MSC (2020): 35M10; 35M12.

Key words: mixed-type equation, unbounded domain, unique solvability, singular integral equation, index of an equation.

Introduction.

The first fundamental studies of equations of mixed elliptic-hyperbolic type were done by F. Tricomi [1]. Following Tricomi's work on mixed-type equations, S. Gellerstedt [2] investigated boundary value problems in which the values of the desired solution in the hyperbolic part of the domain under consideration are prescribed on two internal characteristics of the equation.

The Gellerstedt problem has important applications in the field of transonic gas dynamics. In [3], the Tricomi boundary value problem for the Tricomi equation was studied in a half-strip and a quarter-plane. The Frankl boundary value problem for the Lavrent'ev equation in an unbounded domain was considered in [4]. Works [5-7] are devoted to the study of nonlocal problems for mixed-type equations.

In the elliptic part of the mixed domain, the Tricomi problem for the Gellerstedt equation in an unbounded domain was investigated [8] by employing solutions of the generalized Neumann problem.

In the present paper, we investigate the Tricomi boundary value problem for an equation of mixed elliptic-hyperbolic type in an unbounded domain by utilizing solutions of the Dirichlet problem.

Problem statement.

Consider the following mixed-type equation

$$\operatorname{sgn} y|y|^m u_{xx} + u_{yy} = 0, \qquad m > 0, \tag{1}$$

in the domain $D : (x > 0)$, bounded by the y axis for $y > 0$ and by the following characteristic of equation (1)

$$\Gamma : \quad x - \frac{2}{m+2}(-y)^{\frac{m+2}{2}} = 0.$$

Introduce the notation: D^+ and D^- are the elliptic and hyperbolic parts of the mixed domain D , respectively; I is the semi-infinite interval $0 < x < +\infty$ on the line $y = 0$. For $y > 0$, let $R = \sqrt{x^2 + \frac{4}{(m+2)^2} y^{m+2}}$.

Problem T. Find a function $u = u(x, y)$, which possesses the following properties:

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1) $u(x,y)\in C(\overline{D})\cap C^1(\overline{D}\setminus I)\cap C^2(D\setminus I)$ and satisfies equation (1) in $D^+\cup D^-$;

2)
$$\lim_{R\to\infty} u(x,y)=0,\qquad x\geq 0,\quad y\geq 0; \tag{2}$$

3) $u(x,y)$ satisfies the boundary conditions

$$u(0,y)=\varphi(y),\qquad y\geq 0, \tag{3}$$

$$u(x,y)|_{\Gamma}=\psi(x),\quad x\in[0,\infty), \tag{4}$$

and the conjugation condition
$$\lim_{y\to-0} u_y=\lim_{y\to+0} u_y,\qquad x\in I. \tag{5}$$

Here, $\psi(x)$ and $\varphi(y)$ are given functions satisfying the Holder conditions of orders δ_1 and δ_2, respectively, on $[0,\infty)$.

Theorem. Let $y^{\frac{3m}{4}}\varphi(y)\in L(0,\infty)$, $\varphi(\infty)=0$, $\psi(\infty)=0$. Then problem T is uniquely solvable.

Proof. It is known [7] that the solution to the Dirichlet problem for equation (1) satisfying conditions (2), (3) and condition $u(x,0)=\tau(x)$, $x\in[0,\infty)$ can be represented in the form

$$u(x,y)=k_2y\int\limits_0^{\infty}\tau(t)\left(\left[(t-x)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}-\left[(t+x)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right)dt$$

$$+\frac{2}{m+2}y^{\frac{1}{2}}\int\limits_0^{\infty}t^{\frac{2m+1}{2}}\varphi(t)dt\int\limits_0^{\infty}se^{-sx}J_{\frac{1-2\beta}{2}}\left(\frac{2st^{\frac{m+2}{2}}}{m+2}\right)J_{\frac{1-2\beta}{2}}\left(\frac{2sy^{\frac{m+2}{2}}}{m+2}\right)ds, \tag{6}$$

where
$$k_2=\frac{1}{4\pi}\left(\frac{4}{m+2}\right)^{2-2\beta}\frac{\Gamma^2(\beta)}{\Gamma(2-2\beta)},\qquad \beta=\frac{m}{2(m+2)}.$$

