Spectrum of the Sum of Partial Integral Operators Generated by Incomplete Orthonormal Systems

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Abstract.
In this paper, we study a class of partial integral operators acting in the Hilbert space $L_2(\Omega_1 \times \Omega_2)$, generated by incomplete orthonormal systems in the corresponding L_2 spaces. Using a direct integral decomposition, we obtain an explicit description of the spectra of these operators in terms of the essential ranges of the generating coefficient functions. It is shown that the incompleteness of the underlying orthonormal systems leads to the appearance of an additional spectral component at zero. The main result concerns the spectral analysis of the sum of two such partial integral operators. We provide a precise characterization of the spectrum of the sum operator by exploiting its fiber-wise structure. The obtained results contribute to the spectral theory of non-compact partial integral operators and can be applied to related problems in operator theory and mathematical physics.

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Key words: partial integral operator, spectrum, resolvent, essential range, direct integral, orthonormal system, non-complete system.

Introduction

In recent decades, an important direction in the development of modern operator theory has been the comprehensive study of integral operators of Fredholm and Volterra types. These classes of operators have a wide range of applications from applied mathematics and mechanics to quantum physics and the spectral analysis of Schrödinger operators.

One of the aspects that has received significant attention is the study of the influence of various perturbations on the spectral characteristics of multiplication operators acting via multiplication by a fixed function. In particular, in [1], it is investigated how non-compact partial integral operators affect the spectrum of such operators. The main focus is on identifying those components of the spectrum that change under perturbations and those that retain spectral stability. The theoretical foundation is based on classical results of spectral theory, including Weyl's theorem, which guarantees the invariance of the essential spectrum under compact perturbations. However, in the absence of compactness, new and nontrivial effects arise that influence the structure of the spectrum.

In subsequent works, for example in [2], attention is focused on operators with degenerate kernels having a specific structure-namely, in the form of a product of a positive continuous function with itself, defined on a symmetric domain. Under appropriate integrability conditions on this function, it has been possible to completely characterize the essential spectrum of the corresponding operator and rigorously prove the finiteness of the number of eigenvalues lying below its lower bound.

Of particular interest is the manifestation of the Efimov effect in the context of partial integral operators, which is studied in detail in [3]. In that work, the existence of an infinite sequence of eigenvalues is demonstrated for a certain self-adjoint operator consisting of an unperturbed part and additional integral terms under a special configuration of the kernel. It is shown that the lower bound of the essential spectrum of such a Hamiltonian is equal to zero.

More general cases of degenerate kernels have been analyzed in studies [4,5], where the kernel is represented as the product of two different functions. Under certain positivity conditions and with an appropriate

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choice of test functions, the authors obtained an analytical expression for the lower bound of the essential spectrum of the partial integral operator.

Special interest is also devoted to discrete models of Schrödinger operators describing the behavior of three quantum particles on a three-dimensional lattice. These models are formalized using partial integral operators and possess spectral properties directly related to the Efimov effect and the phenomenon of accumulation of eigenvalues in the discrete part of the spectrum, as shown in [6].

Particular attention in modern literature is paid to the analysis of spectral characteristics of partial integral operators modeling three-particle interactions. In particular, the studies presented in [4,5,10,11] are aimed at a detailed investigation of the structure of the essential and discrete spectra of such operators. These operators belong to the class of partial integral operators of Fredholm type, which naturally arise in problems of quantum mechanics and mathematical physics when describing three-particle interactions.

An analysis of the structure of the essential spectrum of a model operator describing three-particle interaction is carried out in [9]. The existence of negative eigenvalues of this operator is established, and an estimate of their number is obtained.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be infinite-dimensional Hilbert spaces, and let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ denote their tensor product [7,8]. Suppose that bounded self-adjoint operators A and B act in these spaces, respectively. Then the tensor product $A \otimes B$ is also a bounded self-adjoint operator in $\mathcal{H}$. The spectral theory of partial integral operators has been extensively studied under various assumptions, including compactness, kernel regularity, and completeness of the underlying orthonormal systems. In the classical setting, when the generating systems are complete, the corresponding operators admit relatively well-understood spectral descriptions.

