

A THEOREM ON THE EXTENSION OF A HOLOMORPHIC FUNCTION DEFINED ON AN ALGEBRAIC VARIETY TO AN ENTIRE FUNCTION

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ABSTRACT. This paper investigates the properties of the order and type of holomorphic functions defined on a regular compact set K of an algebraic variety A . Furthermore, we prove a theorem on the holomorphic extension of a function defined on K to an entire function on the whole algebraic variety A .

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Key words: regular compact set, holomorphic extension, entire function.

Introduction

Let $A \subset \mathbf{C}^N$ be a subset. A is an algebraic manifold of complex dimension n if for every point $a \in A$ there exists an open neighborhood $U \subset \mathbf{C}^N$ and a finite collection of polynomial functions $f_1, f_2, f_3, \dots, f_{N-n} \in \mathbf{C}[z_1, z_2, \dots, z_N]$ such that:

$$A \cap U = \{z \in U : f_1(z) = f_2(z) = f_3(z) = \dots = f_{N-n}(z) = 0\}$$

and the Jacobian matrix $\left[\frac{\partial f_j}{\partial z_k} \right]_{j=1, \dots, N-n, k=1, \dots, N}$.

Algebraic varieties' problem of extending holomorphic functions while taking into account their growth order and type was also studied by A. Zeriahi [1]. In his work, Zeriahi investigated the relationship between the rate of best polynomial approximation of holomorphic functions on affine algebraic varieties and their growth properties, and established a growth version of the Bernstein-Walsh-Siciak theorem. Using these results, he also considered the problem of extending holomorphic functions defined on an algebraic subvariety to entire functions on the ambient complex space while controlling their growth properties.

In contrast to the approach of Zeriahi [1], the present paper focuses not primarily on estimating the rate of best polynomial approximation, but rather on characterizing directly, by means of pluripotential-theoretic tools, the extension properties of holomorphic functions defined on a compact subset of an algebraic variety. In particular, the holomorphic extension problem is investigated in relation to the growth order and type of the function by using extremal functions and their associated sublevel domains. This approach makes it possible to establish a relationship between the domain of holomorphic extension, the geometric properties of the extremal function, and the growth characteristics of the holomorphic function.

Thus, while in Zeriahi [1] the relationship between the rate of polynomial approximation and the growth properties of holomorphic functions serves as one of the principal tools, in the present work the problem of extending holomorphic functions with prescribed order and type is studied directly in terms of extremal functions. This provides a distinct pluripotential-theoretic characterization of the holomorphic extension problem under consideration.

Example 1. The set $A = \{(z_1, z_2) \in \mathbf{C}^2 : z_1^2 + z_2^2 = 1\}$ is a complex algebraic manifold of complex dimension 1.

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In particular, a function f , defined on a complex manifold A is called holomorphic, $f \in \mathcal{O}(A)$, if it is holomorphic in local coordinates, i.e., the functions $f_j(t^j) = f \circ \varphi_j^{-1}(t^j) \in \mathcal{O}(B)$, $j = 1, 2, \dots$. Here, the ball B lies in the space $\mathbf{C}^n$. A function u defined on a complex manifold A is called plurisubharmonic (psh), $u \in psh(A)$, if it is (psh) in local coordinates, $\varphi_j(t^j) : U_j \rightarrow B \subset \mathbf{C}^n$, $j = 1, 2, \dots$ i.e., the functions $u_j(t^j) = u \circ \varphi_j^{-1}(t^j) \in psh(B)$, $j = 1, 2, \dots$

The algebraic manifold A is a parabolic manifold.

An Algebraic manifold is S -parabolic manifold, i.e there exists a special exhaustion function $\rho(z)$ that satisfies the conditions

- a) $\rho(z) \in psh(A)$, $\{z \in A : \rho(z) \leq c\} \subset\subset A \forall c \in \mathbf{R}$;
- b) ρ is maximal plurisubharmonic function on the out of a compact subset $K \subset\subset A$, i.e. $(dd^c \rho)^n = 0$, on $A \setminus K$.

Let us define the concept of polynomials on A algebraic manifold

Definition 1. If for a function $p(z) \in \mathcal{O}(A)$ there exist positive real numbers c, d such that for all $z \in A$ it holds inequality

$$\ln |p(z)| \leq d \cdot \rho^+(z) + c, \quad (5)$$

where $\rho^+(z) = \max\{0, \rho(z)\}$, then the function p is called ρ -polynomial of degree $\leq d$. The minimal value of d in inequality (9) is called the degree of the polynomial p .

We denote by $\mathcal{P}_\rho^d(A)$ the class of all polynomials ρ on algebraic manifold $A \subset \mathbf{C}^N$ with degree $\deg p \leq d$. Let $\mathcal{P}_\rho(A) = \bigcup_{d=1}^{\infty} \mathcal{P}_\rho^d(A)$ is the class of all polynomials on A .