Differentiating equality (6) with respect to y, we obtain

$$\frac{\partial u}{\partial y}=k_2\int\limits_0^{\infty}\tau(t)\frac{\partial}{\partial y}y\left(\left[(t-x)^2+\frac{4y^{m+2}}{(m+2)^2}\right]^{\beta-1}-\left[(t+x)^2+\frac{4y^{m+2}}{(m+2)^2}\right]^{\beta-1}\right)dt+\frac{\partial\Phi(x,y)}{\partial y}, \tag{7}$$

where
$$\Phi(x,y)=\frac{2}{m+2}y^{\frac{1}{2}}\int\limits_0^{\infty}t^{\frac{2m+1}{2}}\varphi(t)dt\int\limits_0^{\infty}se^{-sx}J_{\frac{1-2\beta}{2}}\left(\frac{2st^{\frac{m+2}{2}}}{m+2}\right)J_{\frac{1-2\beta}{2}}\left(\frac{2sy^{\frac{m+2}{2}}}{m+2}\right)ds.$$

From (7) it is easy to show that

$$\frac{\partial}{\partial y}y\left(\left[(x-t)^2+\frac{4y^{m+2}}{(m+2)^2}\right]^{\beta-1}-\left[(x+t)^2+\frac{4y^{m+2}}{(m+2)^2}\right]^{\beta-1}\right)$$

$$=\frac{m+2}{2}\frac{\partial}{\partial t}\left((x-t)\left[(x-t)^2+\frac{4y^{m+2}}{(m+2)^2}\right]^{\beta-1}+(x+t)\left[(x+t)^2+\frac{4y^{m+2}}{(m+2)^2}\right]^{\beta-1}\right). \tag{8}$$

Taking identity (8) into account, we rewrite equality (7) in the form

$$\frac{\partial u}{\partial y}=k_2\frac{m+2}{2}\int\limits_0^{\infty}\tau(t)\frac{\partial}{\partial t}\left((x-t)\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right.$$

$$\left.+(x+t)\left[(x+t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right)dt+\frac{\partial\Phi(x,y)}{\partial y}. \tag{9}$$

In the integral on the right-hand side of (9) performing integration by parts, without loss of generality assuming $\tau(0)=0, \tau(\infty)=0$, we obtain

$$\frac{\partial u}{\partial y}=-k_2\frac{m+2}{2}\int\limits_0^\infty \tau'(t)\left((x-t)\left[(x-t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right.$$

$$\left.+(x+t)\left[(x+t)^2+\frac{4}{(m+2)^2}y^{m+2}\right]^{\beta-1}\right)dt+\frac{\partial\Phi(x,y)}{\partial y}. \tag{10}$$

Passing to the limit as $y\to+0$ in (10), we have

$$\nu(x)=-k_2\frac{m+2}{2}\int\limits_0^\infty \tau'(t)\left((x-t)|x-t|^{2\beta-2}+(x+t)^{2\beta-1}\right)dt+\Phi_0(x), \quad x\in(0,\infty), \tag{11}$$

where

$$\Phi_0(x)=\lim_{y\to+0}\frac{\partial\Phi(x,y)}{\partial y}=\frac{2}{(m+2)^{\frac{1-2\beta}{2}}\Gamma\left(\frac{1}{2}-\beta\right)}\int\limits_0^\infty \varphi^+(t)t^{\frac{2m+1}{2}}dt\int\limits_0^\infty s^{\frac{3-2\beta}{2}}e^{-sx}J_{\frac{1-2\beta}{2}}\left(\frac{2st^{\frac{m+2}{2}}}{m+2}\right)ds.$$

Formula (11) gives the first functional relation between $\tau(x)$ and $\nu(x)$, extended to I from the elliptic part D^+ of the domain D.