In this paper, we consider a class of partial integral operators generated by incomplete orthonormal systems in $L_2(\Omega_1)$ and $L_2(\Omega_2)$. We focus on the precise description of their spectra in the space $L_2(\Omega_1 \times \Omega_2)$. Using the direct integral approach, we express these operators as measurable families of diagonal operators and describe their spectra in terms of the essential ranges of the corresponding coefficient functions.

The main emphasis is placed on the spectral analysis of the sum of two such operators. Although each operator admits a relatively simple fiber-wise representation, their sum exhibits a more intricate spectral structure. We show that this structure can still be effectively analyzed by exploiting the underlying decomposition and the orthogonality properties of the generating systems.

The results obtained in this paper extend known spectral descriptions to the case of incomplete orthonormal systems and provide a unified framework for studying non-compact partial integral operators. These findings may be useful in further investigations of operator equations and in applications where non-complete expansions naturally arise.

Formulation of the Main Problem of Spectral Analysis

Consider a linear partial integral operator of the form

$$T = T_1 + T_2, \tag{1}$$

where the operator T acts in the Hilbert space $L_2(\Omega_1 \times \Omega_2)$, with $\Omega_1 = [a, b]$ and $\Omega_2 = [c, d]$.

The operators T_1 and T_2 act in $L_2(\Omega_1 \times \Omega_2)$ as partial integral operators with respect to different variables:

$$(T_1 f)(x, y) = \int_{\Omega_1} k_1(x, s, y)\, f(s, y)\, ds,$$

$$(T_2 f)(x, y) = \int_{\Omega_2} k_2(x, t, y)\, f(x, t)\, dt.$$

Here, the kernels k_1 and k_2 are measurable functions defined on $\Omega_1^2 \times \Omega_2$ and $\Omega_1 \times \Omega_2^2$, respectively.

Assume that $\{\varphi_i\}_{i=1}^{\infty}$ is an orthonormal system in $L_2(\Omega_1)$ and $\{\psi_j\}_{j=1}^{\infty}$ is an orthonormal system in $L_2(\Omega_2)$. Let $\{h_i(y)\}_{i=1}^{\infty}$ and $\{g_j(x)\}_{j=1}^{\infty}$ be collections of essentially bounded real-valued functions defined on Ω_2 and Ω_1, respectively. Then the kernels admit the following degenerate representations:

$$k_1(x, s, y) = \sum_{i=1}^{\infty} \varphi_i(x)\, \overline{\varphi_i(s)}\, h_i(y), \tag{2}$$

$$k_2(x,t,y)=\sum_{j=1}^{\infty}\psi_j(y)\,\overline{\psi_j(t)}\,g_j(x). \tag{3}$$

Thus, both operators are partial integral operators with degenerate kernels represented as infinite sums of separable functions. Under appropriate boundedness conditions on the coefficient functions $\{h_i(y)\}$ and $\{g_j(x)\}$, these operators define bounded self-adjoint operators in $L_2(\Omega_1\times\Omega_2)$. The operator T_1+T_2 in the case where h_i and g_j are constant sequences was studied in [12].

Let $\varphi\geq 0$ be a measurable and essentially bounded function defined on a set $\Omega\subset\mathbb{R}^v$. Then

$$\operatorname{ess\,sup}_{\Omega}(\varphi)=\inf\left\{C\in\mathbb{R}:\mu\left(\{\xi\in\Omega:\varphi(\xi)>C\}\right)=0\right\},$$

where $\mu(\cdot)$ denotes the Lebesgue measure. If there exists $\varepsilon>0$ such that

$$\mu\left(\{\xi\in\Omega:\lambda-\varepsilon<\varphi(\xi)<\lambda+\varepsilon\}\right)>0,$$

then the number λ is called an essential value of the function φ. The set of all such values is denoted by Essran(φ).

