Verification of the priority and controllability of analytical solutions determined for active parts in the gravitational field of a spheroidal planet.

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Abstract. This paper presents analytical solutions to the problem of optimal trajectory design for a point mass representing the center of mass of a spacecraft during intermediate-thrust motion in the gravitational field of an axisymmetric spheroidal planet. The mathematical model incorporates the perturbative influence of the second zonal harmonic J_2, accounting for the oblateness of the attracting body. The variational problem is formulated within the framework of optimal control theory, and a class of particular analytical solutions is obtained using the Levi-Civita regularization method. Special attention is devoted to the qualitative analysis of the derived analytical solutions. The obtained program motions are examined with respect to stability and controllability. It is shown that certain solutions exhibit regions of dynamic instability in the neighborhood of the nominal trajectory. The controllability properties of the system are analyzed in the linear approximation, and conditions under which stabilization is achievable are identified. A linear feedback regulator is constructed to ensure asymptotic stability of the investigated program motion. The results provide a theoretical basis for assessing the practical applicability of the derived analytical solutions in spacecraft guidance problems under non-central gravitational perturbations.

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Introduction

Currently, there is a problem of optimizing the motion of a point (the center of mass of a spacecraft) in a gravitational field. This involves selecting, predicting, optimizing, and calculating the trajectories of controlled objects. The variational problem involves determining the controls (magnitude and direction of the reactive force) and optimal trajectories for a point moving with a limited mass flow per second. The availability of reference analytical solutions is necessary for the development of maneuvering systems. Analytical solutions, unlike numerically constructed ones, are not associated with convergence issues, allow for the pre-determination of initial parameter values, ensure the continuity of trajectory parameters when the thrust mode changes, and contain important functional dependencies between spacecraft and trajectory parameters. Knowledge of analytical solutions allows for the analysis of spacecraft parameter behavior and a qualitative assessment of the accuracy of the control algorithm [1]. Thus, analytical solutions are of particular interest due to their clarity and the possibility of comprehensive analysis. This work is devoted to the determination of particular analytical solutions for intermediate thrust sections in the case of axisymmetric gravitational fields, the determination of the magnitude and direction of the reactive force of the spacecraft, solving the variational problem, as well as the stabilization of the found program movements.

Analytical solutions

It is known that the differential equations of the variational problem of the motion of a point (the center of mass of a spacecraft) have the following form(1).

$$\dot{\vec{v}} = \frac{cm}{M}\frac{\vec{\lambda}}{\lambda} + \vec{g}(\vec{r}), \dot{\vec{r}} = \vec{v}, \quad \dot{M} = -m, \tag{1}$$

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$$\dot{\vec{\lambda}} = -\vec{\lambda}_r, \quad \dot{\vec{\lambda}}_r = -\frac{\partial \vec{g}}{\partial \vec{r}}\vec{\lambda}, \quad \dot{\lambda}_M = \frac{cm}{M^2}\lambda,$$

where $\vec{r}$ - is the radius vector of the point (the spacecraft's center of mass), $\vec{v}$ - is its velocity, $M(t)$ - is the mass of the point, c - is the relative exhaust velocity of the combustion products, m - is the mass flow rate per second $(0 \le m \le \bar{m})$, $\vec{g}$ - is the gravitational acceleration, and $\vec{\lambda}, \vec{\lambda}_r, \dot{\lambda}_M$ - are the Lagrange multipliers conjugate to the velocity, radius vector, and mass, respectively (1).

The gravitational field has the following form [2]:

$$U(r,\theta) = f(r) + \frac{\Phi(\theta)}{r^3}, \tag{2}$$

Here we consider the functions $f(r) = \frac{\mu}{r} - \frac{k}{r^3}$ and $\Phi(\theta) = 3ksin^2(\theta)$. where $k = \frac{\mu}{2}J_2R^2$, J_2-is the second zonal harmonic, R is the mean equatorial radius of the planet; μ is the gravitational parameter of the planet.

