

**CYCLIC $\mathcal{R}_2(m)$ LOTKA–VOLTERRA OPERATORS ON SIMPLICES:
CLASSIFICATION ON S^4 AND GLOBAL DYNAMICS FOR ODD m**

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ABSTRACT. We study discrete Lotka–Volterra operators $V : S^{m-1} \rightarrow S^{m-1}$ whose interaction graph is the 2-regular cycle C_m , referred to as $\mathcal{R}_2(m)$ configurations. For $m = 5$ (the simplex S^4) we classify all orientation patterns under the dihedral group D_5 via Burnside’s lemma, obtaining exactly four isomorphism classes (I)–(IV). For each class we determine the $P \rightarrow Q$ repeller–attractor structure, describe invariant polytopes via coordinate signatures, and prove global convergence of trajectories. For general odd $m \geq 3$ we prove that the directed-cycle operator possesses a unique interior fixed point M , construct an explicit Lyapunov function φ_M , and show that every non-fixed interior trajectory converges to the boundary in forward time while its α -limit set is $\{M\}$.

MSC (2020): 37B25; 37C25; 37C27; 05C20.

Key words: discrete Lotka–Volterra operator; simplex; cyclic graph; Burnside’s lemma; coordinate signature; Lyapunov function; trajectory route; α - and ω -limit sets.

Introduction

Lotka–Volterra (LV) mappings on the standard simplex

$$S^{m-1} = \left\{ x \in \mathbb{R}^m : \sum_{i=1}^m x_i = 1, x_i \geq 0 \right\}$$

constitute a fundamental class of nonlinear discrete dynamical systems, widely studied in population dynamics, mathematical biology, and related areas [5,6]. Given a skew-symmetric matrix $A = (a_{ij})$ with $|a_{ij}| \leq 1$, the mapping $V : S^{m-1} \rightarrow S^{m-1}$ is defined by

$$x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad k = 1, \dots, m. \quad (1)$$

Because A is skew-symmetric, V is a homeomorphism of S^{m-1} [1,2]. The interaction graph $\Gamma(A)$ has vertex set $\{1, \dots, m\}$ with a directed edge $i \rightarrow j$ whenever $a_{ji} > 0$.

Among the most natural sparse structures are the *2-regular cyclic* graphs C_m : each vertex has in-degree and out-degree equal to 1. We denote by $\mathcal{R}_2(m)$ the family of LV operators whose underlying undirected graph is C_m . The study of $\mathcal{R}_2(m)$ operators is motivated by cyclic competition models in ecology and by their analogy with continuous Volterra systems [8].

The *trajectory-route method*—partitioning S^{m-1} into convex polytopes via sign-change hyperplanes of the coordinate increments—was developed in [3,4] for bigraph operators. The present work applies and extends this method to cyclic graphs.

Main results. For the five-dimensional population vector corresponding to the simplex S^4 , we classify all cyclic $\mathcal{R}_2(5)$ Lotka–Volterra operators, up to the natural dihedral symmetries of the cycle, into exactly four isomorphism classes. For each class, we describe the global direction of the trajectories and identify the invariant sets that play the roles of repelling and attracting sets.

For cyclic operators associated with an arbitrary odd number $m \geq 3$ of coordinates, we prove the existence and uniqueness of an interior fixed point. We construct an explicit Lyapunov function showing that this fixed

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point is repelling in forward time. Consequently, every non-fixed trajectory starting in the interior approaches the boundary of the simplex in forward time, whereas in backward time it approaches the unique interior fixed point.

Preliminaries

We collect the main definitions and tools used throughout the paper.

Throughout the paper, $\text{ri } S^{m-1}$ denotes the relative interior of the simplex S^{m-1} , while ∂S^{m-1} denotes its boundary.

Definition 1 (Fixed point). A point $x^* \in S^{m-1}$ is called a *fixed point* of the operator $V : S^{m-1} \rightarrow S^{m-1}$ if $V(x^*) = x^*$. The set of all fixed points is denoted $\text{Fix}(V)$. A fixed point x^* is called an *interior fixed point* if $x^* \in \text{ri } S^{m-1}$, i.e. all coordinates $x_k^* > 0$.