In this paper, we study the operator (1) with degenerate kernels of the form (2)-(3). The main goal is to investigate the spectral properties of T, in particular, to describe its essential and discrete spectra and to establish conditions for the existence of eigenvalues.

Spectral properties of the operators T_1+T_2

Let the coefficients $h_i(y)$ and $g_j(x)$ satisfy the following assumptions:

- $h_i\in L_\infty(\Omega_2)$ and $h_i(y)$ is real-valued for all i, with $M_1:=\operatorname{ess\,sup}_{y\in\Omega_2}\sup_i|h_i(y)|<\infty$;
- $g_j\in L_\infty(\Omega_1)$ and $g_j(x)$ is real-valued for all j, with $M_2:=\operatorname{ess\,sup}_{x\in\Omega_1}\sup_j|g_j(x)|<\infty$.

Lemma 1. Let h_i and g_j satisfy the assumptions above. Then the operators T_1 and T_2 on $L_2(\Omega_1\times\Omega_2)$ are bounded, with $\|T_1\|\leq M_1$ and $\|T_2\|\leq M_2$, and they are self-adjoint.

Proof. Using the orthonormality of $\{\varphi_i\}$, for $f\in L_2(\Omega_1\times\Omega_2)$ we can write

$$(T_1f)(x,y)=\sum_{i=1}^{\infty}h_i(y)\,\varphi_i(x)\int_{\Omega_1}f(s,y)\overline{\varphi_i(s)}\,ds.$$

Denote $c_i(y):=(f(\cdot,y),\varphi_i)_{L_2(\Omega_1)}=\int_{\Omega_1}f(s,y)\overline{\varphi_i(s)}\,ds$.

Then $(T_1f)(x,y)=\sum_{i=1}^{\infty}h_i(y)\,\varphi_i(x)\,c_i(y)$.

If $\{\varphi_i\}$ is complete, Parseval's identity gives $\sum_{i=1}^{\infty}|c_i(y)|^2=\|f(\cdot,y)\|^2_{L_2(\Omega_1)}$, while if $\{\varphi_i\}$ is not complete, Bessel's inequality yields $\sum_{i=1}^{\infty}|c_i(y)|^2\leq\|f(\cdot,y)\|^2_{L_2(\Omega_1)}$.

Hence, in both cases,

$$\|T_1f\|^2_{L_2(\Omega_1\times\Omega_2)}=\int_{\Omega_2}\sum_{i=1}^{\infty}|h_i(y)|^2|c_i(y)|^2\,dy\leq\sup_i\|h_i\|^2_{L_\infty(\Omega_2)}\int_{\Omega_2}\sum_{i=1}^{\infty}|c_i(y)|^2\,dy\leq M^2\|f\|^2_{L_2(\Omega_1\times\Omega_2)},$$

where $M:=\sup_i\|h_i\|_{L_\infty(\Omega_2)}<\infty$.

Therefore, T_1 is bounded and $\|T_1\|\leq M$.

If $h_i(y)$ are real-valued, then for all f,g:

$$(T_1f,g)=\int_{\Omega_2}\sum_i h_i(y)c_i(y)\overline{d_i(y)}\,dy=(f,T_1g),$$

where $d_i(y)=(g(\cdot,y),\varphi_i)$. Therefore T_1 is self-adjoint.

Theorem 1. If the orthonormal system $\{\varphi_i\}$ is not complete, then the spectrum of T_1 in $L_2(\Omega_1 \times \Omega_2)$ is given by

$$\sigma(T_1) = \overline{\bigcup_{i=1}^{\infty} \operatorname{ess\,ran}(h_i)} \ \cup\ \{0\}.$$

Proof. For each fixed $y \in \Omega_2$, define the fiber operator

$$A(y) : L_2(\Omega_1) \to L_2(\Omega_1), \quad (A(y)f_y)(x) = \sum_{i=1}^{\infty} h_i(y)\langle f_y, \varphi_i\rangle\, \varphi_i(x),$$

where $f_y(x) := f(x, y)$ is the y-fiber of f.

