

ROTA-BAXTER OPERATORS ON A SIX-DIMENSIONAL SIMPLE LEIBNIZ ALGEBRA

ASKAROV JANDOS*

KARAKALPAK STATE UNIVERSITY NAMED AFTER BERDAKH, NUKUS, UZBEKISTAN
askarovjandos64@gmail.com

KURBANBAEV TUUEL BAY

KARAKALPAK STATE UNIVERSITY NAMED AFTER BERDAKH, NUKUS, UZBEKISTAN
V.I.ROMANOVSKIY INSTITUTE OF MATHEMATICS, UZBEKISTAN ACADEMY OF SCIENCES
tuwelbay2023@gmail.com

ABSTRACT. We study Rota-Baxter operators of weight zero on a six-dimensional simple Leibniz algebra whose associated Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ and whose Leibniz kernel is the three-dimensional irreducible $\mathfrak{sl}_2$ -module $V(2)$. Using a fixed module isomorphism between the Lie part and the Leibniz kernel, we represent an arbitrary linear operator in block form and reduce the Rota-Baxter identity to a system of operator equations on $\mathfrak{sl}_2$. We completely describe the case in which the block mapping the Leibniz kernel to the Lie part vanishes. When the Lie component is zero, we obtain an intrinsic description of all admissible operators in terms of the Killing form. For a nonzero Lie component, we use the known classification of weight-zero Rota-Baxter operators on $\mathfrak{sl}_2$ to determine the compatible remaining blocks and the dimensions of the corresponding solution spaces. Several exceptional parameter values at which these dimensions increase are also identified. The cases in which the block from the Leibniz kernel to the Lie part is nonzero are left for further investigation.

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Key words: Leibniz algebra, simple Leibniz algebra, Rota-Baxter operator, $\mathfrak{sl}_2$ -module, Killing form.

Introduction

Leibniz algebras form a non-antisymmetric generalization of Lie algebras. They were first considered by Blokh and later systematically developed by Loday [3], [4]. Their structure theory is closely related to Lie theory, since the quotient by the ideal generated by squares is a Lie algebra. This makes it possible to combine methods from the theory of Lie algebras and their representations with phenomena that arise specifically in the non-antisymmetric setting.

The structure, representations, derivations, and automorphisms of finite-dimensional Leibniz algebras have been investigated extensively. In particular, simple and semisimple Leibniz algebras admit descriptions in terms of simple Lie algebras and their modules; see, for example, [5], [6], [1], [2]. The case in which the associated Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ is especially suitable for explicit classification problems, since the corresponding Leibniz algebras can be described using finite-dimensional irreducible $\mathfrak{sl}_2$ -modules [6].

Rota-Baxter operators constitute an important class of algebraic operators and are closely connected with classical Yang-Baxter type equations. In the Lie algebra setting, weight-zero Rota-Baxter operators on $\mathfrak{sl}_2(\mathbb{C})$ were completely classified by Pei, Bai and Guo [7]. More recently, Rota-Baxter and relative Rota-Baxter operators have also been studied in the framework of Leibniz algebras. Tang and Sheng established their relation with Leibniz bialgebras, Maurer-Cartan elements, and the classical Leibniz Yang-Baxter equation [8].

In the present paper, we study weight-zero Rota-Baxter operators on the six-dimensional simple Leibniz algebra whose associated Lie algebra is $\mathfrak{sl}_2(\mathbb{C})$ and whose Leibniz kernel is the three-dimensional irreducible module $V(2)$. Since the Lie part and the Leibniz kernel are isomorphic as $\mathfrak{sl}_2$ -modules, a fixed module isomorphism allows us to transfer all blocks of a linear operator to endomorphisms of $\mathfrak{sl}_2$. In this way, the Rota-Baxter identity is reduced to a finite system of operator equations on $\mathfrak{sl}_2$.

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*Corresponding autor

We first consider the case in which the block mapping the Leibniz kernel to the Lie part vanishes. When the Lie component also vanishes, we obtain an intrinsic description of the remaining blocks in terms of the Killing form. For a nonzero Lie component, we use the Pei-Bai-Guo classification of weight-zero Rota-Baxter operators on $\mathfrak{sl}_2$ and determine the corresponding compatible blocks. This yields a complete description of the weight-zero Rota-Baxter operators in the case where the block from the Leibniz kernel to the Lie part is zero.

The remaining cases, in which this block is nonzero, are not treated in the present paper and are left for further investigation.

The paper is organized as follows. We first recall the basic notions concerning Leibniz algebras and Rota-Baxter operators. We then describe the six-dimensional simple Leibniz algebra under consideration and derive the corresponding block equations. The subsequent sections are devoted to the classification in the case where the block from the Leibniz kernel to the Lie part vanishes. Finally, we summarize the obtained results and indicate the remaining open cases.

Preliminaries

Definition 1. A *Lie algebra* is a vector space $\mathfrak{g}$ endowed with a bilinear multiplication $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \rightarrow \mathfrak{g}$ such that

$$\begin{aligned} [x, x] &= 0, \\ [x, [y, z]] &= [[x, y], z] + [y, [x, z]] \end{aligned}$$

for all $x, y, z \in \mathfrak{g}$.