Example 2. Let consider affine algebraic manifold $A \subset \mathbf{C}^N$, $\dim A = n$. By well-known W.Rudin's geometric criterion [2], we can consider (after appropriate affine transformation of coordinates)

$$A \subset \{w = (w', w'') = (w_1, \dots, w_n, w_{n+1}, \dots, w_N) : \|w''\| < C(1 + \|w'\|^k)\}$$

where, C and k are constants. If we put $\rho(w) = \ln \|w'\|$, then restriction $\rho|_A$ is a special exhaustion function for A . Obviously, ρ -polynomials on A are the restriction of ordinary polynomials $p(w', w'')$ to A . Therefore, $\mathcal{P}_\rho(A)$ are dense in $\mathcal{O}(A)$.

Definition 2. A -algebraic manifold is said to be regular if the space of ρ -polynomials $\mathcal{P}_\rho(A)$ is dense in the space of all analytic functions $\mathcal{O}(A)$.

Properties of the Order and Type

Let A be an algebraic variety. It is well known that algebraic varieties constitute a subclass of parabolic varieties. As in the classical theory, the notion of the order of growth of entire functions is introduced on regular parabolic varieties. In this setting, polynomials serve as the principal comparison objects, since among all entire functions they possess the smallest growth rate.

For this purpose, we define the pseudo-balls by

$$B_R = \{z \in A : \rho(z) < \ln R\}, \quad R > 0.$$

For an entire function $f \in \mathcal{O}(A)$, define

$$M(R, f) = \sup\{|f(z)| : z \in \partial B_R\}.$$

Definition 3. The quantity

$$\tau(f) = \limsup_{R \rightarrow \infty} \frac{\ln \ln M(R, f)}{\ln R}$$

is called the *order of growth* of the entire function f .

Clearly,

$$0 \leq \tau(f) \leq +\infty.$$

If $\tau(f) < +\infty$, then the following quantity is defined:

$$\sigma(f) = \limsup_{R \rightarrow \infty} \frac{\ln M(R, f)}{R^{\tau(f)}},$$

which is called the *type* of the entire function f .

Now let $K \subset A$ be a regular compact set. Consider the family of level sets of the extremal function:

$$D_R = \{z \in A : \Phi_A(z, K) < R\}, \quad R > 1,$$

where

$$\Phi_A(z, K) = \sup \left\{ |f(z)|^{1/\deg f} : f \in \mathcal{P}_\rho(A), \|f\|_K \leq 1 \right\}.$$

Furthermore, define

$$M_K(R, f) = \sup\{|f(z)| : z \in \partial D_R\},$$

and

$$\tau_K(f) = \limsup_{R \rightarrow \infty} \frac{\ln \ln M_K(R, f)}{\ln R}.$$

The Green function of B_r relative to A is defined by

$$V_A(z, B_r) = \sup \{u(z) : u \in \mathcal{L}(A), u|_{B_r} \leq 0\}, \quad z \in A,$$

where

$$\mathcal{L}(A) = \{u \in PSH(A) : u(z) \leq \log(1 + \|z\|) + C_u\}$$

is the Lelong class of plurisubharmonic functions on A with logarithmic growth, and C_u is a constant independent of z .

Theorem 1. Let (A, ρ) be an algebraic variety. Then, for every regular compact set $K \subset A$, we have

$$\tau_K(f) = \tau(f).$$

Proof. First, we prove that

$$\tau_K(f) \geq \tau(f).$$

Since K is compact, there exists some $z_0 \in K$ such that

$$r = \rho(z_0).$$

Then $K \subset B_r$. For every polynomial $P_d(z)$ satisfying

$$\|P_d\| \leq 1,$$

we have

$$|P_d(z)|^{1/d} \leq \|P_d\|_{B_r}^{1/d} e^{V_A(z, \overline{B_r})} \leq \sup_{\overline{B_r}} \Phi_A(z, K) e^{\max\{0, \rho(z) - \ln r\}}.$$