By construction, $A(y)$ is diagonal in the orthonormal system $\{\varphi_i\}$. Its spectrum restricted to the span of $\{\varphi_i\}$ is

$$\sigma\big(A(y)\big|_{\operatorname{span}\{\varphi_i\}}\big) = \overline{\{h_i(y)\}_{i=1}^{\infty}}.$$

The operator T_1 can be expressed as a direct integral of the fiber operators:

$$T_1 \cong \int_{\Omega_2}^{\oplus} A(y)\, dy.$$

According to standard results on spectra of direct integrals, we have

$$\sigma(T_1) = \overline{\bigcup_{y\in\Omega_2} \sigma(A(y)\big|_{\operatorname{span}\{\varphi_i\}})}.$$

If $\{\varphi_i\}$ is incomplete in $L_2(\Omega_1)$, then there exists a non-zero subspace $\overline{\operatorname{span}}\{\varphi_i\}_\perp \subset L_2(\Omega_1)$ orthogonal to all φ_i. For any $f_y \in H_\perp$, $(A(y)f_y)(x) = 0$ for all $y \in \Omega_2$. Hence 0 is not in the spectrum of any individual $A(y)$ restricted to $\operatorname{span}\{\varphi_i\}$, but contributes to the spectrum of T_1 when taking the direct integral: $0 \in \sigma(T_1)$.

Using the definition of the essential range, the union over $y \in \Omega_2$ can be rewritten as

$$\overline{\bigcup_{y\in\Omega_2} \overline{\{h_i(y)\}_{i=1}^{\infty}}} = \overline{\bigcup_{i=1}^{\infty} \operatorname{ess\,ran}(h_i)}.$$

Including the zero contribution from the orthogonal subspace, we obtain the complete spectrum:

$$\sigma(T_1) = \overline{\bigcup_{y\in\Omega_2} \overline{\{h_i(y)\}_{i=1}^{\infty}}} \ \cup\ \{0\} = \overline{\bigcup_{i=1}^{\infty} \operatorname{ess\,ran}(h_i)} \ \cup\ \{0\}.$$

Corollary 1. Zero does not belong to the spectrum of T_1 if and only if $\{\varphi_i\}_{i=1}^{\infty}$ forms a complete orthonormal system in $L_2(\Omega_1)$, and $\inf_{i\geq 1,\, y\in\Omega_2} |h_i(y)| > 0$.

Theorem 2. The discrete spectrum of the operator T_1 is empty, i.e.,

$$\sigma_{\text{disc}}(T_1) = \varnothing.$$

Proof. Decompose the space $L_2(\Omega_1)$ as

$$L_2(\Omega_1) = H_1 \oplus H_1^{\perp}, \quad H_1 = \overline{\operatorname{span}}\{\varphi_i\}.$$

Accordingly,

$$L_2(\Omega_1 \times \Omega_2) = (H_1 \otimes L_2(\Omega_2)) \oplus (H_1^{\perp} \otimes L_2(\Omega_2)).$$

Let $\lambda \in \mathbb{R}$ be an eigenvalue of T_1 and let $f \in H_1 \otimes L_2(\Omega_2)$ be a corresponding eigenfunction:

$$T_1 f = \lambda f.$$

Then, for a.e. $y \in \Omega_2$, the fiber $f_y(x) = f(x, y)$ satisfies

$$\sum_{i=1}^{\infty} h_i(y)(f_y, \varphi_i)\varphi_i(x) = \lambda f_y(x).$$

By orthonormality,

$$(h_i(y) - \lambda)(f_y, \varphi_i) = 0 \quad \text{for all } i.$$

Hence

$$f(x, y) = \sum_{i:\, h_i(y)=\lambda} c_i(y)\varphi_i(x), \quad c_i \in L_2(\Omega_2).$$

Therefore, the eigenspace corresponding to λ contains functions of the form

$$\varphi_i(x) \otimes f(y), \quad f \in L_2(\Omega_2),$$

which implies that this eigenspace is infinite-dimensional.

