

LOCAL AND 2-LOCAL ANTI-DERIVATIONS ON NULL-FILIFORM AND FILIFORM LEIBNIZ ALGEBRAS AND THEIR SOLVABLE EXTENSIONS

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ABSTRACT. In this paper, we study anti-derivations, local anti-derivations and 2-local anti-derivations on several finite-dimensional complex Leibniz algebras. More precisely, we consider the null-filiform Leibniz algebra NF_n , the naturally graded non-Lie filiform Leibniz algebras F_n^1 and F_n^2 , and their associated solvable extensions. Here, an anti-derivation is understood as a (-1) -derivation satisfying

$$D([x, y]) = -[D(x), y] - [x, D(y)].$$

We first determine the spaces of anti-derivations for all algebras under consideration. We then obtain complete descriptions of their local anti-derivations. In particular, every local anti-derivation is an anti-derivation precisely for the algebras R , $R(F_n^1)$ and $\mathfrak{L}_2$, whereas NF_n , F_n^1 , F_n^2 and $\mathfrak{L}_1$ admit local anti-derivations which are not anti-derivations. Furthermore, we prove that every 2-local anti-derivation on NF_n and on each of the four solvable extensions is an anti-derivation. For F_n^1 and F_n^2 , we construct families of nonlinear 2-local anti-derivations with values in the center. The members of these families are anti-derivations exactly when the corresponding defining maps are linear.

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Key words: Leibniz algebras, anti-derivations, local anti-derivations, 2-local anti-derivations, filiform algebras.

Introduction

Derivations are useful tools for studying algebraic structures and their symmetries. Their local and 2-local analogues lead to a natural question: when does a mapping that agrees with a derivation at each point, or at each pair of points, necessarily satisfy the derivation identity on the whole algebra? The answer depends on the structure of the algebra and can change when ordinary derivations are replaced by other classes of linear mappings.

The notion of a local derivation was introduced independently by Kadison [18] and by Larson and Sourour [21]. A local derivation is a linear mapping whose value at each element coincides with the value of some derivation, which may depend on that element. Subsequently, Šemrl [22] introduced 2-local derivations: for every pair of elements, one derivation realizes both prescribed values. In the 2-local setting, linearity of the mapping is not assumed. These definitions have motivated the study of when pointwise or pairwise agreement forces a mapping to belong to the original class.

For finite-dimensional Lie algebras over an algebraically closed field of characteristic zero, Ayupov and Kudaybergenov [9] proved that every local derivation on a semisimple Lie algebra is a derivation and constructed nilpotent examples admitting local derivations that are not derivations. Ayupov, Kudaybergenov and Rakhimov [10] established the corresponding result for 2-local derivations on semisimple Lie algebras and showed that every finite-dimensional nilpotent Lie algebra of dimension greater than two admits a 2-local derivation that is not a derivation. The solvable case also exhibits different behaviour: Ayupov and Khudoyberdiyev [7] obtained examples of both possibilities and proved that local derivations are derivations for solvable Lie algebras with model nilradicals and maximal-dimensional complementary spaces.

Related questions have been investigated in infinite-dimensional and non-Lie settings. Local derivations on Witt algebras were studied in [15], while 2-local derivations on generalized Witt algebras were considered in

[11]. For solvable Leibniz algebras, Ayupov, Khudoyberdiyev and Yusupov [8] showed that the nilradical and the dimension of its complement influence whether a local derivation is a derivation. In particular, their results include solvable Leibniz algebras for which every 2-local derivation is a derivation, although local derivations need not be derivations.

Another direction is the study of δ -derivations, developed for Lie algebras by Filippov [16,17]. For a scalar δ , a linear map $D : L \rightarrow L$ is a δ -derivation if

$$D([x, y]) = \delta([D(x), y] + [x, D(y)]) \quad (x, y \in L).$$

The values $\delta = 1$, $\delta = -1$ and $\delta = \frac{1}{2}$ are of particular interest. The first gives ordinary derivations, while a (-1) -derivation is called an anti-derivation. This is the meaning of anti-derivation used throughout the present paper. Local and 2-local δ -derivations were introduced by Khudoyberdiyev and Yusupov [19], who studied the case $\delta = \frac{1}{2}$ on several finite-dimensional Lie algebras. Further results for solvable Lie algebras with a filiform nilradical were obtained by Atajonov, Vaisova and Yusupov [4].

The study of local and 2-local anti-derivations on solvable Lie algebras was initiated by Ayupov, Atajonov and Yusupov [6]. They investigated algebras with filiform, Heisenberg and abelian nilradicals and established both results asserting that local mappings are anti-derivations and examples where this fails. This direction was subsequently developed for solvable Lie algebras with a filiform nilradical [1], the perfect Lie algebras $\mathfrak{L}^m$ [5], and certain Witt-type infinite-dimensional Lie algebras [2]. Analogous questions for null-filiform, filiform and naturally graded quasi-filiform associative algebras were considered in [3].

The present paper contributes to the emerging study of local and 2-local anti-derivations on non-Lie Leibniz algebras. We consider the null-filiform algebra NF_n , the naturally graded non-Lie filiform algebras F_n^1 and F_n^2 , and their maximal solvable extensions. The nilpotent algebras belong to the classes studied by Ayupov and Omirov [12]. The relevant solvable extensions are described in the classification results of Casas, Ladra, Omirov and Karimjanov [13,14], together with the corrigendum by Ladra, Masutova and Omirov [20]. Biderivations of these classes were investigated by Yusupov, Jumaniyozov and Azizov [23]. In that setting, the term anti-derivation is used for the identity $A([x, y]) = [A(x), y] - [A(y), x]$; the present work concerns the $\delta = -1$ identity displayed above.