Using the solution of the Cauchy problem for equation (1) in the domain D^- [7]

$$u(x,y)=\gamma_1\int\limits_0^1 \tau\left[x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}(2t-1)\right]t^{\beta-1}(1-t)^{\beta-1}dt$$

$$+\left(\frac{4}{m+2}\right)^{1-2\beta}\gamma_2 y\int\limits_0^1 \nu\left[x+\frac{2}{m+2}(-y)^{\frac{m+2}{2}}(2t-1)\right]t^{-\beta}(1-t)^{-\beta}dt,$$

where $\gamma_1=\dfrac{\Gamma(2\beta)}{\Gamma^2(\beta)}$, $\gamma_2=\dfrac{1}{2}\left(\dfrac{4}{m+2}\right)^{2\beta}\dfrac{\Gamma(1-2\beta)}{\Gamma^2(1-\beta)}$, by virtue of condition (4) we have

$$\psi(x)=\gamma_1\Gamma(\beta)x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}\tau(x)-\gamma_2\Gamma(1-\beta)D_{0x}^{\beta-1}x^{-\beta}\nu(x), \tag{12}$$

where $D_{0x}^{-\beta}$ is the Riemann–Liouville fractional integration operator [6,8].

Applying the operator

$$D_{0x}^{1-\beta}f(x)=\frac{d}{dx}D_{0x}^{-\beta}f(x)=\frac{1}{\Gamma(\beta)}\frac{d}{dx}\int\limits_0^x\frac{f(t)dt}{(x-t)^{1-\beta}}$$

to both sides of equality (12) and taking into account that $D_{0x}^{1-\beta}D_{0x}^{\beta-1}f\equiv D^0 f$, we obtain

$$\gamma_2\Gamma(1-\beta)x^{-\beta}\nu(x)=\gamma_1\Gamma(\beta)D_{0x}^{1-\beta}x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}\tau(x)-D_{0x}^{1-\beta}\psi(x). \tag{13}$$

It is easy to show that

$$D_{0x}^{1-\beta}x^{1-2\beta}D_{0x}^{-\beta}x^{\beta-1}\tau(x)=x^{-\beta}D_{0x}^{1-2\beta}\tau(x).$$

Then from equality (13) we have

$$\nu(x)=\gamma D_{0x}^{1-2\beta}\tau(x)+\Psi_1(x), \tag{14}$$

where $\gamma=\frac{\gamma_1\Gamma(\beta)}{\gamma_2\Gamma(1-\beta)}$, $\quad \Psi_1(x)=-\frac{x^\beta D_{0x}^{1-\beta}\psi(x)}{\gamma_2\Gamma(1-\beta)}$.

Equality (14) is the second functional relation between $\tau(x)$ and $\nu(x)$, extended to I from the hyperbolic domain D^- of D.

According to (5) eliminating $\nu(x)$ from equalities (11) and (14), we obtain

$$\gamma D_{0x}^{1-2\beta}\tau(x)+\Psi_1(x)$$

$$=-k_2\frac{m+2}{2}\int\limits_0^\infty \tau'(t)\left((x-t)|x-t|^{2\beta-2}+(x+t)^{2\beta-1}\right)dt+\Phi_0(x),\quad x\in(0,\infty). \tag{15}$$

Applying the operator $\Gamma(1-2\beta)D_{0x}^{2\beta-1}$ to both sides of equality (15), we get

$$\gamma\Gamma(1-2\beta)\tau(x)+\Gamma(1-2\beta)D_{0x}^{2\beta-1}\psi_1(x)$$

$$=-k_2\frac{m+2}{2}\Gamma(1-2\beta)D_{0x}^{2\beta-1}\int\limits_0^\infty \tau'(t)\left((x-t)|x-t|^{2\beta-2}+(x+t)^{2\beta-1}\right)dt$$

$$+\Gamma(1-2\beta)D_{0x}^{2\beta-1}\Phi_0(x),\quad x\in(0,\infty). \tag{16}$$

In equation (16), it is easy to show that

$$\Gamma(1-2\beta)D_{0x}^{2\beta-1}\int\limits_0^\infty \tau'(t)(x-t)|x-t|^{2\beta-2}dt=\frac{\pi(1-\cos 2\pi\beta)\tau(x)}{\sin 2\pi\beta}-\int\limits_0^\infty\left(\frac{t}{x}\right)^{2\beta-1}\frac{\tau(t)dt}{t-x}, \tag{17}$$

$$\Gamma(1-2\beta)D_{0x}^{2\beta-1}\int\limits_0^\infty \tau'(t)(x+t)^{2\beta-1}dt=\int\limits_0^\infty\left(\frac{t}{x}\right)^{2\beta-1}\frac{\tau(t)dt}{t+x}. \tag{18}$$