Definition 2. A *right Leibniz algebra* is a vector space $\mathcal{L}$ endowed with a bilinear multiplication $[\cdot, \cdot] : \mathcal{L} \times \mathcal{L} \rightarrow \mathcal{L}$ satisfying the right Leibniz identity

$$[[x, y], z] = [[x, z], y] + [x, [y, z]]$$

for all $x, y, z \in \mathcal{L}$.

In contrast to Lie algebras, the multiplication of a Leibniz algebra need not be skew-symmetric. The ideal

$$\mathcal{I} = \text{span}\{[x, x] \mid x \in \mathcal{L}\}$$

is called the *Leibniz kernel* of $\mathcal{L}$. For a right Leibniz algebra one has $[\mathcal{L}, \mathcal{I}] = 0$, and the quotient algebra $\mathcal{L}/\mathcal{I}$ is a Lie algebra.

Definition 3. A Leibniz algebra $\mathcal{L}$ is called *simple* if $[\mathcal{L}, \mathcal{L}] \neq \mathcal{I}$ and its only ideals are $\{0\}$, $\mathcal{I}$, $\mathcal{L}$.

In particular, if $\mathcal{L}$ is a simple non-Lie Leibniz algebra, then $\mathcal{L}/\mathcal{I}$ is a simple Lie algebra and $\mathcal{I}$ is an irreducible module over this associated Lie algebra.

Definition 4. Let $\mathcal{L}$ be a Leibniz algebra and let $\lambda \in \mathbb{C}$. A linear operator $R : \mathcal{L} \rightarrow \mathcal{L}$ is called a *Rota-Baxter operator of weight λ* if

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)] + \lambda[x, y])$$

for all $x, y \in \mathcal{L}$.

In the present paper, we consider only Rota-Baxter operators of weight zero. Thus R satisfies

$$[R(x), R(y)] = R([R(x), y] + [x, R(y)]) \tag{1}$$

for all $x, y \in \mathcal{L}$.

Remark 1. Every Lie algebra is a right Leibniz algebra. Hence the notion of a Rota-Baxter operator on a Leibniz algebra naturally extends the usual notion of a Rota-Baxter operator on a Lie algebra.

The six-dimensional simple Leibniz algebra

Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) = \text{span}\{e, f, h\}$ with multiplication

$$[e, h] = -[h, e] = 2e, \quad [f, h] = -[h, f] = -2f, \quad [e, f] = -[f, e] = h.$$

Let $\mathcal{I} = \text{span}\{x_0, x_1, x_2\}$ be the three-dimensional irreducible right $\mathfrak{sl}_2$ -module $V(2)$ with nonzero actions

$$\begin{aligned} [x_0, h] &= 2x_0, & [x_2, h] &= -2x_2, \\ [x_0, f] &= x_1, & [x_1, f] &= x_2, \\ [x_1, e] &= -2x_0, & [x_2, e] &= -2x_1. \end{aligned}$$

Consider the vector space $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$ with multiplication $[x+u, y+v] = [x, y]_{\mathfrak{sl}_2} + [u, v]$, where $x, y \in \mathfrak{sl}_2$, $u, v \in \mathcal{I}$.

Equivalently, $[\mathfrak{sl}_2, \mathcal{I}] = 0$, $[\mathcal{I}, \mathcal{I}] = 0$, $[\mathcal{I}, \mathfrak{sl}_2] = \mathcal{I}$.

Thus, with respect to the ordered basis $\{e, f, h, x_0, x_1, x_2\}$, the nonzero products of $\mathcal{L}$ are

$$\begin{aligned} [e, h] &= 2e, & [h, e] &= -2e, & [e, f] &= h, \\ [f, e] &= -h, & [h, f] &= 2f, & [f, h] &= -2f, \\ [x_0, h] &= 2x_0, & [x_2, h] &= -2x_2, & [x_0, f] &= x_1, \\ [x_1, f] &= x_2, & [x_1, e] &= -2x_0, & [x_2, e] &= -2x_1. \end{aligned}$$

The ideal $\mathcal{I}$ coincides with the Leibniz kernel of $\mathcal{L}$, and $\mathcal{L}/\mathcal{I} \simeq \mathfrak{sl}_2$.

Therefore $\mathcal{L}$ is the six-dimensional simple non-Lie Leibniz algebra associated with the irreducible module $V(2)$ [6].

A module isomorphism.

Lemma 1. The linear map $\theta : \mathfrak{sl}_2 \rightarrow \mathcal{I}$ defined by

$$\theta(e) = 2x_0, \quad \theta(f) = x_2, \quad \theta(h) = 2x_1$$

is an isomorphism of right $\mathfrak{sl}_2$ -modules. In particular,

$$[\theta(x), y] = \theta([x, y])$$

for all $x, y \in \mathfrak{sl}_2$.