Definition 2 (Transverse repeller and attractor) Let $\Lambda \subset S^{m-1}$ be a compact invariant set for V . Throughout the paper, $\|\cdot\|$ denotes the standard Euclidean norm on $\mathbb{R}^m$.

The set Λ is called a transverse repeller if there exists a neighbourhood U of Λ in S^{m-1} such that, for every $x \in \Lambda$ and every nonzero vector ν transverse to Λ at x ,

$$\|DV(x)\nu\| > \|\nu\|.$$

Equivalently, all eigenvalues of the Jacobian corresponding to directions transverse to Λ have modulus greater than 1.

Similarly, Λ is called a transverse attractor if all eigenvalues of the Jacobian corresponding to transverse directions have modulus less than 1.

Definition 3 (α - and ω -limit sets). For $x^{(0)} \in S^{m-1}$ define:

$$\omega(x^{(0)}) = \bigcap_{n \geq 0} \overline{\{x^{(k)} : k \geq n\}}, \quad \alpha(x^{(0)}) = \bigcap_{n \geq 0} \overline{\{x^{(-k)} : k \geq n\}},$$

the ω -limit set (forward limit) and the α -limit set (backward limit) of the orbit through $x^{(0)}$.

Definition 4 (Coordinate signature). Let V be an LV operator (1) with skew-symmetric matrix A . The *coordinate signature* of a point $x \in S^{m-1}$ is

$$\sigma(x) = (\text{sgn}(x'_1 - x_1), \dots, \text{sgn}(x'_m - x_m)) = (\text{sgn}x_1(Ax)_1, \dots, \text{sgn}x_m(Ax)_m), \quad \text{sgn}(t) \in \{+, -, 0\}.$$

The closure of the set of all points carrying a fixed admissible signature σ is a convex polytope $F_\sigma \subset S^{m-1}$. The ordered sequence of polytopes traversed by the forward orbit of $x^{(0)}$ is called the *trajectory route* of $x^{(0)}$.

Definition 5 (Invariant regions P and Q). For operator (1) define

$$P = \{x \in S^{m-1} : Ax \geq 0\}, \quad Q = \{x \in S^{m-1} : Ax \leq 0\}.$$

These sets are the *positive* and *negative basins* of V , respectively. Note that $P = F_\sigma$ for $\sigma = (+, \dots, +)$ and $Q = F_\sigma$ for $\sigma = (-, \dots, -)$.

Definition 6 (Graph isomorphism of LV operators). Two $\mathcal{R}_2(m)$ operators V and $\tilde{V}$ with interaction matrices A and $\tilde{A}$, respectively, are called *isomorphic* if there exists a permutation $\pi \in S_m$ such that

$$\tilde{A}_{\pi(i)\pi(j)} = A_{ij} \quad \text{for all } i, j.$$

Equivalently, the directed graphs $\Gamma(A)$ and $\Gamma(\tilde{A})$ are isomorphic as directed graphs. The group of symmetries acting on C_m is the dihedral group D_m of order $2m$.

Burnside's lemma [7]. Let a finite group G act on a finite set X . The number of orbits (isomorphism classes) is

$$N = \frac{1}{|G|} \sum_{g \in G} |X^g|, \quad (2)$$

where $X^g = \{x \in X : g \cdot x = x\}$ is the set of elements fixed by g .

Classification on S^4 : $\mathcal{R}_2(5)$ configurations

Let $V : S^4 \rightarrow S^4$ be an $\mathcal{R}_2(5)$ operator. Each of the five edges of C_5 may be oriented in two ways, giving $2^5 = 32$ orientation patterns encoded by binary vectors $\varsigma = (\varepsilon_1, \dots, \varepsilon_5)$, $\varepsilon_i \in \{0, 1\}$ (clockwise = 1, counter-clockwise = 0).