For (1) and (2), particular analytical solutions were determined using the Levi-Chevita method and are as follows [3].

$$v_1 = 0, v_2^2 = \frac{360k^2\cos^2\theta - 270k^2\cos^4\theta + 36k\mu r^2\cos 2\theta - 2\mu^2 r^2 - 144k^2}{45kr^3 - 105k\sin^2\theta - 3\mu r^5}, v_3 = 0,$$

$$r = const, \varphi = \varphi_0 + \frac{v_2}{rcos(\theta)}t, \theta = const, M = M_0 \exp\left(\frac{N}{c\lambda_1}t\right), N = \frac{df(r)}{dr} - 2\frac{\Phi(\theta)}{r^3} + \frac{1}{r}v_2^2,$$

$$\lambda_1 = \sqrt{1-\lambda_3^2}, \lambda_2 = 0, \lambda_3 = \frac{QP + \sqrt{Q^2+1-P^2}}{Q^2+1}, Q = \tan\theta, P = \frac{a}{2v_2\cos\theta}$$

$$\lambda_4 = 0, \lambda_5 = a, \lambda_6 = 0, \lambda_7 = \frac{c}{M} \tag{3}$$

The analytical solution (3) is defined to represent the motion of the spacecraft. This solution was determined using the Levi-Chevita method, and its detailed determination is presented in this paper [3]. The dependencies of the magnitude and direction of the reactive force on the trajectory plane's position are determined:

$$\tan\alpha = \frac{\lambda_3}{\lambda_1} = \frac{\frac{v_2^2}{r}\tan\theta - \frac{1}{r^4}\frac{d\Phi}{d\theta}}{-\frac{df(r)}{dr} + 3\frac{\Phi(\theta)}{r^4} - \frac{1}{r}v_2^2}, \tag{4}$$

where α - is the angle between the thrust force and the radial direction.

First approximation stability study.

A first-order stability study was conducted for one of the particular solutions. It was shown to be Lyapunov unstable but controllable to a first approximation, and the unperturbed motion can be stabilized by a linear controller. A linear controller was constructed, the addition of which makes the unperturbed motion asymptotically Lyapunov stable [4]. A particular solution to the variational problem for intermediate thrust sections in the case of an axisymmetric gravitational field was found. The initial state of the point can only be realized with a certain error, which may increase or decrease over time. Therefore, it is necessary to study the stability of the obtained solutions.

We write the differential equations of the unperturbed motion of a point in the form.

$$\dot{v}_1 = fe_1 - \frac{df(r)}{dr} - \frac{3\Phi(\theta)}{r^4} + \frac{1}{r}\left(v_3^2 + v_2^2\right)$$

$$\dot{v}_2 = fe_2 - \frac{v_1v_2}{r} + \frac{v_2v_3}{r}\tan\theta$$

$$\dot{v}_3 = fe_3 - \frac{v_1v_3}{r} - \frac{v_2^2}{r}\tan\theta + \frac{1}{r^4}\frac{\partial\Phi}{\partial\theta} \tag{5}$$

$$\dot{r} = v_1, \dot{\phi} = \frac{v_2}{r\cos\theta}, \dot{\theta} = \frac{v_3}{r}, \dot{M} = -m$$

Where $e_1 = \frac{\lambda_1}{\lambda}, e_2 = \frac{\lambda_2}{\lambda}, e_3 = \sqrt{1 - e_1^2 - e_2^2}, f = \frac{cm}{M}$ –jet acceleration.

Thus, the class of particular solutions has the form (3). As the unperturbed motion corresponding to the class of particular solutions under consideration, we take the following:

$$\begin{cases} v_1^* = 0 \\ v_2^* = v_{20} \\ v_3^* = 0 \\ r^* = r_0 \\ \varphi^* = \varphi_0 + \beta t, \beta = \frac{v_2}{r\cos\theta} \\ \theta^* = \theta_0 \end{cases} \tag{6}$$

Let's take as controls:

$$\begin{cases} e_3 = \lambda_{30}, \\ e_2 = 0, \\ e_1 = \sqrt{1 - \lambda_{30}^2}, \\ f^* = \frac{cm^*}{M^*} = \frac{N^*}{\lambda_{10}} > 0. \end{cases}$$

Where $f^* = f_0 = -\frac{N_0}{\lambda_{10}}, m^* = -M^* \frac{N^*}{c\lambda_1^*}$.