We first give complete descriptions of anti-derivations and local anti-derivations on the seven algebras under consideration. Among them, every local anti-derivation is an anti-derivation precisely for the solvable algebras R , $R(F_n^1)$ and $\mathfrak{L}_2$. The algebras NF_n , F_n^1 , F_n^2 and $\mathfrak{L}_1$ admit local anti-derivations which are not anti-derivations. For 2-local anti-derivations, the result is different: every 2-local anti-derivation on NF_n and on each of the four solvable extensions is an anti-derivation. On F_n^1 and F_n^2 , we construct families of nonlinear 2-local anti-derivations with values in the center. These families are defined by homogeneous mappings on the two-dimensional abelianization, and their members are anti-derivations exactly when the defining mappings are linear.

The proofs use the multiplication tables, coefficient comparisons and pointwise interpolation by anti-derivations. After recalling the necessary definitions and algebra presentations, we determine the anti-derivations in two propositions. The two subsequent theorems describe all local anti-derivations and establish the stated results for 2-local anti-derivations.

Preliminaries

All algebras and linear maps are over $\mathbb{C}$. The arguments also work over any field of characteristic zero. We assume $n \geq 2$ for NF_n and R , and $n \geq 4$ in the filiform cases. All products not explicitly listed are zero; skew-symmetry is not assumed. A sum with an empty index set is understood to be zero. A right Leibniz algebra is a vector space $\mathfrak{L}$ equipped with a bilinear multiplication satisfying

$$[x, [y, z]] = [[x, y], z] - [[x, z], y] \quad (x, y, z \in \mathfrak{L}).$$

Throughout the paper, by a Leibniz algebra we mean a right Leibniz algebra.

For a Leibniz algebra $\mathfrak{L}$, consider its lower central and derived series:

$$\begin{aligned} \mathfrak{L}^1 &= \mathfrak{L}, & \mathfrak{L}^{k+1} &= [\mathfrak{L}^k, \mathfrak{L}^1], & k &\geq 1, \\ \mathfrak{L}^{[1]} &= \mathfrak{L}, & \mathfrak{L}^{[s+1]} &= [\mathfrak{L}^{[s]}, \mathfrak{L}^{[s]}], & s &\geq 1. \end{aligned}$$

Definition 1. A Leibniz algebra $\mathfrak{L}$ is called nilpotent (respectively, solvable) if $\mathfrak{L}^p = 0$ (respectively, $\mathfrak{L}^{[q]} = 0$) for some $p, q \in \mathbb{N}$. The least such p (respectively, q) is called the nilpotency index (respectively, the solvability index) of $\mathfrak{L}$.

Note that any Leibniz algebra $\mathfrak{L}$ contains a unique maximal solvable (resp. nilpotent) ideal, called the radical (resp. nilradical) of the algebra. It is known that the maximal nilindex for the n -dimensional Leibniz algebras is equal to $n + 1$, and such Leibniz algebras are called null-filiform. In [12] it is proved that up to isomorphism, there exists a unique null-filiform Leibniz algebra, and the multiplication of this algebra is the following

$$NF_n : [e_i, e_1] = e_{i+1} \quad 1 \leq i \leq n-1.$$

Definition 2. An n -dimensional Leibniz algebra $\mathfrak{L}$ is said to be filiform if $\dim \mathfrak{L}^i = n - i$ for $2 \leq i \leq n$.

In [12], it is shown that up to isomorphism, there exist two non-Lie naturally graded filiform Leibniz algebras

- $F_n^1 : [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1$
- $F_n^2 : [e_1, e_1] = e_3, \quad [e_i, e_1] = e_{i+1}, \quad 3 \leq i \leq n-1.$

In the following theorems, the classification of solvable Leibniz algebras with null-filiform and naturally graded filiform nilradicals is presented.

Theorem 1.[14] Let R be a solvable Leibniz algebra whose nilradical is NF_n . Then there exists a basis $\{e_1, e_2, \dots, e_n, h\}$ of the algebra R such that the multiplication table of R with respect to this basis has the following form:

$$R : \begin{cases} [e_i, e_1] = e_{i+1}, & 1 \leq i \leq n-1, \\ [h, e_1] = -e_1, \\ [e_i, h] = ie_i, & 1 \leq i \leq n \end{cases}$$

Theorem 2.[20] Let $R(F_n^1)$ be $(n+2)$ -dimensional solvable Leibniz algebra with nilradical F_n^1 . Then it is isomorphic to the Leibniz algebra with a basis $\{e_1, e_2, \dots, e_n, h_1, h_2\}$ and the multiplication table:

$$\begin{aligned} [e_i, e_1] &= e_{i+1}, & 2 \leq i \leq n-1, & \quad [e_1, h_2] = e_1, \\ [e_i, h_1] &= e_i, & 2 \leq i \leq n, & \quad [e_i, h_2] = (i-1)e_i, \quad 2 \leq i \leq n, \\ & & & \quad [h_2, e_1] = -e_1. \end{aligned}$$

Although the product $[e_1, e_1]$ does not occur in this presentation, the nilradical is isomorphic to the standard algebra F_n^1 . Indeed, putting

$$f_1 = e_1 + e_2, \quad f_2 = e_2, \quad f_i = e_i \quad (i \geq 3),$$

we obtain

$$[f_1, f_1] = f_3, \quad [f_i, f_1] = f_{i+1} \quad (2 \leq i < n).$$

Thus the difference is only a choice of basis.