Theorem 1. *Under the action of D_5 the 32 orientations of C_5 split into exactly four isomorphism classes.*

Proof. $|D_5| = 10$. The identity fixes all 32 patterns. Each of the four nontrivial rotations ($k = 1, 2, 3, 4$, $\gcd(k, 5) = 1$) acts as a single 5-cycle on the edges and fixes only the two constant vectors $(0, 0, 0, 0, 0)$ and $(1, 1, 1, 1, 1)$, giving $|X^g| = 2$ in each case. Since C_5 is an odd cycle, every reflection fixes exactly one edge and reverses the orientation of the remaining four; no pattern with a fixed single edge can be fixed globally under such a reflection, so $|X^g| = 0$ for all five reflections. By Burnside's lemma (2):

$$N = \frac{1}{10}(32 + 4 \cdot 2 + 5 \cdot 0) = \frac{40}{10} = 4. \quad \square$$

The four classes are distinguished by their source-sink structure:

- (I) $(1, 1, 1, 1, 1)$ — directed cycle, 0 sources, 0 sinks;
- (II) $(1, 1, 1, 1, 0)$ — 1 source, 1 sink, arc lengths $\{4, 1\}$;
- (III) $(1, 1, 1, 0, 0)$ — 1 source, 1 sink, arc lengths $\{3, 2\}$;
- (IV) $(1, 1, 0, 1, 0)$ — 2 sources, 2 sinks, arc lengths $\{2, 1, 1, 1\}$.

Class (I): directed cycle $(1, 1, 1, 1, 1)$. The representative orientation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \rightarrow 1$ gives the operator

$$\begin{aligned} x'_1 &= x_1(1 - a_1x_2 + a_5x_5), \\ x'_2 &= x_2(1 + a_1x_1 - a_2x_3), \\ x'_3 &= x_3(1 + a_2x_2 - a_3x_4), \\ x'_4 &= x_4(1 + a_3x_3 - a_4x_5), \\ x'_5 &= x_5(1 + a_4x_4 - a_5x_1), \end{aligned} \tag{3}$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & a_3 & 0 & -a_4 \\ -a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}.$$

The fixed-point set is

$$\text{Fix}(V) = \Gamma_{13} \cup \Gamma_{24} \cup \Gamma_{35} \cup \Gamma_{14} \cup \Gamma_{25} \cup \{M\},$$

where

$$\Gamma_{ij} = \text{co}\{e_i, e_j\}$$

denotes the edge of S^4 joining the vertices e_i and e_j , and the unique interior fixed point is

$$M = \left(\frac{a_2a_4}{\Delta}, \frac{a_3a_5}{\Delta}, \frac{a_1a_4}{\Delta}, \frac{a_2a_5}{\Delta}, \frac{a_1a_3}{\Delta} \right), \quad \Delta = a_2a_4 + a_3a_5 + a_1a_4 + a_2a_5 + a_1a_3.$$

Theorem 2. *For operator (3) the interior fixed point M is a transverse repeller for V and a transverse attractor for V^{-1} . In particular, $\omega(x^{(0)}) \subset \partial S^4$ for every $x^{(0)} \in \text{ri}S^4 \setminus \{M\}$.*

Proof. For $m = 5$, we apply the Lyapunov-function argument developed in Theorem 6. The corresponding Lyapunov function takes the explicit form