Since $H_1^{\perp} \perp \{\varphi_i\}$, we have

$$T_1 f = 0 \quad \text{for all } f \in H_2 \otimes L_2(\Omega_2).$$

Thus, 0 is an eigenvalue and its eigenspace contains the whole subspace $H_2 \otimes L_2(\Omega_2)$, which is infinite-dimensional.

In both cases, every eigenvalue of T_1 has an infinite-dimensional eigenspace. Hence, T_1 has no eigenvalues of finite multiplicity, and therefore its discrete spectrum is empty:

$$\sigma_{\text{disc}}(T_1) = \varnothing.$$

Theorem 3. Let $\lambda \notin \sigma(T_1)$ and $\mu \notin \sigma(T_2)$. Then the resolvents of T_1 and T_2 are given by

$$(T_1 - \lambda I)^{-1} f = \sum_{i=1}^{\infty} \frac{\varphi_i(x)}{h_i(y) - \lambda} \int_{\Omega_1} \overline{\varphi_i(s)} f(s, y)\, ds - \frac{1}{\lambda} f_1^{\perp}(x, y),$$

$$(T_2 - \mu I)^{-1} f = \sum_{j=1}^{\infty} \frac{\psi_j(y)}{g_j(x) - \mu} \int_{\Omega_2} \overline{\psi_j(t)} f(x, t)\, dt - \frac{1}{\mu} f_2^{\perp}(x, y). \tag{5}$$

where the orthogonal projections are defined by

$$f_1^{\perp}(\cdot, y) = f(\cdot, y) - \sum_{i=1}^{\infty} \varphi_i(\cdot) \int_{\Omega_1} \overline{\varphi_i(s)} f(s, y)\, ds, \quad f_2^{\perp}(x, \cdot) = f(x, \cdot) - \sum_{j=1}^{\infty} \psi_j(\cdot) \int_{\Omega_2} \overline{\psi_j(t)} f(x, t)\, dt.$$

Here $f_1^{\perp}$ and $f_2^{\perp}$ are the orthogonal projections of f onto $\mathcal{H}_1^{\perp}$ and $\mathcal{H}_2^{\perp}$, respectively.

Proof. We use the direct integral decomposition

$$L_2(\Omega_1 \times \Omega_2) \simeq \int_{\Omega_2}^{\oplus} L_2(\Omega_1)\, dy.$$

For each fixed y, define the operator

$$A(y) : L_2(\Omega_1) \to L_2(\Omega_1), \quad A(y)u = \sum_{i=1}^{\infty} h_i(y)(u, \varphi_i)\varphi_i.$$

Then

$$(T_1 f)(\cdot, y) = A(y) f(\cdot, y), \quad T_1 = \int_{\Omega_2}^{\oplus} A(y)\, dy.$$

Hence,

$$(T_1 - \lambda I)^{-1} = \int_{\Omega_2}^{\oplus} (A(y) - \lambda I)^{-1} dy,$$

provided $\lambda \notin \sigma(T_1)$.

Now fix y. Since $\{\varphi_i\}$ is orthonormal, we have the orthogonal decomposition

$$L_2(\Omega_1) = \overline{\operatorname{span}\{\varphi_i\}} \oplus \mathcal{H}_1^{\perp}.$$

On these subspaces:

$$A(y)\varphi_i = h_i(y)\varphi_i, \quad A(y)u_{\perp} = 0.$$

Therefore,

$$(A(y) - \lambda I)\varphi_i = (h_i(y) - \lambda)\varphi_i, \quad (A(y) - \lambda I)u_{\perp} = -\lambda u_{\perp}.$$

If $\lambda \notin \operatorname{ess\,ran}(h_i)$ and $\lambda \neq 0$, then

$$(A(y) - \lambda I)^{-1}\varphi_i = \frac{1}{h_i(y) - \lambda}\varphi_i, \quad (A(y) - \lambda I)^{-1}u_{\perp} = -\frac{1}{\lambda}u_{\perp}.$$

Applying this fiber-wise to $f(\cdot, y)$ yields (5).