Theorem 3. [13] For $s \in \{1, 2\}$, put $\varepsilon_1 = 1, \varepsilon_2 = 0$. After interchanging h_1 and h_2 in the standard presentation, the algebra $\mathfrak{L}_s$ has basis $\{e_1, \dots, e_n, h_1, h_2\}$ and nonzero products

$$\mathfrak{L}_s : \begin{cases} [e_1, e_1] = e_3, \\ [e_i, e_1] = e_{i+1}, & 3 \leq i \leq n-1, \\ [e_1, h_2] = e_1, & [h_2, e_1] = -e_1, \\ [e_i, h_2] = (i-1)e_i, & 3 \leq i \leq n, \\ [e_2, h_1] = e_2, & [h_1, e_2] = -\varepsilon_s e_2. \end{cases}$$

Definition 3. Let $(\mathfrak{L}, [-, -])$ be an algebra over a field $\mathbb{F}$. A linear map $D : \mathfrak{L} \rightarrow \mathfrak{L}$ is called a δ -derivation if it satisfies

$$D([x, y]) = \delta([D(x), y] + [x, D(y)]),$$

for all $x, y \in \mathfrak{L}$, where $\delta \in \mathbb{F}$ is fixed.

The values $\delta = 1$, $\delta = -1$, and $\delta = \frac{1}{2}$ are especially significant. The case $\delta = 1$ gives the usual derivations, whereas $\delta = -1$ leads to anti-derivations.

For a fixed δ , we denote by $\text{Der}_\delta(\mathfrak{L})$ the set of all δ -derivations of $\mathfrak{L}$.

Remark 1. In the literature on Leibniz biderivations, anti-derivation also denotes a map satisfying $A([x, y]) = [A(x), y] - [A(y), x]$ for a right Leibniz algebra; see [23, Definition 2.2]. That identity is different from

$$D([x, y]) = -([D(x), y] + [x, D(y)]). \quad (1)$$

All the results below concern precisely (1).

Definition 4. A linear map Δ is called a local δ -derivation, if for any $x \in \mathfrak{L}$, there exists a δ -derivation $D_x : \mathfrak{L} \rightarrow \mathfrak{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local δ -derivations on $\mathfrak{L}$ we denote by $\text{LocDer}_\delta(\mathfrak{L})$.

Definition 5. A map $\nabla : \mathfrak{L} \rightarrow \mathfrak{L}$ (not necessarily linear) is called a 2-local δ -derivation, if for any $x, y \in \mathfrak{L}$, there exists a δ -derivation $D_{x,y} \in \text{Der}_\delta(\mathfrak{L})$ such that

$$\nabla(x) = D_{x,y}(x), \quad \nabla(y) = D_{x,y}(y).$$

It should be noted that a 2-local δ -derivation is not necessarily linear. However, for any $x \in \mathfrak{L}$ and any scalar λ , we have

$$\nabla(\lambda x) = D_{x,\lambda x}(\lambda x) = \lambda D_{x,\lambda x}(x) = \lambda \nabla(x).$$

Main part

In this section, we describe anti-derivations on null-filiform and filiform Leibniz algebras and their solvable extensions. We then apply these descriptions to local and 2-local anti-derivations.

Proposition 1. Let L be one of the complex Leibniz algebras NF_n , F_n^1 , and F_n^2 , with basis $\{e_1, \dots, e_n\}$. A linear map $D : L \rightarrow L$ is an anti-derivation if and only if it has the form specified below for the corresponding algebra. In each case, the parameters are arbitrary and independent.

(i)

$$\text{NF}_n : \begin{cases} D(e_1) = \sum_{j=1}^n a_j e_j, \\ D(e_i) = \kappa_i a_1 e_i + (-1)^{i-1} \sum_{j=i+1}^n a_{j-i+1} e_j, \quad 2 \leq i \leq n. \end{cases}$$

where

$$\kappa_i = \begin{cases} 1, & i \text{ is odd,} \\ -2, & i \text{ is even.} \end{cases} \quad (2)$$

In particular, $\dim \text{Der}_{-1}(\text{NF}_n) = n$.

(ii)

$$F_n^1 : \begin{cases} D(e_1) = \sum_{j=1}^n a_j e_j, \\ D(e_2) = (a_1 + a_2)e_2 + \sum_{j=3}^{n-1} a_j e_j + b e_n, \\ D(e_i) = \theta_i e_i + (-1)^i \sum_{j=i+1}^n a_{j-i+2} e_j, \quad 3 \leq i \leq n, \end{cases}$$

where

$$\theta_i = \begin{cases} a_1 + a_2, & i \text{ is even,} \\ -2a_1 - a_2, & i \text{ is odd.} \end{cases}$$

In particular, $\dim \text{Der}_{-1}(F_n^1) = n + 1$.

(iii)

$$F_n^2 : \begin{cases} D(e_1) = \sum_{j=1}^n a_j e_j, \\ D(e_2) = b e_2 + c e_n, \\ D(e_i) = \kappa_{i-1} a_1 e_i + (-1)^i \sum_{j=i+1}^n a_{j-i+2} e_j, \quad 3 \leq i \leq n, \end{cases}$$

where κ_i is defined by (2). In particular, $\dim \text{Der}_{-1}(F_n^2) = n + 2$.