The disturbed motion has the form:

$$\begin{cases} v_1 = v_1^* + x_1, \\ v_2 = v_2^* + x_2, \\ v_3 = v_3^* + x_3, \\ r = r^* + x_4, \\ \varphi = \varphi^* + x_5, \\ \theta = \theta^* + x_6, \end{cases} \qquad \begin{cases} e_3 = e_3^* + u_1, \\ e_2 = e_2^* + u_2, \\ f = f^* + u_3. \end{cases}$$

This yields the following.

$$\begin{cases} v_1 = x_1, \\ v_2 = v_{20} + x_2, \\ v_3 = x_3, \\ r = r_0 + x_4, \\ \varphi = \varphi_0 + \beta t + x_5, \\ \theta = \theta_0 + x_6, \end{cases} \qquad \begin{cases} e_3 = \lambda_{30} + u_1, \\ e_2 = u_2, \\ f = f_0 + u_3. \end{cases}$$

Let us formulate the differential equations of the disturbed motion.

$$\dot{x}_1 = (f_0 + u_3)\sqrt{1 - (u_1 + \lambda_{30})^2 - u_2^2} + \frac{df(r_0 + x_4)}{d(r_0 + x_4)} - 3\frac{\Phi(\theta_0 + x_6)}{(r_0 + x_4)^4} - \frac{(x_3^2 + (v_{20} + x_2)^2)}{(r_0 + x_4)}$$

$$\dot{x}_2 = (f_0 + u_3)u_2 - \frac{x_1(v_{20} + x_2)}{r_0 + x_4} + \frac{(x_2 + v_{20})x_3}{r_0 + x_4}\tan(\theta_0 + x_6)$$

$$\dot{x}_3 = (f_0 + u_3)(\lambda_{30} + u_1) - \frac{x_1 x_3}{r_0 + x_4} - \frac{(v_{20} + x_2)^2}{(r_0 + x_4)^4}\tan(\theta_0 + x_6) + \frac{1}{(r_0 + x_4)^4}\frac{d\Phi(\theta_0 + x_6)}{d(\theta_0 + x_6)}$$

$$\dot{x}_4 = x_1$$

$$\dot{x}_5 + \beta = \frac{v_{20} + x_2}{(r_0 + x_4)(\cos(\theta_0 + x_6))},$$

$$\dot{x}_6 = \frac{x_3}{r_0 + x_4}.$$

Let's isolate the first-approximation equations. Expand the right-hand sides of the equations into a series using the formula:

$$\dot{x}_i = F(0)+(\frac{\partial F}{\partial x_1})_0 x_1+(\frac{\partial F}{\partial x_2})_0 x_2+(\frac{\partial F}{\partial x_3})_0 x_3+(\frac{\partial F}{\partial x_4})_0 x_4+(\frac{\partial F}{\partial x_5})_0 x_5+(\frac{\partial F}{\partial x_6})_0 x_6+(\frac{\partial F}{\partial u_1})_0 u_1+(\frac{\partial F}{\partial u_2})_0 u_2+(\frac{\partial F}{\partial u_3})_0 u_3+...$$

$$\dot{x}_1 = \frac{2v_{20}}{r_0}x_2 + (\left[\frac{\partial}{\partial x_4}\left(\frac{df(r)}{d(r)}\right)_{r=r_0+x_4}\right]_0 + 6\frac{\Phi(\theta_0)}{r_0^4} - \frac{v_{20}^2}{r_0^2})x_4 - \frac{2}{r_0^3}\frac{\partial(\Phi(\theta_0+x_6))}{x_6}x_6 - \frac{2f_0\lambda_{30}u_1}{\sqrt{1-\lambda_{30}^2}} + \sqrt{1-\lambda_{30}^2}u_3$$

$$\dot{x}_2 = f_0u_2 - \frac{v_{20}}{r_0}x_1 + \frac{v_{20}}{r_0}\tan(\theta_0)x_3$$

$$\dot{x}_3 = f_0u_1 + \lambda_{30}u_3 - \frac{2v_{20}}{r_0}\tan(\theta_0)x_2 + \left(\frac{v_{20}^2}{r_0^2}\tan(\theta_0) - \frac{3}{r_0^4}\left(\frac{d\Phi(\theta)}{d\theta}\right)_{\theta=\theta_0}\right)x_4+$$

$$+\left(-\frac{v_{20}^2}{r_0}\cos^2\theta_0 + \frac{1}{r_0^3}\frac{\partial}{\partial x_6}\left(\frac{\partial\Phi(\theta)}{\partial\theta}\right)_{\theta=\theta_0+x_6}\right)x_6$$