Hence,

$$\Phi_A(z, K) \leq L \cdot \max \left\{ 1, \frac{e^{\rho(z)}}{r} \right\}.$$

It follows that, for sufficiently large R ,

$$B_{\frac{Rr}{L}} \subset D_R,$$

and therefore

$$M\left(\frac{Rr}{L}, f\right) \leq M_K(R, f).$$

This proves that

$$\tau_K(f) \geq \tau(f).$$

Now we prove the reverse inequality

$$\tau_K(f) \leq \tau(f).$$

Since $K \subset B_r$, by the properties of the Green function,

$$V_A(z, B_r) \leq V_A(z, K).$$

Consequently,

$$\max\{0, \rho(z) - \ln r\} \leq V_A(z, K) = \ln \Phi_A(z, K).$$

Therefore, consider the set

$$D_{\frac{R}{r}} = \left\{ z \in A : \Phi_A(z, K) \leq \frac{R}{r} \right\}.$$

This set is contained in B_R , since

$$\rho(z) - \ln r \leq \ln \frac{R}{r}$$

implies

$$\rho(z) \leq \ln R.$$

Hence,

$$D_{\frac{R}{r}} \subset B_R.$$

As a result,

$$M_K\left(\frac{R}{r}, f\right) \leq M(R, f).$$

Therefore,

$$\tau_K(f) \leq \tau(f).$$

Combining both inequalities, we conclude that

$$\tau_K(f) = \tau(f).$$

Since this proves that

$$\tau_K(f) \leq \tau(f),$$

combining the two inequalities, we obtain

$$\tau_K(f) = \tau(f).$$

Main part

Definition 4. Let $f \in \mathcal{O}(A)$. If there exists a constant $C > 0$ such that

$$|f(z)| \leq Ce^{\sigma e^{\tau \rho(z)}} \quad \text{for all } z \in A.$$

then the holomorphic function f is said to be of *type* σ and *order* τ on the algebraic variety A .

$e_d(f, K)$ denotes the error of best polynomial approximation of f on K , defined by

$$e_d(f, K) = \inf_{p \in \mathcal{P}_d} \|f - p\|_K,$$

where $\mathcal{P}_d$ denotes the space of polynomials of degree at most d , and

$$\|f - p\|_K = \sup_{z \in K} |f(z) - p(z)|.$$

Theorem 2. Let $f \in \mathcal{O}(A)$ be a function defined on a regular compact set $K \subset A$. Suppose that

$$\overline{\lim}_{d \rightarrow \infty} d^{1/\tau} e_d(f, K)^{1/d} \leq (e\tau\sigma)^{1/\tau}.$$

Then f admits an extension to an entire function on A whose order does not exceed τ and whose type does not exceed σ .

Proof. In this part, we prove that a function f satisfying

$$\overline{\lim}_{d \rightarrow \infty} d^{1/\tau} e_d(f, K)^{1/d} \leq (e\tau\sigma)^{1/\tau}$$

extends to an entire function on A of order at most τ and type at most σ .

To this end, we proceed as follows. In particular, the above inequality also holds in the case

$$\lim_{d \rightarrow \infty} e_d(f, K)^{1/d} = 0.$$

Hence, by the Bernstein–Walsh theorem, the function f admits a holomorphic extension to A .

It remains to prove that

$$\lim_{R \rightarrow \infty} \frac{\ln \ln M_K(R, f)}{\ln R} \leq \tau.$$

Thus, There exists a sequence of polynomials $\{p_d\}$ such that

$$d^{1/\tau} \|f - p_d\|_K^{1/d} \leq (e^{1+\varepsilon}\tau\sigma)^{1/\tau}.$$

By the definition of the upper limit, for every $\varepsilon > 0$, there exists $d_0 \in \mathbb{N}$ such that for all $d > d_0$,

$$d^{1/\tau} \|f - p_d\|_K^{1/d} \leq (e^{1+\varepsilon}\tau\sigma)^{1/\tau}.$$

Equivalently,

$$\|f - p_d\|_K^{1/d} \leq \left(\frac{e^{1+\varepsilon}\tau\sigma}{d} \right)^{1/\tau}.$$

From this inequality it immediately follows that

$$\|p_{d+1} - p_d\|_K^{1/d} \leq 2 \left(\frac{e^{1+\varepsilon}\tau\sigma}{d} \right)^{1/\tau}.$$

Now consider the series

$$f = p_1 + \sum_{k=1}^{\infty} (p_{k+1} - p_k).$$

Next, we estimate $M_K(R, f)$:

$$\begin{aligned} M_K(R, f) &= \|f\|_{\overline{D_R}} \\ &= \|f - p_d + p_d\|_{\overline{D_R}} \\ &\leq \|f - p_d\|_{\overline{D_R}} + \|p_d\|_{\overline{D_R}} \\ &= \left\| \sum_{k=d}^{\infty} (p_{k+1} - p_k) \right\|_{\overline{D_R}} + \|p_d\|_{\overline{D_R}} \\ &\leq \sum_{k=d}^{\infty} 2 \left(\frac{e^{1+\varepsilon}\tau\sigma R^\tau}{k} \right)^{k/\tau} + C R^{td}. \end{aligned}$$

Hence,

$$M_K(R, f) \leq L e^{\sigma R^\tau},$$

where L is a positive constant.

This inequality is equivalent to

$$\lim_{R \rightarrow \infty} \frac{\ln \ln M_K(R, f)}{\ln R} \leq \tau.$$

This completes the proof.

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