Proof. It is enough to verify the identity on the basis $\{e, f, h\}$. For example,

$$\begin{aligned} [\theta(e), h] &= [2x_0, h] = 4x_0 = \theta(2e) = \theta([e, h]), \\ [\theta(f), h] &= [x_2, h] = -2x_2 = \theta(-2f) = \theta([f, h]), \\ [\theta(h), e] &= [2x_1, e] = -4x_0 = \theta(-2e) = \theta([h, e]). \end{aligned}$$

The remaining identities are verified similarly. Since both $\mathfrak{sl}_2$ and $\mathcal{I}$ are three-dimensional and θ maps the chosen basis of $\mathfrak{sl}_2$ onto a basis of $\mathcal{I}$, the map θ is an isomorphism. $\square$

Remark 2. Since $\mathcal{I} = \text{Leib}(\mathcal{L})$, the ideal $\mathcal{I}$ is characteristic. Hence every automorphism of $\mathcal{L}$ induces an automorphism of the quotient $\mathcal{L}/\mathcal{I} \simeq \mathfrak{sl}_2$.

The automorphism group of this simple Leibniz algebra was described in [1]. Since the explicit form of the automorphisms is not needed in the subsequent classification of Rota-Baxter operators, we do not reproduce it here.

Matrix convention. Throughout the paper, matrices of linear operators on $\mathfrak{sl}_2$ are written with respect to the ordered basis $\{e, f, h\}$ and act on coordinate column vectors.

Our convention for $\mathfrak{sl}_2$ is

$$[e, h] = 2e, \quad [f, h] = -2f, \quad [e, f] = h,$$

whereas the Pei-Bai-Guo classification [?] is written with respect to the convention

$$[h, e] = 2e, \quad [h, f] = -2f, \quad [e, f] = h.$$

The two bases are related by

$$e_{\text{PBG}} = e, \quad f_{\text{PBG}} = -f, \quad h_{\text{PBG}} = -h.$$

Hence, if

$$J = \text{diag}(1, -1, -1),$$

the corresponding change of basis is represented by J .

Moreover, the matrices P_i in the Pei-Bai-Guo classification are written in the row convention. Therefore, in our column convention, the corresponding operators are represented by

$$A_i = JP_i^T J, \quad 1 \leq i \leq 22.$$

In what follows, A_i always denotes the Pei-Bai-Guo normal form transported to the convention used in the present paper. All matrices appearing below are written in this convention.

Block equations for Rota-Baxter operators. Let $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$ be the six-dimensional simple Leibniz algebra introduced above. Let $R : \mathcal{L} \rightarrow \mathcal{L}$ be a linear operator. With respect to the vector space decomposition $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$, write

$$R = \begin{pmatrix} A & B \\ C & D \end{pmatrix},$$

where $A : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$, $B : \mathcal{I} \rightarrow \mathfrak{sl}_2$, $C : \mathfrak{sl}_2 \rightarrow \mathcal{I}$, $D : \mathcal{I} \rightarrow \mathcal{I}$.

Using the fixed $\mathfrak{sl}_2$ -module isomorphism $\theta : \mathfrak{sl}_2 \rightarrow \mathcal{I}$, we transfer all four blocks to $\mathfrak{sl}_2$ and write

$$B = G\theta^{-1}, \quad C = \theta F, \quad D = \theta S\theta^{-1},$$

where $F, G, S \in \text{End}(\mathfrak{sl}_2)$. Thus $R = \begin{pmatrix} A & G\theta^{-1} \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$. For $x, z \in \mathfrak{sl}_2$ we have $R(x) = A(x) + \theta(F(x))$ and $R(\theta(z)) = G(z) + \theta(S(z))$. Recall that $[\mathfrak{sl}_2, \mathcal{I}] = 0$, $[\mathcal{I}, \mathcal{I}] = 0$, and $[\theta(x), y] = \theta([x, y])$, $x, y \in \mathfrak{sl}_2$.

Theorem 1. Let $R = \begin{pmatrix} A & G\theta^{-1} \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$ be a linear operator on $L = \mathfrak{sl}_2 \ltimes \mathcal{I}$. Then R is a Rota-Baxter operator of weight zero if and only if the following identities hold for all $x, y, z, w \in \mathfrak{sl}_2$:

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]) + G([F(x), y]), \quad (2)$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)]) + S([F(x), y]), \quad (3)$$

$$[A(x), G(z)] = A([x, G(z)]), \quad (4)$$

$$[F(x), G(z)] = F([x, G(z)]), \quad (5)$$

$$[G(z), A(x)] = A([G(z), x]) + G([S(z), x] + [z, A(x)]), \quad (6)$$

$$[S(z), A(x)] = F([G(z), x]) + S([S(z), x] + [z, A(x)]), \quad (7)$$

$$[G(z), G(w)] = G([z, G(w)]), \quad (8)$$

$$[S(z), G(w)] = S([z, G(w)]). \quad (9)$$

Proof. Since $\mathcal{L} = \mathfrak{sl}_2 \oplus \mathcal{I}$, as a vector space, it is sufficient, by bilinearity, to verify the Rota-Baxter identity on the following four types of pairs:

$$(\mathfrak{sl}_2, \mathfrak{sl}_2), \quad (\mathfrak{sl}_2, \mathcal{I}), \quad (\mathcal{I}, \mathfrak{sl}_2), \quad (\mathcal{I}, \mathcal{I}).$$

Let $x, y, z, w \in \mathfrak{sl}_2$.