The proof for T_2 is analogous, with the roles of x and y interchanged and φ_i replaced by ψ_j.

Let

$$\mathcal{H}_1 = \overline{\operatorname{span}}\{\varphi_i\}, \quad \mathcal{H}_2 = \overline{\operatorname{span}}\{\psi_j\}.$$

Consider the block decomposition of $L_2(\Omega_1 \times \Omega_2)$:

$$\begin{aligned} L_2(\Omega_1 \times \Omega_2) = & \underbrace{\overline{\operatorname{span}\{\varphi_i \otimes \psi_j\}}}_{H_{11}} \oplus \underbrace{(\mathcal{H}_1^{\perp} \otimes \overline{\operatorname{span}\{\psi_j\}})}_{H_{21}} \\ & \oplus \underbrace{(\overline{\operatorname{span}\{\varphi_i\}} \otimes \mathcal{H}_2^{\perp})}_{H_{12}} \oplus \underbrace{(\mathcal{H}_1^{\perp} \otimes \mathcal{H}_2^{\perp})}_{H_{22}}. \end{aligned}$$

Theorem 4. Let $\{\varphi_i\}$ and $\{\psi_j\}$ be incomplete orthonormal systems. Then

$$\sigma(T_1 + T_2) = \overline{\bigcup_{i,j} (\operatorname{ess\,ran}(h_i) + \operatorname{ess\,ran}(g_j))} \cup \overline{\bigcup_{j} \operatorname{ess\,ran}(g_j)} \cup \overline{\bigcup_{i} \operatorname{ess\,ran}(h_i)} \cup \{0\}.$$

Proof. We use the orthogonal decomposition

$$L_2(\Omega_1 \times \Omega_2) = \mathcal{H}_{11} \oplus \mathcal{H}_{12} \oplus \mathcal{H}_{21} \oplus \mathcal{H}_{22},$$

Each subspace is invariant under T_1 and T_2, hence also under $T_1 + T_2$. Therefore,

$$\sigma(T_1 + T_2) = \bigcup_{k=1}^{4} \sigma\big((T_1 + T_2)|_{\mathcal{H}_k}\big).$$

On $\mathcal{H}_{11}$, the operator admits a direct integral representation. For each (x, y)-fiber, it acts diagonally with respect to the basis $\{\varphi_i(x)\psi_j(y)\}$, and the corresponding fiber operators have spectra

$$\{h_i(y) + g_j(x) : i, j \geq 1\}.$$

Hence,

$$\sigma(\mathcal{H}_{11}) = \overline{\bigcup_{i,j} (\operatorname{ess\,ran}(h_i) + \operatorname{ess\,ran}(g_j))}.$$

On the $\mathcal{H}_{21}$ subspace, $T_1 = 0$, hence

$$(T_1 + T_2)|_{\mathcal{H}_{21}} = T_2.$$

Therefore,

$$\sigma(\mathcal{H}_{21}) = \overline{\bigcup_j \operatorname{ess\,ran}(g_j)}.$$

Similarly, $T_2 = 0$ on $\mathcal{H}_{12}$ subspace, hence

$$(T_1 + T_2)|_{\mathcal{H}_{12}} = T_1,$$

and

$$\sigma(\mathcal{H}_{12}) = \overline{\bigcup_i \operatorname{ess\,ran}(h_i)}.$$

On $\mathcal{H}_{22}$, both T_1 and T_2 vanish, so

$$(T_1 + T_2)|_{\mathcal{H}_{22}} = 0,$$

and hence

$$\sigma(\mathcal{H}_{22}) = \{0\}.$$

Combining all four blocks, we obtain

$$\sigma(T_1 + T_2) = \overline{\bigcup_{i,j} (\operatorname{ess\,ran}(h_i) + \operatorname{ess\,ran}(g_j))} \cup \overline{\bigcup_j \operatorname{ess\,ran}(g_j)} \cup \overline{\bigcup_i \operatorname{ess\,ran}(h_i)} \cup \{0\}.$$

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