$$\dot{x}_4 = x_1,$$

$$\dot{x}_5 = \left(\frac{1}{r_0\cos\theta_0}\right)x_2 - \frac{v_{20}}{r_0^2\cos\theta_0}x_4 + \frac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0}x_6,$$

$$\dot{x}_6 = \frac{1}{r_0}x_3,$$

Let's introduce the following notations:

$$R = \left[\frac{\partial}{\partial x_4}\left(\frac{df(r)}{d(r)}\right)\right]_{\theta=r_0+x_4} + 6\frac{\Phi(\theta_0)}{r_0^4} - \frac{v_{20}^2}{r_0^2},$$

$$T = \left(\frac{v_{20}^2\tan(\theta_0)}{r_0^2}\right) - \frac{3}{r_0^4}\left(\frac{d\Phi(\theta)}{d\theta}\right)_{\theta=\theta_0},$$

$$K = -\frac{v_{20}^2}{r_0\cos^2\theta_0} + \frac{1}{r_0^3}\left(\frac{\partial}{\partial x_6}\left(\frac{\partial\Phi(\theta)}{\partial\theta}\right)\right)_{\theta=\theta_0+x_6},$$

$$A = \frac{\partial\Phi(\theta_0+x_6)}{\partial x_6}.$$

We obtain the equations of the first approximation:

$$\dot{x}_1 = \frac{2v_{20}}{r_0}x_2 + Rx_4 - \frac{2}{r_0^3}Ax_6 - \frac{2f_0\lambda_{30}}{\sqrt{1-\lambda_{30}^2}}u_1 + \sqrt{1-\lambda_{30}^2}u_3,$$

$$\dot{x}_2 = f_0u_2 - \frac{v_{20}}{r_0}x_1 + \frac{v_{20}}{r_0}\tan(\theta_0)x_3$$

$$\dot{x}_3 = f_0u_1 + \lambda_{30}u_3 - \frac{2v_{20}}{r_0}\tan(\theta_0)x_2 + Tx_4 + Kx_6,$$

$$\dot{x}_4 = x_1,$$

$$\dot{x}_5 = \left(\frac{1}{r_0\cos\theta_0}\right)x_2 - \frac{v_{20}}{r_0^2\cos\theta_0}x_4 + \frac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0}x_6,$$

$$\dot{x}_6 = \frac{1}{r_0} x_3,$$

We will write the system of equations of the disturbed motion in matrix form:

$$\frac{d\vec{x}}{dt} = W\vec{x} + B\vec{u} + \vec{g}(x,u)$$

$$\vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{pmatrix}, \quad \vec{u} = \begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad \vec{g}(x,u)$$

The matrix W has the form:

$$[W = \begin{pmatrix} 0 & \dfrac{2v_{20}}{r_0} & 0 & R & 0 & -\dfrac{2A}{r_0^3} \\ -\dfrac{v_{20}}{r_0} & 0 & \dfrac{v_{20}}{r_0}\operatorname{tg}\theta_0 & 0 & 0 & 0 \\ 0 & -\dfrac{2v_{20}}{r_0}\operatorname{tg}\theta_0 & 0 & T & 0 & K \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \dfrac{1}{r_0\cos\theta_0} & 0 & -\dfrac{v_{20}^2}{r_0^2\cos\theta_0} & 0 & \dfrac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0} \\ 0 & 0 & \dfrac{1}{r_0} & 0 & 0 & 0 \end{pmatrix}$$

For $u = 0$, we have a problem of Lyapunov stability [4] of unperturbed motion$(x_i = 0, i = 1..6)$:

$$\dot{\vec{x}} = W\vec{x} + \vec{g}(x,u)$$

Characteristic equation of the first approximation system:

$$[W - SE] = 0$$

$$[W - SE] = \begin{pmatrix} -S & \dfrac{2v_{20}}{r_0} & 0 & R & 0 & -\dfrac{2A}{r_0^3} \\ -\dfrac{v_{20}}{r_0} & -S & \dfrac{v_{20}}{r_0}\operatorname{tg}\theta_0 & 0 & 0 & 0 \\ 0 & -\dfrac{2v_{20}}{r_0}\operatorname{tg}\theta_0 & -S & T & 0 & K \\ 1 & 0 & 0 & -S & 0 & 0 \\ 0 & \dfrac{1}{r_0\cos\theta_0} & 0 & -\dfrac{v_{20}^2}{r_0^2\cos\theta_0} & -S & \dfrac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0} \\ 0 & 0 & \dfrac{1}{r_0} & 0 & 0 & -S \end{pmatrix} \tag{7}$$