Case 1: $X = x$, $Y = y$. We have $R(x) = A(x) + \theta(F(x))$, $R(y) = A(y) + \theta(F(y))$. Since $[\mathfrak{sl}_2, \mathcal{I}] = [\mathcal{I}, \mathcal{I}] = 0$, the left-hand side of the Rota-Baxter identity is

$$[R(x), R(y)] = [A(x), A(y)] + [\theta(F(x)), A(y)].$$

Using $[\theta(u), v] = \theta([u, v])$, we obtain $[R(x), R(y)] = [A(x), A(y)] + \theta([F(x), A(y)])$. Furthermore, $[R(x), y] + [x, R(y)] = [A(x), y] + [x, A(y)] + \theta([F(x), y])$.

Applying R , we obtain

$$R([R(x), y] + [x, R(y)]) = A([A(x), y] + [x, A(y)]) + G([F(x), y]) + \theta(F([A(x), y] + [x, A(y)]) + S([F(x), y])).$$

Comparing the $\mathfrak{sl}_2$ and $\mathcal{I}$ -components gives (2) and (3).

Case 2: $X = x, Y = \theta(z)$. We have $R(\theta(z)) = G(z) + \theta(S(z))$. Hence

$$[R(x), R(\theta(z))] = [A(x), G(z)] + \theta([F(x), G(z)]).$$

On the other hand, $[R(x), \theta(z)] = 0$ because $[\mathfrak{sl}_2, \mathcal{I}] = [\mathcal{I}, \mathcal{I}] = 0$, whereas $[x, R(\theta(z))] = [x, G(z)]$. Therefore

$$R([x, G(z)]) = A([x, G(z)]) + \theta(F([x, G(z)])).$$

Comparison of the two components yields (4) and (5).

Case 3: $X = \theta(z), Y = x$. The left-hand side is

$$\begin{aligned} [R(\theta(z)), R(x)] &= [G(z) + \theta(S(z)), A(x) + \theta(F(x))] \\ &= [G(z), A(x)] + \theta([S(z), A(x)]). \end{aligned}$$

Furthermore,

$$[R(\theta(z)), x] = [G(z), x] + \theta([S(z), x])$$

and

$$[\theta(z), R(x)] = [\theta(z), A(x)] = \theta([z, A(x)]).$$

Consequently,

$$\begin{aligned} R([R(\theta(z)), x] + [\theta(z), R(x)]) &= \\ &= A([G(z), x] + [S(z), x] + [z, A(x)]) + \theta(F([G(z), x] + S([S(z), x] + [z, A(x)]))). \end{aligned}$$

Comparing components gives (6) and (7).

Case 4: $X = \theta(z), Y = \theta(w)$. Since $[\mathcal{I}, \mathcal{I}] = 0$, we have

$$\begin{aligned} [R(\theta(z)), R(\theta(w))] &= [G(z) + \theta(S(z)), G(w) + \theta(S(w))] \\ &= [G(z), G(w)] + \theta([S(z), G(w)]). \end{aligned}$$

Moreover, $[R(\theta(z)), \theta(w)] = 0$ and $[\theta(z), R(\theta(w))] = [\theta(z), G(w)] = \theta([z, G(w)])$. Hence

$$R(\theta([z, G(w)])) = G([z, G(w)]) + \theta(S([z, G(w)])).$$

Comparison of components yields (8) and (9).

Thus the Rota-Baxter identity is equivalent to (2)–(9). $\square$

Corollary 1. Let $R = \begin{pmatrix} A & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$. Then R is a Rota-Baxter operator of weight zero on $\mathcal{L}$ if and only if

$$\begin{aligned} [A(x), A(y)] &= A([A(x), y] + [x, A(y)]), \\ [F(x), A(y)] &= F([A(x), y] + [x, A(y)]) + S([F(x), y]), \\ [S(z), A(x)] &= S([S(z), x] + [z, A(x)]) \end{aligned}$$

for all $x, y, z \in \mathfrak{sl}_2$.

Corollary 2. Let $R = \begin{pmatrix} A & 0 \\ \theta F & 0 \end{pmatrix}$. Then R is a Rota-Baxter operator of weight zero if and only if

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]),$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)])$$

for all $x, y \in \mathfrak{sl}_2$.

In particular, A is a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$.

For a weight-zero Rota-Baxter operator $A \in \text{End}(\mathfrak{sl}_2)$, put

$$\mathcal{F}_0(A) = \{F \in \text{End}(\mathfrak{sl}_2) \mid [F(x), A(y)] = F([A(x), y] + [x, A(y)])\}.$$

Lemma 2. Let A be a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$. Then $\mathbb{C}A \subseteq \mathcal{F}_0(A)$. In other words, for every $\mu \in \mathbb{C}$, $F = \mu A$ satisfies $[F(x), A(y)] = F([A(x), y] + [x, A(y)])$.