$$\begin{aligned}
 \varphi_M(Vx) &= \prod_{i=1}^5 x_i^{M_i} = (x_1(1 - a_1x_2 + a_5x_5))^{\frac{a_2a_4}{\Delta}} (x_2(1 + a_1x_1 - a_2x_3))^{\frac{a_3a_5}{\Delta}} \cdot \\
 &\cdot (x_3(1 + a_2x_2 - a_3x_4))^{\frac{a_1a_4}{\Delta}} (x_4(1 + a_3x_3 - a_4x_5))^{\frac{a_2a_5}{\Delta}} (x_5(1 + a_4x_4 - a_5x_1))^{\frac{a_1a_3}{\Delta}} = \\
 &= x_1^{\frac{a_2a_4}{\Delta}} x_2^{\frac{a_3a_5}{\Delta}} x_3^{\frac{a_1a_4}{\Delta}} x_4^{\frac{a_2a_5}{\Delta}} x_5^{\frac{a_1a_3}{\Delta}} \left((1 - a_1x_2 + a_5x_5)^{\frac{a_2a_4}{\Delta}} (1 + a_1x_1 - a_2x_3)^{\frac{a_3a_5}{\Delta}} \cdot \right. \\
 &\cdot (1 + a_2x_2 - a_3x_4)^{\frac{a_1a_4}{\Delta}} (1 + a_3x_3 - a_4x_5)^{\frac{a_2a_5}{\Delta}} (1 + a_4x_4 - a_5x_1)^{\frac{a_1a_3}{\Delta}} \Big) \leq \\
 &\leq \varphi_M(x) \left((1 - a_1x_2 + a_5x_5) \cdot \frac{a_2a_4}{\Delta} + (1 + a_1x_1 - a_2x_3) \cdot \frac{a_3a_5}{\Delta} + \right. \\
 &\left. + (1 + a_2x_2 - a_3x_4) \cdot \frac{a_1a_4}{\Delta} + (1 + a_3x_3 - a_4x_5) \cdot \frac{a_2a_5}{\Delta} + (1 + a_4x_4 - a_5x_1) \cdot \frac{a_1a_3}{\Delta} \right) \leq \varphi_M(x)
 \end{aligned}$$

satisfies $\varphi_M(Vx) \leq \varphi_M(x)$ with equality only at $x = M$ (Lemma 1), so forward orbits move away from M and accumulate on ∂S^4 . $\square$

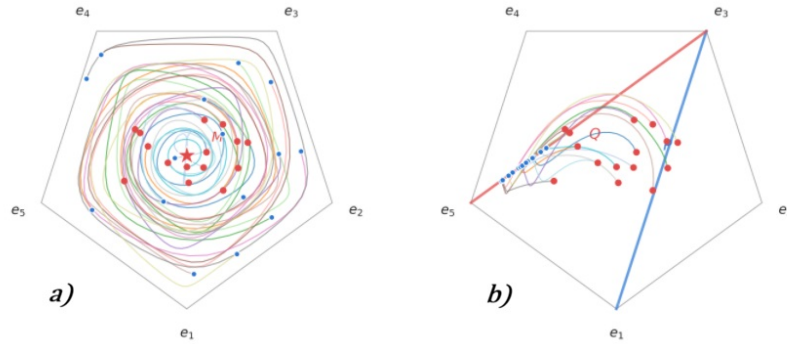


Figure 1: Trajectory dynamics of $\mathcal{R}_2(5)$ operators on S^4 (pentagon projection): *a)* Class (I) — all interior orbits spiral away from the interior repeller M (star) toward ∂S^4 ; *b)* Class (II) — blue segment $P = [e_1, M_1]$ (repeller), red segment $Q = [e_5, M_3]$ (attractor); all orbits follow $P \rightarrow Q$.

Class (II): one source, arc lengths $\{4, 1\}$, representative $(1, 1, 1, 1, 0)$.

This class contains 5 graphs; a representative is the orientation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \rightarrow 5 \leftarrow 1$ (vertex 1 is the source, vertex 5 is the sink):

$$\begin{aligned}
 x'_1 &= x_1(1 - a_1x_2 - a_5x_5), \\
 x'_2 &= x_2(1 + a_1x_1 - a_2x_3), \\
 x'_3 &= x_3(1 + a_2x_2 - a_3x_4), \\
 x'_4 &= x_4(1 + a_3x_3 - a_4x_5), \\
 x'_5 &= x_5(1 + a_4x_4 + a_5x_1),
 \end{aligned} \tag{4}$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & -a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & a_3 & 0 & -a_4 \\ a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}.$$

Solving $V(x) = x$ on the boundary gives fixed points

$$M_1 = \left(\frac{a_2}{a_1 + a_2}, 0, \frac{a_1}{a_1 + a_2}, 0, 0 \right), \quad M_2 = \left(0, \frac{a_3}{a_2 + a_3}, 0, \frac{a_2}{a_2 + a_3}, 0 \right), \quad M_3 = \left(0, 0, \frac{a_4}{a_3 + a_4}, 0, \frac{a_3}{a_3 + a_4} \right).$$