Proof. We first obtain a common reduction for the three algebras. Define

$$\phi \left(\sum_{j=1}^n x_j e_j \right) = x_1, \quad Tx = [x, e_1].$$

Their multiplication tables give $[x, y] = \phi(y)Tx$, with $T \neq 0$. Consequently, (1) is equivalent to

$$\phi(y)(DT + TD)x + \phi(Dy)Tx = 0 \quad (x, y \in L).$$

For $y \in \ker \phi$, choosing x with $Tx \neq 0$ yields $\phi(Dy) = 0$. If $D(e_1) = \sum_{j=1}^n a_j e_j$, it follows that $\phi D = a_1 \phi$. Taking $y = e_1$ then gives the second identity in

$$\phi D = a_1 \phi, \quad DT + TD = -a_1 T. \quad (3)$$

Conversely, these identities imply (1) for every ordered pair x, y , including all pairs with zero product. Thus they are necessary and sufficient. In particular, $D(e_j)$ has no e_1 component for $j \geq 2$. Whenever $Te_i = e_{i+1}$, they give

$$D(e_{i+1}) = -TD(e_i) - a_1 e_{i+1}. \quad (4)$$

Case (i): $L = \text{NF}_n$ Here $Te_i = e_{i+1}$ for $i < n$ and $Te_n = 0$. Applying (4) to e_1 gives

$$D(e_2) = -2a_1 e_2 - \sum_{j=3}^n a_{j-1} e_j.$$

If d_i denotes the coefficient of e_i in $D(e_i)$, then $d_1 = a_1$ and $d_{i+1} = -d_i - a_1$. Hence $d_i = \kappa_i a_1$. Under (4), each remaining term changes sign and moves one step along the chain. Induction therefore gives the formula in (i).

Case (ii): $L = F_n^1$ Here $Te_1 = Te_2 = e_3$, $Te_i = e_{i+1}$ for $3 \leq i < n$, and $Te_n = 0$. Write

$$D(e_1) = \sum_{j=1}^n a_j e_j, \quad D(e_2) = \sum_{j=2}^n b_j e_j.$$

Applying (3) to e_1 and e_2 gives, respectively,

$$\begin{aligned} D(e_3) &= -(2a_1 + a_2)e_3 - \sum_{j=3}^{n-1} a_j e_{j+1}, \\ D(e_3) &= -(b_2 + a_1)e_3 - \sum_{j=3}^{n-1} b_j e_{j+1}. \end{aligned}$$

Comparing coefficients, we obtain

$$b_2 = a_1 + a_2, \quad b_j = a_j \quad (3 \leq j \leq n-1).$$

Since $Te_n = 0$, the coefficient b_n remains free; denote it by b . For $3 \leq i < n$, relation (4) determines $D(e_{i+1})$. Its diagonal coefficient alternates between $-2a_1 - a_2$ and $a_1 + a_2$, starting with $D(e_3)$. The other coefficients change sign and shift one place at each step. This proves all formulas in (ii).

Case (iii): $L = F_n^2$ In this case,

$$Te_1 = e_3, \quad Te_2 = 0, \quad Te_i = e_{i+1} \quad (3 \leq i < n), \quad Te_n = 0.$$

Thus $\ker T = \text{span}\{e_2, e_n\}$. Applying (3) to e_2 gives $TD(e_2) = 0$, and hence $D(e_2) = be_2 + ce_n$. Applying it to e_1 gives

$$D(e_3) = -2a_1e_3 - \sum_{j=3}^{n-1} a_j e_{j+1}.$$

Relation (4) now determines the remaining values. The diagonal coefficients of $D(e_3), D(e_4), \dots$ alternate between $-2a_1$ and a_1 , while the other terms change sign and shift along the chain. This gives (iii).

We verify sufficiency for the three cases together. Every displayed map satisfies $\phi D = a_1 \phi$. In (i), relation (4) verifies the second identity of (3) on $e_1, \dots, e_{n-1}$. In (ii), the two expressions for $D(e_3)$ verify it on e_1, e_2 , and the recurrence verifies it on $e_3, \dots, e_{n-1}$. In (iii), the expression for $D(e_3)$ and the recurrence verify it along the chain, while $D(e_2) \in \ker T$ verifies it on e_2 . Finally, in all three cases $D(e_n)$ is a scalar multiple of e_n , so the identity holds on e_n as well. By the reduction above, every displayed map is an anti-derivation.

The parameters are independent: $D(e_1)$ determines $a_1, \dots, a_n$; in (ii), the e_n coefficient of $D(e_2)$ determines b ; in (iii), the two coefficients of $D(e_2)$ determine b, c . The dimension assertions follow.

Proposition 2. Let L be one of the solvable Leibniz algebras $R, R(F_n^1), \mathfrak{L}_1$, and $\mathfrak{L}_2$ described above. A linear map $D : L \rightarrow L$ is an anti-derivation if and only if it has the corresponding form below.

- (i) $\text{Der}_{-1}(R) = \{0\}, \quad \dim \text{Der}_{-1}(R) = 0.$
- (ii) $\text{Der}_{-1}(R(F_n^1)) = \{0\}, \quad \dim \text{Der}_{-1}(R(F_n^1)) = 0.$
- (iii)

$$\mathfrak{L}_s : \begin{cases} D(e_2) = \alpha e_2, \\ D(h_1) = -2\alpha h_1 + \beta e_2, \\ D(h_2) = 0, \\ D(e_i) = 0, \quad 1 \leq i \leq n, \quad i \neq 2, \end{cases}$$

where $\alpha \in \mathbb{C}$ is arbitrary, while $\beta \in \mathbb{C}$ is arbitrary for $s = 1$ and $\beta = 0$ for $s = 2$. Consequently,

$$\dim \text{Der}_{-1}(\mathfrak{L}_1) = 2, \quad \dim \text{Der}_{-1}(\mathfrak{L}_2) = 1.$$

Proof. Equation (1) immediately gives $D([L, L]) \subseteq [L, L]$. For every algebra in the statement, $[L, L] = N = \text{span}\{e_1, \dots, e_n\}$; hence $D(N) \subseteq N$.