Proof. Since A is a Rota-Baxter operator, $[A(x), A(y)] = A([A(x), y] + [x, A(y)])$. Multiplying this identity by μ gives $[\mu A(x), A(y)] = \mu A([A(x), y] + [x, A(y)])$, which is precisely the required identity for $F = \mu A$. $\square$

Corollary 3. For every linear map $F \in \text{End}(\mathfrak{sl}_2)$, the operator $R_F = \begin{pmatrix} 0 & 0 \\ \theta F & 0 \end{pmatrix}$ is a Rota-Baxter operator of weight zero on $\mathcal{L}$. Consequently, $\text{End}(\mathfrak{sl}_2) \simeq \text{Hom}(\mathfrak{sl}_2, \mathcal{I})$ gives a nine-dimensional linear family of Rota-Baxter operators on $\mathcal{L}$. Moreover, $R_F^2 = 0$ and $\text{rank } R_F = \text{rank } F \leq 3$.

Rota-Baxter operators with $A = B = 0$. We first consider Rota-Baxter operators whose $\mathfrak{sl}_2$ -component vanishes.

$$\text{Let } R = \begin{pmatrix} 0 & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}, F, S \in \text{End}(\mathfrak{sl}_2).$$

By Corollary 1, the Rota-Baxter identity is equivalent to

$$S([F(x), y]) = 0, \quad S([S(z), x]) = 0 \tag{10}$$

for all $x, y, z \in \mathfrak{sl}_2$.

Lemma 3. Let $0 \neq w \in \mathfrak{sl}_2$. Then $\dim[w, \mathfrak{sl}_2] = 2$ and $C_{\mathfrak{sl}_2}(w) = \{x \in \mathfrak{sl}_2 \mid [w, x] = 0\} = \mathbb{C}w$. Consequently, $[w, \mathfrak{sl}_2]^\perp = \mathbb{C}w$ with respect to the Killing form.

Proof. Since $\mathfrak{sl}_2$ has rank one, the centralizer of every nonzero element is one-dimensional. Hence $\dim C_{\mathfrak{sl}_2}(w) = 1$. By the rank-nullity theorem applied to $\text{ad}_w : \mathfrak{sl}_2 \rightarrow \mathfrak{sl}_2$, we obtain

$$\dim[w, \mathfrak{sl}_2] = 3 - \dim C_{\mathfrak{sl}_2}(w) = 2.$$

Finally, by invariance of the Killing form, $\kappa(w, [x, y]) = \kappa([w, x], y)$, and therefore $[w, \mathfrak{sl}_2]^\perp = C_{\mathfrak{sl}_2}(w) = \mathbb{C}w$. $\square$

Theorem 2. Let $R = \begin{pmatrix} 0 & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$ be a linear operator on $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$. Then R is a Rota-Baxter operator of weight zero if and only if one of the following two alternatives holds.

(i) $S = 0$, while $F \in \text{End}(\mathfrak{sl}_2)$ is arbitrary.

(ii) There exist $0 \neq w \in \mathfrak{sl}_2$, $\lambda \in \mathbb{C}^*$, $\ell \in \mathfrak{sl}_2^*$ such that $S(x) = \lambda \kappa(w, x)w$, and $F(x) = \ell(x)w$ for all $x \in \mathfrak{sl}_2$, where κ denotes the Killing form.

Equivalently, $S = \lambda w \otimes \kappa(w, \cdot)$, $F = w \otimes \ell$.

Proof. By (10), the Rota-Baxter identity is equivalent to $S([F(x), y]) = 0$ and $S([S(z), x]) = 0$ for all $x, y, z \in \mathfrak{sl}_2$. If $S = 0$, both identities are automatically satisfied and F is arbitrary. This gives case (i).

Assume now that $S \neq 0$. Put $U = \text{Im } S$. Equation (10) implies $[U, \mathfrak{sl}_2] \subseteq \ker S$.

We first show that $\text{rank } S = 1$. Indeed, if $\text{rank } S = 3$, then $\ker S = 0$, and hence $[U, \mathfrak{sl}_2] = 0$. Since $U = \mathfrak{sl}_2$, this contradicts $Z(\mathfrak{sl}_2) = 0$. Suppose that $\text{rank } S = 2$. Then $\dim \ker S = 1$. Choose $0 \neq u \in U$. By Lemma 3, $\dim[u, \mathfrak{sl}_2] = 2$. On the other hand, $[u, \mathfrak{sl}_2] \subseteq [U, \mathfrak{sl}_2] \subseteq \ker S$, which is impossible because $\dim \ker S = 1$. Therefore $\text{rank } S = 1$.

Hence there exist $0 \neq w \in \mathfrak{sl}_2$ and $0 \neq \varphi \in \mathfrak{sl}_2^*$ such that $S(x) = \varphi(x)w$. Substituting this into (10) gives $\varphi(z)\varphi([w, x])w = 0$ for all $x, z \in \mathfrak{sl}_2$. Since $\varphi \neq 0$, we obtain $\varphi([w, x]) = 0$ for all $x \in \mathfrak{sl}_2$. Therefore $\varphi \in [w, \mathfrak{sl}_2]^\perp$.

By Lemma 3, $[w, \mathfrak{sl}_2]^\perp = \mathbb{C}w$ under the identification of $\mathfrak{sl}_2$ with its dual via the Killing form. Thus $\varphi(x) = \lambda \kappa(w, x)$ for some $\lambda \in \mathbb{C}^*$. Consequently, $S(x) = \lambda \kappa(w, x)w$.

