THE DIRICHLET PROBLEM FOR THE WAVE EQUATION IN A SPHERICAL DOMAIN

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ABSTRACT. In this paper, an ill-posed Dirichlet boundary value problem for a second-order hyperbolic equation in a spherical domain is studied. The solution is constructed using the method of separation of variables. The main focus is on determining the conditions that ensure the existence, uniqueness, and stability of the solution. In the proofs, a crucial role is played by the analysis of small denominators, based on Liouville's theorem and K. Roth's results on Diophantine approximations of algebraic numbers.

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Key words: Dirichlet problem; hyperbolic equation; small denominators; algebraic numbers; Fourier method; ill-posed boundary value problems.

Introduction

In recent years, there has been a significant increase in interest in the study of ill-posed boundary value problems for equations of mathematical physics, in particular the Dirichlet boundary value problem for the wave equation in a spherical domain. As noted in the present work, the representation of solutions gives rise to the problem of small denominators, which leads to convergence difficulties of the associated series.

Main part

We consider the problem

$$u_{tt}(r,t) = a^2\Delta u(r,t), \quad 0 < r < R,\ 0 < t < T. \tag{1.1}$$

$$u(r,0) = \varphi(r), \quad u_t(r,0) = 0, \quad 0 \le r \le R. \tag{1.2}$$

$$u(R,t) = 0, \quad 0 \le t \le T. \tag{1.3}$$

The solvability of the problem is closely related to the arithmetic properties of the parameter aT/R, which arises in the analysis of the convergence of the series appearing in the solution.

We seek the solution in the form

$$u(r,t) = V(r)H(t).$$

Substituting this representation into equation (1.1) and separating the variables, we obtain

$$\frac{H''(t)}{a^2H(t)} = \frac{\Delta V(r)}{V(r)}.$$

For the function V, we arrive at the following equations:

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \lambda^2 V = 0,$$

$$V'' + \frac{2}{r}V' + \lambda^2 V = 0.$$

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Introducing the substitution

$$y(r) = rV(r),$$

we obtain

$$y'' + \lambda^2 y = 0,$$

whose general solution is

$$y(r) = C_1 \cos(\lambda r) + C_2 \sin(\lambda r),$$

or equivalently,

$$V(r) = \frac{C_1 \cos(\lambda r) + C_2 \sin(\lambda r)}{r}.$$

The regularity condition at the origin ($V(0) < \infty$) implies that $C_1 = 0$. The boundary condition (1.3) gives

$$\frac{C_2 \sin(\lambda R)}{R} = 0,$$

which implies

$$\lambda R = \pi k, \quad k = 1, 2, \dots$$

hence

$$\lambda_k = \frac{\pi k}{R}.$$

Therefore, the eigenfunctions are given by

$$R_k(r) = \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right), \quad k = 1, 2, \dots.$$

For the H, we have

$$H_k'' + a^2 \lambda_k^2 H_k = 0,$$
$$H_k(t) = A_k \cos(\lambda_k a t) + B_k \sin(\lambda_k a t).$$

Thus, the solution can be represented as

$$u(r,t) = \sum_{k=1}^{\infty} \left(A_k \cos(\lambda_k a t) + B_k \sin(\lambda_k a t)\right) \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right).$$

Applying the initial conditions (1.2) yields

$$\sum_{k=1}^{\infty} A_k \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right) = \varphi(r),$$

hence

$$A_k = \frac{2}{R} \int_0^R r\varphi(r) \sin\left(\frac{\pi k}{R} r\right) \, dr = \varphi_k, \quad k = 1, 2, \dots.$$

From the (1.2) condition, we obtain

$$\sum_{k=1}^{\infty} \left(A_k \cos(\lambda_k a T) + B_k \sin(\lambda_k a T)\right) \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right) = 0,$$

hence

$$B_k = -\frac{A_k \cos(\lambda_k a T)}{\sin(\lambda_k a T)} = -\frac{\varphi_k \cos(\lambda_k a T)}{\sin(\lambda_k a T)}.$$

Consequently, the formal solution of problem (1.1)вЂ“(1.3) is

$$u(r,t) = \sum_{k=1}^{\infty} \left(\varphi_k \cos(\lambda_k a t) - \frac{\varphi_k \cos(\lambda_k a T)}{\sin(\lambda_k a T)} \sin(\lambda_k a t)\right) \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right),$$

or equivalently,

$$u(r,t) = \sum_{k=1}^{\infty} \frac{\varphi_k \sin(\lambda_k a(T-t))}{\sin(\lambda_k aT)} \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right). \tag{1.4}$$

It remains to prove that the series (1.4), as well as the derivatives of this solution involved in equation (1.1), converge uniformly. In particular, we consider

$$u_{tt}(r,t) = -\left(\frac{\pi a}{R}\right)^2 \sum_{k=1}^{\infty} \frac{k^2 \varphi_k \sin(\lambda_k a(T-t))}{\sin(\lambda_k aT)} \frac{1}{r} \sin\left(\frac{\pi k}{R} r\right). \tag{1.5}$$