$$S^2\left[S^4 + \left(-R - \frac{1}{r_0}K + \frac{2v_{20}^2}{r_0^2\cos^2\theta_0}\right)S^2 + \left(\frac{2TA}{r_0^4} + 4\frac{v_{20}^2 A\tan\theta_0}{r_0^6} + \frac{RK}{r_0} - \frac{2v_{20}^2 T\tan\theta_0}{r_0} - \frac{2v_{20}^2 K}{r_0} - \frac{2v_{20}^2\tan^2\theta_0}{r_0^2}\right)\right] = 0$$

The characteristic equation has two zero roots. Therefore, if at least one of the roots has a positive real part, then, according to Lyapunov's instability theorem, to a first approximation, the unperturbed motion will be unstable. For this to occur, it is sufficient that the term at S^2 the characteristic equation be negative[5].

Stabilization of programmatic movement.

We will stabilize the unperturbed motion (5), that is, we will choose a controller such that, when substituted into (7), the unperturbed motion will be asymptotically Lyapunov stable. We will verify that system (7) is controllable. The following controllability and stabilization criterion exists to a first approximation [6]:

1) System $\frac{d\vec{x}}{dt} = W\vec{x} + B\vec{u}$ is fully controllable if matrix $V = \left|B, WB, ..W^{n-1}B\right|$ has rank n, where n is the order of the system (n=6).
2) If the rank of matrix V is n, then a linear controller $\vec{u} = P\vec{x}$ exists.

Let's construct a matrix V:

$$B = \begin{pmatrix} \dfrac{2f_o\lambda_{30}}{\sqrt{1-\lambda_{30}^2}} & 0 & \sqrt{1-\lambda_{30}^2} \\ 0 & f_o & 0 \\ f_o & 0 & \lambda_{30} \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

The rank of the matrix W is equal to 6. Consequently, the unperturbed motion (5) is stabilized by the linear controller $\vec{u} = P\vec{x}$, regardless of the nonlinear terms of the $g(\vec{x},u)$, the issue of stabilization is resolved using the linear approximation [7].

The constant real matrix P must be chosen such that the unperturbed motion of the system

$$\frac{d\vec{x}}{dt} = (W + BP)\vec{x} + g(\vec{x},u)$$

is asymptotically stable, that is, so that the real parts of all eigenvalues of the matrix $(W + BP)$ are negative. Moreover, the matrix P must be as simple as possible. For example, the following matrix satisfies this requirement[8].

$$P = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & p_{22} & 0 & p_{24} & p_{25} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

that is $u_1 = 0$,$u_3 = 0$. Then the characteristic equation of the first approximation will have the form:

$$[W + BP - SE] = 0$$

or

$$\det\begin{vmatrix} -S & \dfrac{2v_{20}}{r_0} & 0 & R & 0 & -\dfrac{2A}{r_0^3} \\ -\dfrac{v_{20}}{r_0} & -S + f_0p_{22} & \dfrac{v_{20}\tan\theta_0}{r_0} & f_0p_{24} & f_0p_{25} & 0 \\ 0 & -\dfrac{2v_{20}\tan\theta_0}{r_0} & -S & T & 0 & K \\ 1 & 0 & 0 & -S & 0 & 0 \\ 0 & \dfrac{1}{r_0\cos\theta_0} & 0 & -\dfrac{v_{20}}{r_0^2\cos\theta_0} & -S & \dfrac{v_{20}\sin\theta_0}{r_0\cos^2\theta_0} \\ 0 & 0 & \dfrac{1}{r_0} & 0 & 0 & -S \end{vmatrix} = 0 \qquad (8)$$