It remains to determine F . Equation (10) becomes $\kappa(w, [F(x), y]) = 0$ for all $x, y \in \mathfrak{sl}_2$. By invariance of the Killing form, $\kappa([w, F(x)], y) = 0$. Since κ is nondegenerate, $[w, F(x)] = 0$. Hence $F(x) \in C_{\mathfrak{sl}_2}(w) = \mathbb{C}w$. Therefore there exists $\ell \in \mathfrak{sl}_2^*$ such that $F(x) = \ell(x)w$. Conversely, suppose that $S(x) = \lambda\kappa(w, x)w$, $F(x) = \ell(x)w$. Then $S([S(z), x]) = \lambda^2\kappa(w, z)\kappa(w, [w, x])w = 0$ and $S([F(x), y]) = \lambda\ell(x)\kappa(w, [w, y])w = 0$, because $\kappa(w, [w, u]) = \kappa([w, w], u) = 0$.

Thus both (10) hold, and R is a Rota-Baxter operator. $\square$

Corollary 4. Let $R = \begin{pmatrix} 0 & 0 \\ C & D \end{pmatrix}$ be a Rota-Baxter operator of weight zero on $\mathcal{L}$. Then either $D = 0$ and $C : \mathfrak{sl}_2 \rightarrow \mathcal{I}$ is arbitrary, or there exist $0 \neq w \in \mathfrak{sl}_2$, $\lambda \in \mathbb{C}^*$, $\ell \in \mathfrak{sl}_2^*$ such that $C(x) = \ell(x)\theta(w)$ and $D(\theta(x)) = \lambda\kappa(w, x)\theta(w)$.

In particular, $\text{rank } C \leq 1$, $\text{rank } D = 1$ whenever $D \neq 0$.

Remark 3. Normalize the invariant bilinear form by $\kappa(e, f) = 1$, $\kappa(h, h) = 2$.

Let $w = pe + qf + rh$. Then $\kappa(w, e) = q$, $\kappa(w, f) = p$, $\kappa(w, h) = 2r$. Therefore $S = \lambda w \otimes \kappa(w, \cdot)$ has matrix

$$S = \lambda \begin{pmatrix} pq & p^2 & 2pr \\ q^2 & pq & 2qr \\ qr & pr & 2r^2 \end{pmatrix}$$

with respect to the ordered basis $\{e, f, h\}$.

If $\ell(e) = \alpha$, $\ell(f) = \beta$, $\ell(h) = \gamma$, then $F = w \otimes \ell$ has matrix

$$F = \begin{pmatrix} p\alpha & p\beta & p\gamma \\ q\alpha & q\beta & q\gamma \\ r\alpha & r\beta & r\gamma \end{pmatrix}.$$

Corollary 5. For every nonzero operator S occurring in Theorem 2, $S^2 = \lambda\kappa(w, w)S$.

Consequently:

1. if w is nilpotent, then $\kappa(w, w) = 0$ and $S^2 = 0$;
2. if w is semisimple, then $\kappa(w, w) \neq 0$, and after multiplication by a nonzero scalar, S becomes an idempotent operator.

Rota-Baxter operators with $B = D = 0$. Let $R = \begin{pmatrix} A & 0 \\ \theta F & 0 \end{pmatrix}$, $A, F \in \text{End}(\mathfrak{sl}_2)$. By Corollary 2, the operator R is a Rota-Baxter operator of weight zero if and only if

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]), \quad (11)$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)]) \quad (12)$$

for all $x, y \in \mathfrak{sl}_2$.

For a weight-zero Rota-Baxter operator A on $\mathfrak{sl}_2$, define

$$\mathcal{F}_0(A) = \{F \in \text{End}(\mathfrak{sl}_2) \mid [F(x), A(y)] = F([A(x), y] + [x, A(y)])\}.$$

Thus the classification of Rota-Baxter operators with $B = D = 0$ reduces to the determination of the spaces $\mathcal{F}_0(A)$.

Theorem 3. Let $R = \begin{pmatrix} A & 0 \\ \theta F & 0 \end{pmatrix}$ be a Rota-Baxter operator of weight zero on $\mathcal{L} = \mathfrak{sl}_2 \times \mathcal{I}$.

Then A is a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$. Let $A_1, \dots, A_{22}$ denote the Pei-Bai-Guo normal forms transported to the convention of the present paper as described above.

Then the corresponding spaces $\mathcal{F}_0(A_i)$ satisfy

$$\dim \mathcal{F}_0(A_i) = \begin{cases} 9, & i = 2, \\ 1, & i \in \{4, 7, 8, 10, 12, 13, 15, 16, 17, 19, 20\}, \\ 2, & i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}. \end{cases}$$

More precisely, we obtain $\mathcal{F}_0(A_2) = \text{End}(\mathfrak{sl}_2)$, whereas $\mathcal{F}_0(A_i) = \mathbb{C}A_i$ for each $i \in \{4, 7, 8, 10, 12, 13, 15, 16, 17, 19, 20\}$.