Case (i): $L = R$ Write

$$D(e_1) = \sum_{j=1}^n a_j e_j, \quad D(h) = \beta h + \sum_{j=1}^n u_j e_j.$$

The zero product $[h, h] = 0$ gives

$$0 = [D(h), h] + [h, D(h)] = \sum_{j=2}^n j u_j e_j.$$

Therefore $D(h) = \beta h + \gamma e_1$. Applying (1) to $[e_1, h] = e_1$ and $[h, e_1] = -e_1$ gives

$$\begin{aligned} D(e_1) &= - \sum_{j=1}^n j a_j e_j - \beta e_1 - \gamma e_2, \\ -D(e_1) &= (\beta + a_1)e_1 - \gamma e_2. \end{aligned}$$

Comparing coefficients, we obtain

$$2a_1 = -\beta, \quad 3a_2 = -\gamma, \quad a_2 = \gamma, \quad a_j = 0 \quad (3 \leq j \leq n).$$

Thus $a_2 = \gamma = 0$, and

$$D(e_1) = a_1 e_1, \quad D(h) = -2a_1 h.$$

Since $[e_1, e_1] = e_2$, we have $D(e_2) = -2a_1 e_2$. Applying (1) to $[e_2, h] = 2e_2$ now gives

$$-4a_1 e_2 = -[-2a_1 e_2, h] - [e_2, -2a_1 h] = 8a_1 e_2.$$

Hence $12a_1 = 0$, so $a_1 = 0$. We have $D(h) = D(e_1) = 0$; induction along $e_{i+1} = [e_i, e_1]$ shows that D vanishes everywhere. The zero map satisfies (1).

Case (ii): $L = R(F_n^1)$ Initially write

$$D(h_1) = ah_1 + bh_2 + \sum_{i=1}^n u_i e_i,$$

$$D(h_2) = ch_1 + dh_2 + \sum_{i=1}^n v_i e_i.$$

The zero products $[h_1, h_1] = [h_2, h_2] = 0$ yield

$$\sum_{i=2}^n u_i e_i = 0, \quad \sum_{i=2}^n (i-1)v_i e_i = 0.$$

Next, $[h_1, h_2] = 0$ gives $u_1 = 0$. Consequently,

$$D(h_1) = ah_1 + bh_2, \quad D(h_2) = ch_1 + dh_2 + ve_1.$$

Write $D(e_1) = \sum_{i=1}^n p_i e_i$. Applying (1) to $[e_1, h_1] = 0$ gives

$$0 = \sum_{i=2}^n p_i e_i + be_1.$$

Thus $b = 0$ and $D(e_1) = pe_1$. For each $2 \leq i \leq n$, the product $[e_i, h_1] = e_i$ yields

$$D(e_i) = -[D(e_i), h_1] - ae_i.$$

The e_1 coefficient on the right is zero. Comparing all other coefficients then gives

$$D(e_i) = qe_i \quad (2 \leq i \leq n), \quad q = -\frac{a}{2}.$$

The product $[e_1, h_2] = e_1$ gives $d = -2p$. Next, $[e_2, h_2] = e_2$ gives

$$qe_2 = -(q + c + d)e_2 - ve_3,$$

so $v = 0$ and $c + d = -2q$. Using $[e_3, h_2] = 2e_3$ then gives

$$2qe_3 = -(2q + c + 2d)e_3,$$

and hence $c + 2d = -4q$. It follows that $d = -2q$, $c = 0$, and $p = q$. Finally, $[e_2, e_1] = e_3$ gives

$$qe_3 = D(e_3) = -[qe_2, e_1] - [e_2, pe_1] = -(p + q)e_3.$$

Thus $p + 2q = 0$, and therefore $p = q = 0$. All coefficients of $D(h_1), D(h_2)$ vanish as well, so $D = 0$. Again, sufficiency is immediate.

Case (iii): $L = \mathfrak{L}_s$, $s = 1, 2$ Set

$$A = \text{span}\{h_2, e_1, e_3, \dots, e_n\}, \quad B_s = \text{span}\{h_1, e_2\}.$$

The multiplication table gives a direct sum of ideals $\mathfrak{L}_s = A \oplus B_s$, with $[A, B_s] = [B_s, A] = 0$. Under the relabelling

$$z_1 = e_1, \quad z_j = e_{j+1} \quad (2 \leq j \leq n-1), \quad h = h_2,$$

the algebra A has the table in (i), with null-filiform nilradical of dimension $n-1$. Thus $\text{Der}_{-1}(A) = 0$ by Case (i).

We justify that D preserves both summands. Their two-sided centers are zero. Indeed, if $z = th_2 + \sum_{j=1}^{n-1} t_j z_j \in Z(A)$, then

$$[z, h_2] = \sum_{j=1}^{n-1} j t_j z_j = 0$$

forces all $t_j = 0$, and $[th_2, z_1] = -tz_1 = 0$ forces $t = 0$. Likewise, for $w = uh_1 + ve_2 \in Z(B_s)$, the equalities $[e_2, w] = ue_2 = 0$ and $[w, h_1] = ve_2 = 0$ give $u = v = 0$.