The invariant segments are

$$P = [e_1, M_1] \subset \Gamma_{13}, \quad Q = [e_5, M_3] \subset \Gamma_{35}.$$

Theorem 3. For operator (4): every point of riP is a transverse repeller, every point of riQ a transverse attractor, and for every $x^{(0)} \in riS^4$,

$$\alpha(x^{(0)}) \subset P = [e_1, M_1], \quad \omega(x^{(0)}) \subset Q = [e_5, M_3].$$

The global dynamics follows the repeller–attractor scheme $P \rightarrow Q$.

Proof. *Transversality.* For $x \in riP$ the characteristic polynomial of $DV(x)$ factors as

$$(1 - \lambda)^2(1 + a_1k - \lambda) \left(1 + \frac{a_2a_3(1 - k)}{a_1 + a_2} - \lambda\right) \left(1 + a_5k + \frac{a_2a_5(1 - k)}{a_1 + a_2} - \lambda\right), \quad 0 < k < 1,$$

so $\lambda_1 = \lambda_2 = 1$ along P and $\lambda_3, \lambda_4, \lambda_5 > 1$ (repeller). For $x \in riQ$ the analogous computation gives all transverse eigenvalues less than 1 (attractor).

Global dynamics. For $p = (p_1, 0, p_2, 0, p_3) \in P$ (so $Ap \geq 0$), the function $\varphi_p(x) = x_1^{p_1} x_3^{p_2} x_5^{p_3}$ satisfies, by the weighted AM–GM inequality,

$$\varphi_p(Vx) \leq \varphi_p(x)(1 - \langle Ap, x \rangle) \leq \varphi_p(x),$$

since $\langle Ap, x \rangle \geq 0$ on S^4 . Forward orbits are thus driven into Q . Here,

$$\langle Ap, x \rangle = \sum_{i=1}^m (Ap)_i x_i$$

denotes the standard Euclidean inner product in $\mathbb{R}^m$. The symmetric argument for V^{-1} gives $\alpha(x^{(0)}) \subset P$. $\square$

The 8 admissible coordinate signatures for operator (4) are

$$\sigma_1 = (-++++), \sigma_2 = (-++-+), \sigma_3 = (-+---), \sigma_4 = (-+-++), \sigma_5 = (--+++), \sigma_6 = (--+-+),$$

$$\sigma_7 = (----+), \sigma_8 = (---++),$$

with convex polytopes (where $C = \frac{1}{D}(a_2a_4, 0, a_1a_4, 0, a_1a_3)$, $D = a_1a_3 + a_1a_4 + a_2a_4$):

$$\begin{aligned} F_1 &= co\{e_1, e_2, M_1, M_2, C\}, & F_2 &= co\{e_1, e_2, e_5, M_2, C\}, & F_3 &= co\{e_1, e_4, e_5, M_2, C\}, \\ F_4 &= co\{e_1, e_4, M_1, M_2, C\}, & F_5 &= co\{e_2, e_3, M_1, M_2, M_3, C\}, & F_6 &= co\{e_2, e_5, M_2, M_3, C\}, \\ F_7 &= co\{e_4, e_5, M_2, M_3, C\}, & F_8 &= co\{e_3, e_4, M_1, M_2, M_3, C\}. \end{aligned}$$

The transitions across common facets (verified by sign of the corresponding increment) yield an acyclic route diagram terminating at Q . The direction of the transition across each common facet is set by the sign of the corresponding increment; for example, at $T = F_2 \cap F_6 = \{e_2, e_5, M_2, C\}$, $x_2'' = x_2' \left(1 - \frac{a_2a_4k_1k_4}{D}(a_1^2 + a_2^2) - \frac{a_1^2a_3a_4K_3k_4}{D(a_2+a_3)} - \frac{a_1a_2a_3a_4^2a_5k_2k_4}{D^2}\right) < x_2'$, so $T \subset F_6$ and the transition is $F_2 \rightarrow F_6$. Working through all adjacent pairs produces the route diagram, which terminates at Q .