For $i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}$, there exists an operator $\widehat{Q}_i$ such that $\mathcal{F}_0(A_i) = \mathbb{C}A_i \oplus \mathbb{C}\widehat{Q}_i$. The operators $\widehat{Q}_i$ are determined by $\widehat{Q}_i = JM_iJ$, where the matrices M_i are listed in Table 1.

Proof. Since $B = D = 0$, the Rota-Baxter identity reduces to (11) and (12). Equation (11) means precisely that A is a Rota-Baxter operator of weight zero on $\mathfrak{sl}_2$. Therefore the classification of Pei-Bai-Guo applies.

Let $A = A_i$ be one of the transported normal forms. Write

$$F = \begin{pmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{pmatrix}$$

with respect to the ordered basis $\{e, f, h\}$. For a fixed A_i , equation (12) is homogeneous and linear in the nine coefficients f_{jk} . Hence it is sufficient to substitute $(x, y) \in \{e, f, h\} \times \{e, f, h\}$. Solving the resulting finite linear systems gives

$$\dim \mathcal{F}_0(A_i) = 9, 1, \text{ or } 2,$$

according to the three cases stated in the theorem. For $A_2 = 0$, equation (12) is identically satisfied; therefore

$$\mathcal{F}_0(A_2) = \text{End}(\mathfrak{sl}_2).$$

By Lemma 2, $\mathbb{C}A_i \subseteq \mathcal{F}_0(A_i)$ for every i . For $i \in \{4, 7, 8, 10, 12, 13, 15, 16, 17, 19, 20\}$, the corresponding linear system has rank eight. Consequently, $\dim \mathcal{F}_0(A_i) = 1$, and therefore $\mathcal{F}_0(A_i) = \mathbb{C}A_i$.

For $i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}$, the system has rank seven. Hence $\dim \mathcal{F}_0(A_i) = 2$. Choosing a second generator linearly independent of A_i gives $\mathcal{F}_0(A_i) = \mathbb{C}A_i \oplus \mathbb{C}\widehat{Q}_i$, where the operators $\widehat{Q}_i$ are listed in Table 1. $\square$

Corollary 6. Let $A \neq 0$ be a weight-zero Rota-Baxter operator on $\mathfrak{sl}_2$ belonging to one of the Pei-Bai-Guo normal forms. Then, in the case $B = D = 0$, $1 \leq \dim \mathcal{F}_0(A) \leq 2$.

More precisely, $\dim \mathcal{F}_0(A) = 1$ if and only if every admissible F is proportional to A , whereas $\dim \mathcal{F}_0(A) = 2$ if and only if there exists an admissible operator F which is not proportional to A .

Remark 4. The preceding theorem shows that the extension from $\mathfrak{sl}_2$ to the simple Leibniz algebra $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$ introduces only a small amount of additional freedom in the case $B = D = 0$. Indeed, for every nonzero Rota-Baxter operator A on $\mathfrak{sl}_2$, the compatible block $C = \theta F$ depends on at most two independent parameters. For eleven normal forms, it is necessarily of the form $C = \lambda \theta A$. For the remaining ten nonzero normal forms, exactly one additional independent direction occurs.

Rota-Baxter operators with $B = 0$ and $D \neq 0$. Let $R = \begin{pmatrix} A & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$ be a Rota-Baxter operator of weight zero on $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$, where $A, F, S \in \text{End}(\mathfrak{sl}_2)$, $S \neq 0$.

By Corollary 1, the Rota-Baxter identity is equivalent to

$$[A(x), A(y)] = A([A(x), y] + [x, A(y)]), \quad (13)$$

$$[S(z), A(x)] = S([S(z), x] + [z, A(x)]), \quad (14)$$

$$[F(x), A(y)] = F([A(x), y] + [x, A(y)]) + S([F(x), y]) \quad (15)$$

for all $x, y, z \in \mathfrak{sl}_2$.

Equation (13) means that A is a weight-zero Rota-Baxter operator on $\mathfrak{sl}_2$. Thus, up to the Pei-Bai-Guo classification and the change of convention fixed above, we may assume $A = A_i$, $1 \leq i \leq 22$. For a fixed pair (A, S) , define

$$\mathcal{F}(A, S) = \{F \in \text{End}(\mathfrak{sl}_2) \mid (15) \text{ holds}\}.$$

Theorem 4. Let $R = \begin{pmatrix} A & 0 \\ \theta F & \theta S\theta^{-1} \end{pmatrix}$, $A \neq 0$, $S \neq 0$, be a Rota-Baxter operator of weight zero on $\mathcal{L} = \mathfrak{sl}_2 \ltimes \mathcal{I}$. Let $A = A_i$ be one of the transported Pei-Bai-Guo normal forms. Then the structure of the solution set for S and the dimensions of the corresponding spaces $\mathcal{F}(A_i, S)$ are summarized in Table 2.

Conversely, for every pair (A_i, S) occurring in Table 2 and every $F \in \mathcal{F}(A_i, S)$, the operator $R = \begin{pmatrix} A_i & 0 \\ \theta F & \theta S \theta^{-1} \end{pmatrix}$ is a Rota-Baxter operator of weight zero on $\mathcal{L}$.

Remark 5. The case $A = A_2 = 0$ is not included in Table 2, since it was completely described in Theorem 2.