For $x \in A$ and $y \in B_s$, decompose

$$D(x) = D_A(x) + f(x), \quad D(y) = g(y) + D_B(y),$$

where $D_A(x), g(y) \in A$ and $f(x), D_B(y) \in B_s$. Applying (1) to $[x, y] = 0$ gives

$$0 = [f(x), y] + [x, g(y)].$$

The two terms lie in distinct direct summands, so each is zero. Applying the identity to $[y, x] = 0$ also gives $[y, f(x)] = [g(y), x] = 0$. Since these identities hold for every $x \in A$ and $y \in B_s$, we obtain $f(x) \in Z(B_s) = 0$ and $g(y) \in Z(A) = 0$. Thus D preserves A, B_s , and its restrictions are anti-derivations. Since $\text{Der}_{-1}(A) = 0$, only the restriction to B_s remains to be determined.

Write

$$D(e_2) = \alpha e_2 + \rho h_1, \quad D(h_1) = \beta e_2 + \sigma h_1.$$

The product $[e_2, h_1] = e_2$ yields

$$\alpha e_2 + \rho h_1 = -(\alpha + \sigma)e_2.$$

Hence $\rho = 0$ and $\sigma = -2\alpha$. The zero product $[h_1, h_1] = 0$ gives

$$0 = [D(h_1), h_1] + [h_1, D(h_1)] = (1 - \varepsilon_s)\beta e_2.$$

Therefore β is arbitrary for $s = 1$ and equals zero for $s = 2$.

For sufficiency, the two identities just used hold whenever these conditions are satisfied. For the remaining ordered basis pairs in B_s , both sides of (1) vanish for $[e_2, e_2] = 0$, and

$$D([h_1, e_2]) = -\varepsilon_s \alpha e_2 = -[D(h_1), e_2] - [h_1, D(e_2)].$$

Thus all four ordered basis pairs satisfy (1). Extending these maps by zero on A preserves the identity because all mixed products vanish. This proves the stated classifications. The free parameters α, β for $\mathfrak{L}_1$ and α for $\mathfrak{L}_2$ are independent, giving the dimensions 2 and 1.

Theorem 4. Let L be one of the complex Leibniz algebras considered in Propositions 1 and 2, with the bases and dimension assumptions used there. A linear map $\Delta : L \rightarrow L$ is a local anti-derivation if and only if it has the corresponding form below.

(i)

$$\text{NF}_n : \quad \Delta(e_i) = \sum_{j=i}^n b_{ji} e_j \quad (1 \leq i \leq n),$$

where all $b_{ji} \in \mathbb{C}$ are arbitrary. In particular, $\dim \text{LocDer}_{-1}(\text{NF}_n) = n(n+1)/2$.

(ii)

$$F_n^1 : \begin{cases} \Delta(e_1) = \sum_{j=1}^n a_j e_j, \\ \Delta(e_2) = (a_1 + a_2)e_2 + \sum_{j=3}^{n-1} a_j e_j + b e_n, \\ \Delta(e_i) = \sum_{j=i}^n c_{ji} e_j \quad (3 \leq i \leq n), \end{cases}$$

where $a_1, \dots, a_n, b$ and all c_{ji} are arbitrary and independent. In particular, $\dim \text{LocDer}_{-1}(F_n^1) = n(n-1)/2 + 2$.

(iii)

$$F_n^2 : \begin{cases} \Delta(e_1) = \sum_{j=1}^n a_j e_j, \\ \Delta(e_2) = b e_2 + c e_n, \\ \Delta(e_i) = \sum_{j=i}^n d_{ji} e_j \quad (3 \leq i \leq n), \end{cases}$$

where $a_1, \dots, a_n, b, c$ and all d_{ji} are arbitrary and independent. In particular, $\dim \text{LocDer}_{-1}(F_n^2) = n(n-1)/2 + 3$.

(iv) For $L = R$ or $L = R(F_n^1)$, we have $\Delta = 0$. Thus the corresponding spaces of local anti-derivations have dimension zero.

(v)

$$\mathfrak{L}_s : \begin{cases} \Delta(e_2) = \alpha e_2, \\ \Delta(h_1) = \beta e_2 + \gamma h_1, \\ \Delta(h_2) = 0, \\ \Delta(e_i) = 0 \quad (1 \leq i \leq n, i \neq 2), \end{cases}$$

where $s \in \{1, 2\}$ and $\alpha, \beta, \gamma \in \mathbb{C}$ are arbitrary for $s = 1$, whereas $\beta = 0$ and $\gamma = -2\alpha$ for $s = 2$. Consequently,

$$\dim \text{LocDer}_{-1}(\mathfrak{L}_1) = 3, \quad \dim \text{LocDer}_{-1}(\mathfrak{L}_2) = 1.$$

Among these seven algebras, every local anti-derivation is an anti-derivation precisely for R , $R(F_n^1)$, and $\mathfrak{L}_2$.

Proof. For an algebra L and $x \in L$, write

$$\text{Der}_{-1}(L)x = \{D(x) : D \in \text{Der}_{-1}(L)\}.$$

By definition, a linear map Δ is a local anti-derivation if and only if $\Delta(x) \in \text{Der}_{-1}(L)x$ for every x .

Case (i): $L = \text{NF}_n$. Put $E_k = \text{span}\{e_k, \dots, e_n\}$. We claim that, for every nonzero $x = \sum_{i=k}^n x_i e_i$ with $x_k \neq 0$,

$$\text{Der}_{-1}(\text{NF}_n)x = E_k.$$

The inclusion into E_k follows from Proposition 1(i). Conversely, prescribe $y = \sum_{j=k}^n y_j e_j \in E_k$ and use the parameters $a_1, \dots, a_n$ from that proposition. The coefficient of e_k in $D(x)$ is $\kappa_k a_1 x_k$, so we can choose a_1 to obtain y_k . For each $j > k$, the coefficient of e_j is

$$\kappa_j a_1 x_j + \sum_{i=k}^{j-1} (-1)^{i-1} x_i a_{j-i+1}.$$

The new parameter a_{j-k+1} occurs with nonzero coefficient $(-1)^{k-1} x_k$. Choosing these parameters successively gives $D(x) = y$. This proves the claim, including $k = n$.