We will receive:

$$S^6 - S^5 f_0 p_{22} + S^4 \left[\frac{f_0 p_{25}}{r_0 \cos\theta_0} - \frac{2v_{20}^2}{r_0^2} - \frac{2v_0^2 \tan^2\theta_0}{r_0^2} - R + \frac{K}{r_0}\right] + S^3 \left[R f_0 p_{22} - \frac{2 f_0 v_0 p_{24}}{r_0} - \frac{K f_0 p_{22}}{r_0}\right] -$$
$$-S^2 \left(\frac{2v_0^2 K}{r_0^3} + \frac{2v_{20}^2 A \tan\theta_0}{r_0^6} - \frac{K f_0 p_{25}}{r_0^2 \cos\theta_0} - \frac{f_0 p_{25} v_{20}^2 \sin^2\theta_0}{r_0^3 \cos^3\theta_0} + \frac{2Rv_0^2 \tan^2\theta_0}{r_0^2} + \right)$$
$$\left(+ \frac{2v_{20}^2 T \tan\theta_0}{r_0^2} - \frac{R f_0 p_{25}}{r_0 \cos\theta_0} + \frac{2 f_0 p_{25} v_{20}^2}{r_0^3 \cos\theta_0} - \frac{RK}{r_0} + 2\frac{AT}{r_0^2}\right) + \tag{9}$$
$$+S \left[\frac{2 f_0 p_{25} RK}{r_0} - \frac{4Av_{20} f_0 p_{24} \tan\theta_0}{r_0^3} + \frac{2AT f_0 p_{22}}{r_0^2} + \frac{2Kv_{20} f_0 p_{24}}{r_0^2}\right] +$$
$$+\left(-\frac{2AT f_0 p_{25}}{r_0^5 \cos\theta_0} - \frac{RK f_0 p_{25}}{r_0^2 \cos\theta_0} - \frac{2K f_0 p_{25} A v_{20}^2}{r_0^4 \cos\theta_0} - \frac{4 f_0 p_{25} A v_{20}^2 \sin\theta_0}{r_0^7 \cos^2\theta_0} - \frac{2T f_0 p_{25} v_{20}^2 \sin\theta_0}{r_0^3 \cos^2\theta_0} - \frac{2R f_0 p_{25} v_{20}^2 \sin^2\theta_0}{r_0 \cos^3\theta_0}\right) = 0$$

The characteristic equation was reduced to the form:

$$b_0 S^6 + b_1 S^5 + b_2 S^4 + b_3 S^3 + b_4 S^2 + b_5 S^1 + b_6 S^0 = 0$$

where the coefficients of the b_i $(i = 1..6)$ are as follows:

$$b_0 = 1$$

$$b_1 = -f_0 p_{22}$$
$$b_2 = \frac{f_0 p_{25}}{r_0 \cos\theta_0} - \frac{2v_{20}^2}{r_0^2} - \frac{2v_0^2 \tan^2\theta_0}{r_0^2} - R + \frac{K}{r_0}$$
$$b_3 = R f_0 p_{22} - \frac{2 f_0 v_0 p_{24}}{r_0} - \frac{K f_0 p_{22}}{r_0}$$
$$b_4 = \left(\frac{2v_0^2 K}{r_0^3} + \frac{2v_{20}^2 A \tan\theta_0}{r_0^6} - \frac{K f_0 p_{25}}{r_0^2 \cos\theta_0} - \frac{f_0 p_{25} v_{20}^2 \sin^2\theta_0}{r_0^3 \cos^3\theta_0} + \frac{2Rv_0^2 \tan^2\theta_0}{r_0^2} + \right) \tag{10}$$
$$\left(+ \frac{2v_{20}^2 T \tan\theta_0}{r_0^2} - \frac{R f_0 p_{25}}{r_0 \cos\theta_0} + \frac{2 f_0 p_{25} v_{20}^2}{r_0^3 \cos\theta_0} - \frac{RK}{r_0} + 2\frac{AT}{r_0^2}\right)$$
$$b_5 = \frac{2 f_0 p_{25} RK}{r_0} - \frac{4Av_{20} f_0 p_{24} \tan\theta_0}{r_0^3} + \frac{2AT f_0 p_{22}}{r_0^2} + \frac{2Kv_{20} f_0 p_{24}}{r_0^2}$$
$$b_6 = \left(-\frac{2AT f_0 p_{25}}{r_0^5 \cos\theta_0} - \frac{RK f_0 p_{25}}{r_0^2 \cos\theta_0} - \frac{2K f_0 p_{25} A v_{20}^2}{r_0^4 \cos\theta_0} - \frac{4 f_0 p_{25} A v_{20}^2 \sin\theta_0}{r_0^7 \cos^2\theta_0} - \frac{2T f_0 p_{25} v_{20}^2 \sin\theta_0}{r_0^3 \cos^2\theta_0} - \frac{2R f_0 p_{25} v_{20}^2 \sin^2\theta_0}{r_0 \cos^3\theta_0}\right)$$