Proof (Proof of Theorem 4). For $B = 0$, the Rota-Baxter identity is equivalent to (13)–(15). Equation (13) implies that A is a weight-zero Rota-Baxter operator on $\mathfrak{sl}_2$. Therefore, we may use the transported Pei-Bai-Guo normal forms $A = A_i$.

Fix one such A_i . For each fixed A_i , substitution of the basis elements e, f, h into equation (14) yields a finite quadratic system in the entries of S . Solving this system gives the branches listed in Table 2. For every admissible S , equation (15) is a homogeneous linear system in the entries of F , whose nullity gives $\dim \mathcal{F}(A_i, S)$.

Conversely, direct substitution into equations (14) and (15) shows that every listed pair (S, F) satisfies the required identities. This completes the proof. $\square$

Proposition 1. For the family A_{22} put $\Delta = 16ab^3 - 1$. If $\Delta \neq 0$, the corresponding equation for S has four nonzero branches, and the dimensions of the associated spaces $\mathcal{F}(A_{22}, S)$ are 2, 0, 4, 2.

If $\Delta = 0$, then $S = \lambda A_{22}$, $\lambda \in \mathbb{C}^*$. Moreover, $\dim \mathcal{F}(A_{22}, \lambda A_{22}) = \begin{cases} 2, & \lambda \neq -1, \\ 4, & \lambda = -1. \end{cases}$ Thus the hypersurface $16ab^3 = 1$ is an exceptional parameter locus at which the structure of the solution space changes.

Corollary 7. Let $A \neq 0$, $B = 0$, $D \neq 0$. Then, for every admissible pair (A, S) , $\dim \mathcal{F}(A, S) \in \{0, 1, 2, 4\}$. In particular, $\dim \mathcal{F}(A, S) \leq 4$. The maximal value $\dim \mathcal{F}(A, S) = 4$ occurs only on special branches of the classification.

Remark 6. The remaining cases are $A = 0$, $B \neq 0$, and $A \neq 0$, $B \neq 0$. A complete description of Rota-Baxter operators in these cases is left as an open problem.

For $i \in \{1, 3, 5, 6, 9, 11, 14, 18, 21, 22\}$, write

$$\widehat{Q}_i = JM_iJ, \quad J = \text{diag}(1, -1, -1).$$

The matrices M_i are given in Table 1.

Table 1

i	M_i	i	M_i
1	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	11	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & a \end{pmatrix}$
3	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$	14	$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 2a \\ a & 0 & 2a^2 \end{pmatrix}$
5	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$	18	$\begin{pmatrix} -2a & 0 & 1 \\ 0 & 0 & 0 \\ -2a^2 & 0 & a \end{pmatrix}$
6	$\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & a \end{pmatrix}$	21	$\begin{pmatrix} 4a^2 & 1 & 0 \\ 16a^4 & 4a^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}$
9	$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	22	$\begin{pmatrix} \frac{1}{-2b} & 0 & 1 \\ 2a & 0 & -4ab \\ -\frac{d}{8b^2} & 0 & \frac{d}{4b} \end{pmatrix}$

where $d = 1 + 16ab^3$.

For each normal form A_i , the admissible operators S are obtained by solving

$$[S(z), A_i(x)] = S([S(z), x] + [z, A_i(x)]), \quad x, z \in \mathfrak{sl}_2.$$

Depending on A_i , the solution set consists either of a one-parameter family or of several distinct branches. To avoid lengthy repetitions of explicit matrices, we record in Table 2 only the number of branches, their parameter dependence, and the dimensions of the corresponding spaces $\mathcal{F}(A_i, S)$.

Table 2

A_i	admissible S	$\dim \mathcal{F}(A_i, S)$
A_1	four solution branches	2, 0, 4, 2
A_3	$S = \lambda A_3, \lambda \neq 0$	2; 4 if $\lambda = -1$
A_4	$S = \lambda A_4, \lambda \neq 0$	1
A_5	$S = \lambda A_5, \lambda \neq 0$	2; 4 if $\lambda = -1$
A_6	four solution branches	0, 4, 2, 2
A_7	$S = \lambda A_7, \lambda \neq 0$	1
A_8	one-parameter family	1
A_9	four solution branches	2, 0, 4, 2
A_{10}	one-parameter family	1
A_{11}	four solution branches	0, 4, 2, 2
A_{12}	one-parameter family	1
A_{13}	$S = \lambda A_{13}, \lambda \neq 0$	1
A_{14}	four solution branches	2, 4, 0, 2
A_{15}	one-parameter family	1

A_i	admissible S	$\dim \mathcal{F}(A_i, S)$
A_{16}	one-parameter family	1
A_{17}	$S = \lambda A_{17}, \lambda \neq 0$	1
A_{18}	three solution branches	2, 0, 2
A_{19}	one-parameter family	1
A_{20}	one-parameter family	1
A_{21}	four solution branches	2, 0, 4, 2
$A_{22}, \Delta \neq 0$	four solution branches	2, 0, 4, 2
$A_{22}, \Delta = 0$	$S = \lambda A_{22}, \lambda \neq 0$	2; 4 if $\lambda = -1$

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