Class (III): one source, arc lengths $\{3, 2\}$, representative $(1, 1, 1, 0, 0)$.

The orientation $1 \rightarrow 2 \rightarrow 3 \rightarrow 4 \leftarrow 5 \leftarrow 1$ (single source 1, single sink 4) gives

$$\begin{aligned} x_1' &= x_1(1 - a_1x_2 - a_5x_5), & x_2' &= x_2(1 + a_1x_1 - a_2x_3), \\ x_3' &= x_3(1 + a_2x_2 - a_3x_4), & x_4' &= x_4(1 + a_3x_3 + a_4x_5), \\ x_5' &= x_5(1 - a_4x_4 + a_5x_1), \end{aligned} \tag{5}$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & -a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & -a_3 & 0 \\ 0 & 0 & a_3 & 0 & a_4 \\ a_5 & 0 & 0 & -a_4 & 0 \end{pmatrix}.$$

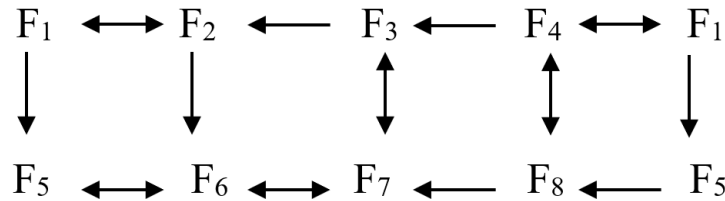


Figure 2: trajectory route for class (II)

Theorem 4. For operator (5): (i) $P = [e_1, M_1] \subset \Gamma_{13}$ and $Q = [e_4, M_2] \subset \Gamma_{24}$, where

$$M_1 = \left(\frac{a_2}{a_1 + a_2}, 0, \frac{a_1}{a_1 + a_2}, 0, 0 \right), \quad M_2 = \left(0, \frac{a_3}{a_2 + a_3}, 0, \frac{a_2}{a_2 + a_3}, 0 \right);$$

(ii) every point of riP is a transverse repeller and every point of riQ a transverse attractor; (iii) for every $x^{(0)} \in riS^4$, $\alpha(x^{(0)}) \subset P$ and $\omega(x^{(0)}) \subset Q$.

Proof.

Step 1 (two constant signs). The increments $(Ax)_1 = -a_1x_2 - a_5x_5 < 0$ and $(Ax)_4 = a_3x_3 + a_4x_5 > 0$ hold for all $x \in riS^4$, so x_1 strictly decreases and x_4 strictly increases along every interior orbit; both converge.

Step 2 (invariant segments). From $Ax \geq 0$: $(Ax)_1 \geq 0$ forces $x_2 = x_5 = 0$; then $(Ax)_3 = -a_3x_4 \geq 0$ forces $x_4 = 0$; and $(Ax)_2 \geq 0$ gives $x_1 \geq a_2/(a_1 + a_2)$. Hence $P = [e_1, M_1] \subset \Gamma_{13}$. From $Ax \leq 0$: $(Ax)_4 \leq 0$ forces $x_3 = x_5 = 0$; then $(Ax)_2 = a_1x_1 \leq 0$ forces $x_1 = 0$; and $(Ax)_3 \leq 0$ gives $x_2 \leq a_3/(a_2 + a_3)$. Hence $Q = [e_4, M_2] \subset \Gamma_{24}$. Thus P and Q lie in distinct 2-faces of S^4 .

Step 3 (absorbing region). On the boundary $\ell_5 : \{a_5x_1 = a_4x_4\}$ one has $x'_5 = x_5$ and $(Ax')_5 < 0$, so the region $\{a_4x_4 \geq a_5x_1\}$ is forward-invariant; since x_4 increases and x_1 decreases, every interior trajectory eventually enters it.