Substituting (17) and (18) into (16), we obtain

$$\left(\gamma\Gamma(1-2\beta)+k_2\frac{(m+2)\pi}{2\sin 2\pi\beta}(1-\cos 2\pi\beta)\right)\tau(x)$$

$$-k_2\frac{m+2}{2}\int\limits_0^\infty\left(\frac{x}{t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{t+x}\right)\tau(t)dt=\Phi_1(x),\qquad x\in(0,\infty), \tag{19}$$

where $\Phi_1(x)=\Gamma(1-2\beta)D_{0x}^{2\beta-1}(\Phi_0(x)-\Psi_1(x))$.

After straightforward calculations, equality (19) takes the following form

$$\tau(x)-\lambda\int\limits_0^\infty\left(\frac{x}{t}\right)^{1-2\beta}\left(\frac{1}{t-x}-\frac{1}{t+x}\right)\tau(t)dt=\Phi_1(x),\quad x\in(0,\infty), \tag{20}$$

where $\lambda=\dfrac{\cos\pi\beta}{\pi(1+\sin\pi\beta)}$.

In equation (20), by letting $x^{2\beta-1}\tau(x)=\rho(x)$, $x^{2\beta-1}\Phi_1(x)=\Phi_2(x)$, we get

$$\rho(x)-\lambda\int\limits_0^\infty\left(\frac{1}{t-x}-\frac{1}{t+x}\right)\rho(t)dt=\Phi_2(x),\quad x\in(0,\infty). \tag{21}$$

In equation (21), we make the change of variables $t^2=\dfrac{s}{1-s}$, $x^2=\dfrac{\xi}{1-\xi}$, then it takes the form

$$\rho\left(\sqrt{\frac{\xi}{1-\xi}}\right)-\lambda\int\limits_0^1\rho\left(\sqrt{\frac{s}{1-s}}\right)\frac{\sqrt{\xi}\sqrt{1-\xi}ds}{\sqrt{s}\sqrt{1-s}(s-\xi)}=\Phi_2\left(\sqrt{\frac{\xi}{1-\xi}}\right). \tag{22}$$

Multiplying both sides of equation (22) by $\dfrac{1}{\sqrt{\xi(1-\xi)}}$ and denoting $\dfrac{1}{\sqrt{\xi(1-\xi)}}\rho\left(\sqrt{\frac{\xi}{1-\xi}}\right) = \mu(\xi)$, $\dfrac{1}{\sqrt{\xi(1-\xi)}}\Phi_2\left(\sqrt{\frac{\xi}{1-\xi}}\right) = \Phi_3(\xi)$, we obtain the singular integral equation with the Cauchy kernel

$$\mu(\xi) - \lambda\int\limits_0^1 \frac{\mu(s)ds}{s-\xi} = \Phi_3(\xi), \quad \xi \in (0,1). \tag{23}$$

We seek the solution of equation (23) in the class of $h(1)$ functions. $\mu(\xi) \in H(0,1)$ bounded as $\xi \to 1$ and having an infinity of order less than $1-2\beta$ at the point $\xi = 0$. The index χ of equation (23) in this class is equal to zero.

Using the result of work [9], we write the explicit form of the solution to equation (23)

$$\mu(\xi) = \frac{1+\sin(\beta\pi)}{2}\Phi_3(\xi) + \frac{\cos(\pi\beta)}{2\pi}\left(\frac{\xi}{1-\xi}\right)^{-\frac{1}{4}(1-2\beta)}\int\limits_0^1\left(\frac{s}{1-s}\right)^{\frac{1}{4}(1-2\beta)}\frac{\Phi_3(s)ds}{s-\xi}.$$

As a result, returning to the original variables and functions, we have

$$\tau(x) = \frac{1+\sin(\beta\pi)}{2}\Phi_1(x) + \frac{\cos(\pi\beta)}{2\pi}\int\limits_0^\infty\left(\frac{x}{t}\right)^{\frac{1}{2}(1-2\beta)}\left(\frac{1}{t-x} - \frac{1}{t+x}\right)\Phi_1(t)dt.$$

The proof of the theorem is finished.

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