It suffices to establish the convergence of the series (1.5). A corresponding majorant series is given by

$$|u_{tt}(r,t)| \le C \sum_{k=1}^{\infty} \frac{k^3 |\varphi_k|}{|\sin(\lambda_k aT)|}. \tag{1.6}$$

We proceed by estimating the denominator of series (1.6). Note that

$$\sin\left(\frac{\pi k}{R} aT\right) = 0,$$

$$\frac{\pi k}{R} aT = \pi m.$$

Introduce the notation

$$\delta_{mk} = \frac{\pi k}{R} aT - \pi m,$$

or

$$\frac{\pi k}{R} aT = \delta_{mk} + \pi m. \tag{1.7}$$

Without loss of generality, one may assume that $|\delta_{mk}| < \frac{\pi}{2}$. From (1.7) we obtain

$$\frac{aT}{R} = \frac{\delta_{mk}}{\pi k} + \frac{m}{k}. \tag{1.8}$$

As noted by D. Borzhin and R. Duffin [1] (see also [2]), when this parameter is an algebraic number, it is possible to obtain estimates that prevent the denominator from turning to zero.

In [3]-[4], Sh. A. Alimov studied the Dirichlet problem for the wave equation in a rectangular domain and demonstrated that the existence and uniqueness of the solution depend on the arithmetic properties of the parameter $\theta = T/a$. We shall similarly examine the conditions for the existence and uniqueness of the solution in a spherical domain.

Lemma. *Let the number aT/R be an algebraic number of degree 2. Then the following estimate holds*

$$|\delta_{mk}| \ge \frac{C}{k}. \tag{1.9}$$

Proof. By assumption, aT/R – is an algebraic number of degree 2. According to Liouville's theorem [6], there exists a constant $C > 0$ such that for any natural numbers m and k, the inequality

$$\left|\frac{aT}{R} - \frac{m}{k}\right| \ge \frac{C}{k^2}.$$

holds. From this, we immediately obtain

$$|\delta_{mk}| = \pi k \left|\frac{aT}{R} - \frac{m}{k}\right| \ge \frac{C}{k}.$$

The lemma is thus proved.

Next, we consider the convergence of series (1.6). Taking into account (1.8), we have

$$\sin\left(\frac{\pi k}{R}aT\right) = \sin(\delta_{mk} + \pi m) = (-1)^m \sin(\delta_{mk}).$$

Since $|\delta_{mk}| < \pi/2$ and using estimate (1.9), it follows that

$$\left|\sin\left(\frac{\pi k}{R}aT\right)\right| = |\sin(\delta_{mk})| \geq \frac{2C}{\pi k}.$$

Therefore, estimate (1.6) can be rewritten as:

$$|u_{tt}(r,t)| \leq C\sum_{k=1}^{\infty}\frac{k^3|\varphi_k|}{|\sin(\lambda_k aT)|} \leq C_1\sum_{k=1}^{\infty}k^4|\varphi_k|. \tag{1.10}$$

Theorem 1. *Let aT/R be an algebraic number of degree 2. Then, for any function $\varphi \in W_2^4(|r| < R)$, the problem (1.1)-(1.3) possesses a unique solution.*

Proof. As discussed above, it remains to prove the convergence of series (1.5). Using estimate (1.10), we obtain

$$|u_{tt}(r,t)|^2 \leq C_1^2\sum_{k=1}^{\infty}k^8|\varphi_k|^2 \leq C_1^2\sum_{k=1}^{\infty}(1+k^2)^4|\varphi_k|^2.$$

The last series converges due to the assumptions on the function φ. The theorem is proved.

We remark that the solvability of the problem (1.1)-(1.3) also holds when aT/R is an arbitrary algebraic number, not necessarily of degree 2.

Theorem 2. *Let aT/R be an algebraic number of degree greater than 2. Then, for any function $\varphi \in W_2^{5+\varepsilon}(|r| < R)$, $\varepsilon > 0$, problem (1.1)-(1.3) possesses a unique solution.*

Proof. The proof is analogous to that of Theorem 1. Instead of estimate (1.9), one uses the bound obtained by K. Roth [5]:

$$\left|\frac{aT}{R} - \frac{m}{k}\right| \geq \frac{C(\varepsilon)}{k^{2+\varepsilon}}, \quad \varepsilon > 0.$$

which yields the following estimate for δ_{mk}:

$$|\delta_{mk}| = \pi k\left|\frac{aT}{R} - \frac{m}{k}\right| \geq \frac{C}{k^{1+\varepsilon}}, \quad \varepsilon > 0.$$

Consequently, estimate (1.10) takes the form

$$|u_{tt}(r,t)| \leq C\sum_{k=1}^{\infty}\frac{k^5|\varphi_k|}{|\sin(\lambda_k aT)|} \leq C_1\sum_{k=1}^{\infty}k^{6+\varepsilon}|\varphi_k|.$$

The uniform convergence of the latter series follows from the assumptions of Theorem 2.

Remark.

The result of Theorem 2 is valid not only for algebraic numbers aT/R, but, according to A. Ya. Khinchin [7], for almost all real values of aT/R in the sense of Lebesgue measure.

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