Step 4 (limit sets). On the ω -limit set Ω , x_4 is constant (positive supremum), so $(Ay)_4 = a_3y_3 + a_4y_5 = 0$ forces $y_3 = y_5 = 0$, i.e. $\Omega \subset \Gamma_{124}$. On Γ_{124} the residual dynamics drives $x_1 \rightarrow 0$, so $\Omega \subset \Gamma_{24}$. On Γ_{24} the transverse eigenvalues satisfy $\lambda_1 = 1 - a_1x_2 < 1$, $\lambda_5 = 1 - a_4x_4 < 1$, and $\lambda_3 = 1 + a_2x_2 - a_3x_4 < 1$ iff $x_2 < a_3/(a_2 + a_3)$; hence the attracting part is exactly $[e_4, M_2] = Q$. The symmetric argument for V^{-1} gives $\alpha(x^{(0)}) \subset P$.

Step 5 (acyclic route $P \rightarrow Q$). The simplex riS^4 is partitioned into at most four signature cells by the boundaries $\ell_2 : \{a_1x_1 = a_2x_3\}$ and $\ell_3 : \{a_2x_2 = a_3x_4\}$. Since $\sigma_1 \equiv -$ and $\sigma_4 \equiv +$ (Step 1), the only varying signs are $\sigma_2 = \text{sgn}(a_1x_1 - a_2x_3)$ and $\sigma_3 = \text{sgn}(a_2x_2 - a_3x_4)$. On ℓ_2 one verifies $(Ax')_2 = a_1x'_1(1 + \dots) - a_2x'_3(\dots) < 0$, so the cell $\{a_1x_1 \leq a_2x_3\}$ is forward-invariant once entered. On ℓ_3 , since x_4 strictly increases and $x_2 \rightarrow 0$ (forced by $x_1 \rightarrow 0$ and the residual dynamics), the sign of $a_2x_2 - a_3x_4$ eventually becomes negative and stays so. Hence the route graph has no directed cycles: the four cells can only be traversed in the order dictated by the monotone decrease of x_1 and increase of x_4 , terminating at Q . $\square$

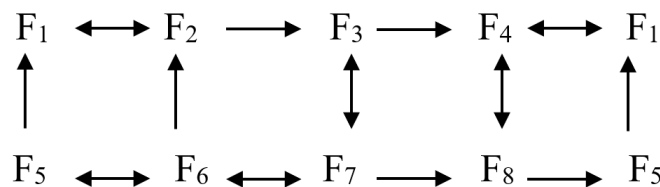


Figure 3: trajectory route for class (III)

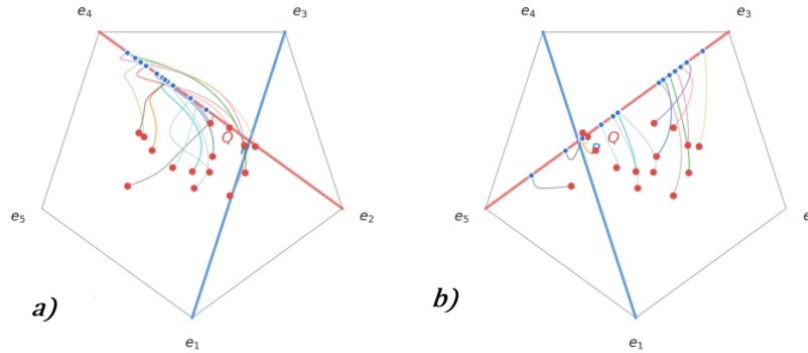


Figure 4: Trajectory dynamics of $\mathcal{R}_2(5)$ operators on S^4 (pentagon projection): a) Class (III) — $P = [e_1, M_1] \subset \Gamma_{13}$ (repeller, blue), $Q = [e_4, M_2] \subset \Gamma_{24}$ (attractor, red); b) Class (IV) — $P = [e_1, e_4]$ (repeller, blue), $Q = [e_3, e_5]$ (attractor, red); single transition $F_+ \rightarrow F_-$ yields terminal signature $(-, -, +, -, +)$.

Class (IV): two sources/two sinks, representative $(1, 1, 0, 1, 0)$.