Locality at each e_i now forces $\Delta(e_i) \in E_i$. Conversely, a linear map with these values sends every $x \in E_k$ into E_k , and the claim provides an anti-derivation agreeing with it at x . This proves (i).

Case (ii): $L = F_n^1$. Use the auxiliary basis

$$u = e_1 - e_2, \quad v_j = e_{j+1} \quad (1 \leq j \leq m), \quad m = n - 1.$$

The nonzero products in this basis are $[v_j, u] = v_{j+1}$ for $1 \leq j < m$. Proposition 1(ii), expressed in this basis, gives

$$D(u) = au + rv_m, \quad D(v_j) = \lambda_j v_j + (-1)^{j-1} \sum_{\ell=j+1}^m c_{\ell-j+1} v_\ell,$$

where $a, r, b, c_2, \dots, c_m$ are arbitrary and

$$\lambda_j = \begin{cases} b, & j \text{ odd}, \\ -a - b, & j \text{ even}. \end{cases}$$

Indeed, one may choose $D(v_1) = bv_1 + \sum_{\ell=2}^m c_\ell v_\ell$ arbitrarily; the other values follow from $D(v_{j+1}) = -[D(v_j), u] - av_{j+1}$.

Let $W_k = \text{span}\{v_k, \dots, v_m\}$. Locality at u and at the v_j implies

$$\Delta(u) \in \text{span}\{u, v_m\}, \quad \Delta(v_j) \in W_j \quad (1 \leq j \leq m).$$

These conditions are also sufficient. To see this, write $\Delta(u) = au + rv_m$ and take $x = tu + w$. If $w = 0$, any anti-derivation with this value on u agrees with Δ at x . If $w = \sum_{j=k}^m w_j v_j$ with $w_k \neq 0$, then $\Delta(w) \in W_k$. Keep a, r fixed and choose b so that the v_k coefficient of $D(w)$ equals that of $\Delta(w)$. This is possible because the coefficient of b there is $(-1)^{k-1} w_k \neq 0$. For each $\ell > k$, the new parameter $c_{\ell-k+1}$ occurs in the v_ℓ coefficient with nonzero coefficient $(-1)^{k-1} w_k$. Successive choices therefore give $D(w) = \Delta(w)$ and hence $D(x) = \Delta(x)$.

Returning to the original basis, write $\Delta(e_1) = \sum a_j e_j$ and $\Delta(e_2) = \sum_{j=2}^n b_j e_j$. The condition on $\Delta(u) = \Delta(e_1 - e_2)$ gives

$$b_2 = a_1 + a_2, \quad b_j = a_j \quad (3 \leq j < n),$$

while b_n is free. The other conditions are $\Delta(e_i) \in \text{span}\{e_i, \dots, e_n\}$ for $i \geq 3$. These are exactly the formulas in (ii).

Case (iii): $L = F_n^2$. Write

$$F_n^2 = A \oplus \mathbb{C}z, \quad A = \text{span}\{e_1, e_3, \dots, e_n\}, \quad z = e_2.$$

Here z is central and A is a null-filiform algebra of dimension $n - 1$, with ordered basis $e_1, e_3, \dots, e_n$. Proposition 1(iii) shows that every anti-derivation has the form

$$D(x + tz) = D_A(x) + cte_n + (ax_1 + bt)z,$$

where $D_A \in \text{Der}_{-1}(A)$, the parameters a, b, c are arbitrary, and x_1 is the coefficient of e_1 in $x \in A$. In particular, locality at the basis vectors implies all the conditions in (iii).

Conversely, a map with the stated values can be written as

$$\Delta(x + tz) = P(x) + cte_n + (ax_1 + bt)z,$$

where $P : A \rightarrow A$ preserves every tail of the ordered basis of A . By Case (i), P is a local anti-derivation of A . For any fixed $x + tz$, choose D_A with $D_A(x) = P(x)$ and use the same a, b, c in the preceding formula for D . Then $D(x + tz) = \Delta(x + tz)$, proving sufficiency.

Case (iv): $L = R$ or $L = R(F_n^1)$. By Proposition 2, these algebras have only the zero anti-derivation. Locality therefore gives $\Delta(x) = 0$ for every $x \in L$.

Case (v): $L = \mathfrak{L}_s$. Proposition 2 and locality at the basis vectors first give

$$\Delta(e_2) = \alpha e_2, \quad \Delta(h_1) = \beta e_2 + \gamma h_1,$$

and zero values on h_2 and all e_i with $i \neq 2$. For $s = 1$, there are no further conditions. Indeed, write $x = x_0 + te_2 + uh_1$, where $x_0 \in \text{span}\{h_2, e_1, e_3, \dots, e_n\}$. An anti-derivation with parameters p, q from Proposition 2 has

$$D(x) = (pt + qu)e_2 - 2puh_1.$$

If $u \neq 0$, take

$$p = -\frac{\gamma}{2}, \quad q = \beta + \left(\alpha + \frac{\gamma}{2}\right) \frac{t}{u}.$$

If $u = 0$, take $p = \alpha$ and $q = 0$. In either case $D(x) = \Delta(x)$.