The matrix P was chosen to be as simple as possible. If, for example, $u_1 = 0$, in addition to counting $u_2 = 0$ (for $u_3 \neq 0$), then the free term of the characteristic equation (8) vanishes, and one root of this equation becomes zero. Let us construct the Hurwitz matrix from the coefficients $b_i (i = 1..6)$ of equation (8).

$$\begin{pmatrix} b_1 & b_3 & b_5 & 0 & 0 & 0 \\ b_0 & b_2 & b_4 & b_6 & 0 & 0 \\ 0 & b_1 & b_3 & b_5 & 0 & 0 \\ 0 & b_0 & b_2 & b_4 & b_6 & 0 \\ 0 & 0 & b_1 & b_3 & b_5 & 0 \\ 0 & 0 & b_0 & b_2 & b_4 & b_6 \end{pmatrix}$$

Hurwitz's theorem.

In order for all roots of the algebraic equation (8) with real coefficients and a positive coefficient at the leading term to have negative real parts, it is necessary and sufficient that all main diagonal minors of the matrix (9) are positive [4].

$$b_1 > 0$$

$$\Delta_2 = \det\begin{pmatrix} b_1 & b_3 \\ b_0 & b_2 \end{pmatrix} = b_1b_2 - b_0b_3 > 0$$

$$\Delta_3 = \begin{vmatrix} b_1 & b_3 & b_5 \\ b_0 & b_2 & b_4 \\ 0 & b_1 & b_3 \end{vmatrix} = b_3\Delta_2 + b_0b_1b_5 - b_1^2b_4 = b_3\Delta_2 + b_1b_5 - b_1^2b_4 > 0$$

$$\Delta_4 = \begin{vmatrix} b_1 & b_3 & b_5 & 0 \\ b_0 & b_2 & b_4 & b_6 \\ 0 & b_1 & b_3 & b_5 \\ 0 & b_0 & b_2 & b_4 \end{vmatrix} = b_4\Delta_3 - b_6(b_3\Delta_2 - b_1^2b_4) + b_0(b_4b_5 - b_3b_6 - b_0b_3b_5 - b_0b_6b_3) =$$

$$= b_4\Delta_3 - b_5(b_3\Delta_2 - b_1^2b_4) + b_1(b_4b_5 - b_6b_3) - b_5^2 > 0$$

$$\Delta_5 = \begin{vmatrix} b_1 & b_3 & b_5 & 0 & 0 \\ b_0 & b_2 & b_4 & b_6 & 0 \\ 0 & b_1 & b_3 & b_5 & 0 \\ 0 & b_0 & b_2 & b_4 & b_6 \\ 0 & 0 & b_1 & b_3 & b_5 \end{vmatrix} = b_5\Delta_4 - b_6\big(b_3\Delta_3 - b_1(b_3\Delta_2 - b_1^2b_4)\big) =$$

$$= b_5\Delta_4 - b_6\Delta_3 + b_1b_6(b_3\Delta_2 - b_1^2b_4) + b_0b_6(b_3 - b_1^2b_6) > 0$$

$\Delta_6 = b_6\Delta_5 > 0$, that is $b_6 > 0$

Thus, if the parameters p_{22}, p_{24}, p_{25} satisfy the above conditions, where b_i are determined from (10), then the unperturbed motion (5) is asymptotically stable: Therefore, the stabilizing control has been found.

$$\begin{cases} u_1 = 0 \\ u_2 = p_{22}x_2 + p_{24}x_4 + p_{25}x_5 \\ u_3 = 0 \end{cases}$$

where the coefficients p_{22}, p_{24}, p_{25} satisfy the conditions found above.

Conclusion

A variational problem of the motion of a point with variable mass in intermediate momentum sections in the gravitational field of spheroidal planets is considered. The differential equations of the variational problem are written in a spherical coordinate system. The feasibility of using the Levi-Civita method to find certain solutions is demonstrated, and a specific class of solutions is found using the Levi-Civita method. One specific solution is shown to be Lyapunov unstable but can be controlled to a first approximation, and the unperturbed motion can be stabilized using a linear controller. A linear controller is constructed, the addition of which makes the unperturbed motion asymptotically Lyapunov stable. It is shown that to stabilize unstable motion to a first approximation, only perturbations in three phase coordinates need be considered.

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