For $s = 2$, locality at h_1 forces $\beta = 0$. Locality at $e_2 + h_1$ then gives a scalar p such that

$$\alpha e_2 + \gamma h_1 = pe_2 - 2ph_1,$$

so $\gamma = -2\alpha$. Conversely, each map with these values is already an anti-derivation by Proposition 2.

Finally, the dimensions follow by counting the free parameters. In (ii) and (iii), the free triangular part contributes $(n-2)(n-1)/2$ parameters. Thus the dimensions are, respectively, $n+1+(n-2)(n-1)/2 = n(n-1)/2+2$ and $n+2+(n-2)(n-1)/2 = n(n-1)/2+3$. Comparison with the dimensions in Propositions 1 and 2 proves the last assertion.

Theorem 5. For the algebras in Propositions 1 and 2, the following statements hold.

- (i) Every 2-local anti-derivation on NF_n , R , $R(F_n^1)$, $\mathfrak{L}_1$, or $\mathfrak{L}_2$ is an anti-derivation.
- (ii) Let $L = F_n^1$ or $L = F_n^2$, and let $\Psi : \mathbb{C}^2 \rightarrow Z(L)$ satisfy

$$\Psi(\lambda s, \lambda t) = \lambda \Psi(s, t) \quad (\lambda, s, t \in \mathbb{C}),$$

where

$$Z(F_n^1) = \mathbb{C}e_n, \quad Z(F_n^2) = \text{span}\{e_2, e_n\}.$$

Then the map

$$\nabla_\Psi \left(\sum_{i=1}^n x_i e_i \right) = \Psi(x_1, x_2)$$

is a 2-local anti-derivation. It is an anti-derivation if and only if Ψ is linear. In particular, both F_n^1 and F_n^2 admit nonlinear 2-local anti-derivations.

Proof. We first make an elementary observation. Suppose that an algebra L has an element v for which

$$D \in \text{Der}_{-1}(L), \quad D(v) = 0 \implies D = 0.$$

Let ∇ be a 2-local anti-derivation. Choose an anti-derivation D_0 with $\nabla(v) = D_0(v)$. For any $x \in L$, 2-locality provides an anti-derivation $D_{v,x}$ satisfying

$$D_{v,x}(v) = \nabla(v) = D_0(v), \quad D_{v,x}(x) = \nabla(x).$$

The anti-derivation $D_{v,x} - D_0$ vanishes at v , so it is zero. Hence $\nabla(x) = D_0(x)$ for every x , and $\nabla = D_0$. No linearity assumption on ∇ is needed.

For NF_n , Proposition 1(i) shows that $v = e_1$ has this property, because $D(e_1)$ determines every parameter of D . For $\mathfrak{L}_1$ and $\mathfrak{L}_2$, take $v = h_1$: the values $D(h_1) = -2\alpha h_1 + \beta e_2$ and $D(h_1) = -2\alpha h_1$, respectively, determine all parameters. For R and $R(F_n^1)$, Proposition 2 gives $\text{Der}_{-1}(L) = \{0\}$, so every 2-local anti-derivation is zero. This proves (i).

To prove (ii), observe that for both filiform algebras

$$[L, L] = \text{span}\{e_3, \dots, e_n\}.$$

Their multiplication tables also give the stated two-sided centers. For arbitrary $p, q \in Z(L)$, define a linear map by

$$D_{p,q}(e_1) = p, \quad D_{p,q}(e_2) = q, \quad D_{p,q}(e_i) = 0 \quad (3 \leq i \leq n).$$

This map annihilates $[L, L]$ and takes its values in $Z(L)$. Consequently, both sides of

$$D_{p,q}([x, y]) = -[D_{p,q}(x), y] - [x, D_{p,q}(y)]$$

vanish for all x, y , so $D_{p,q}$ is an anti-derivation.

Now fix $x = \sum x_i e_i$ and $y = \sum y_i e_i$. We seek $p, q \in Z(L)$ satisfying

$$x_1 p + x_2 q = \Psi(x_1, x_2), \quad y_1 p + y_2 q = \Psi(y_1, y_2).$$

If $x_1 y_2 - x_2 y_1 \neq 0$, this system has a unique solution. If the two coordinate pairs are dependent and one is nonzero, homogeneity of Ψ makes the two equations consistent, and one can solve the equation corresponding to the nonzero pair. If both pairs are zero, choose $p = q = 0$, using $\Psi(0, 0) = 0$. Thus in every case an anti-derivation $D_{p,q}$ satisfies

$$D_{p,q}(x) = \nabla_{\Psi}(x), \quad D_{p,q}(y) = \nabla_{\Psi}(y).$$

This proves 2-locality, including pairs with zero or dependent images in $L/[L, L]$.

If Ψ is linear, then $\Psi(s, t) = s\Psi(1, 0) + t\Psi(0, 1)$, and $\nabla_{\Psi} = D_{\Psi(1,0), \Psi(0,1)}$ is an anti-derivation. Conversely, if ∇_{Ψ} is an anti-derivation, its linearity on $\text{span}\{e_1, e_2\}$ implies that Ψ is linear.

For an explicit nonlinear example on either algebra, take

$$\Psi(s, t) = \begin{cases} \frac{t^2}{s} e_n, & s \neq 0, \\ 0, & s = 0. \end{cases}$$

This map satisfies the required homogeneity for all complex scalars. The resulting 2-local anti-derivation satisfies

$$\nabla_{\Psi}(e_1) = \nabla_{\Psi}(e_2) = 0, \quad \nabla_{\Psi}(e_1 + e_2) = e_n,$$

so it is not additive and hence is not an anti-derivation.

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