The orientation $1 \rightarrow 2 \rightarrow 3 \leftarrow 4 \rightarrow 5 \leftarrow 1$ (sources $\{1, 4\}$, sinks $\{3, 5\}$) gives

$$\begin{aligned} x'_1 &= x_1(1 - a_1x_2 - a_5x_5), & x'_2 &= x_2(1 + a_1x_1 - a_2x_3), \\ x'_3 &= x_3(1 + a_2x_2 + a_3x_4), & x'_4 &= x_4(1 - a_3x_3 - a_4x_5), \\ x'_5 &= x_5(1 + a_4x_4 + a_5x_1), \end{aligned} \quad (6)$$

with skew-symmetric matrix

$$A = \begin{pmatrix} 0 & -a_1 & 0 & 0 & -a_5 \\ a_1 & 0 & -a_2 & 0 & 0 \\ 0 & a_2 & 0 & a_3 & 0 \\ 0 & 0 & -a_3 & 0 & -a_4 \\ a_5 & 0 & 0 & a_4 & 0 \end{pmatrix}.$$

Theorem 5. For operator (6): (i) $P = [e_1, e_4]$ and $Q = [e_3, e_5]$; (ii) every point of riP is a transverse repeller and every point of riQ a transverse attractor; (iii) for every $x^{(0)} \in riS^4$, $\alpha(x^{(0)}) \subset P$ and $\omega(x^{(0)}) \subset Q$.

Proof.

Step 1 (four constant signs). For $x \in riS^4$ the two sources and two sinks have definite signs: $(Ax)_1 = -a_1x_2 - a_5x_5 < 0$, $(Ax)_4 = -a_3x_3 - a_4x_5 < 0$ (sources x_1, x_4 decrease), and $(Ax)_3 = a_2x_2 + a_3x_4 > 0$, $(Ax)_5 = a_4x_4 + a_5x_1 > 0$ (sinks x_3, x_5 increase). Only $\sigma_2 = \text{sgn}(a_1x_1 - a_2x_3)$ varies, producing two cells $F_{\pm} = \{a_1x_1 \gtrless a_2x_3\}$ meeting along $\ell_2 : \{a_1x_1 = a_2x_3\}$.

Step 2 (invariant edges). $Ax \geq 0$ forces $x_2 = x_3 = x_5 = 0$ (all sink and mixed coordinates zero), so $P = [e_1, e_4]$. $Ax \leq 0$ forces $x_1 = x_2 = x_4 = 0$, so $Q = [e_3, e_5]$.

Step 3 (transversality). On riP ($x_2 = x_3 = x_5 = 0$): $\lambda_2 = 1 + a_1x_1 > 1$, $\lambda_3 = 1 + a_3x_4 > 1$, $\lambda_5 = 1 + a_4x_4 + a_5x_1 > 1$ (repeller). On riQ ($x_1 = x_2 = x_4 = 0$): $\lambda_1 = 1 - a_5x_5 < 1$, $\lambda_2 = 1 - a_2x_3 < 1$, $\lambda_4 = 1 - a_3x_3 - a_4x_5 < 1$ (attractor).

Step 4 (route $P \rightarrow Q$). On ℓ_2 one verifies $(Ax')_2 = -a_1^2x_1x_2 - a_1a_5x_1x_5 - a_2^2x_2x_3 - a_2a_3x_3x_4 < 0$, so $F_- = \{a_1x_1 \leq a_2x_3\}$ is forward-invariant; since $x_1 \rightarrow 0$ and x_3 increases, every trajectory enters F_- and the single transition $F_+ \rightarrow F_-$ occurs, leaving terminal signature $(-, -, +, -, +)$.

Step 5 (limit sets). On $\omega(x^{(0)})$, x_3 and x_5 are constant, so $(Ay)_5 = a_4y_4 + a_5y_1 = 0$ forces $y_1 = y_4 = 0$ and $(Ay)_3 = a_2y_2 = 0$ forces $y_2 = 0$; hence $\omega(x^{(0)}) \subset [e_3, e_5] = Q$. Symmetrically $\alpha(x^{(0)}) \subset [e_1, e_4] = P$. $\square$

General odd- m cyclic operators

Let $m \geq 3$ be odd and let $V : S^{m-1} \rightarrow S^{m-1}$ be the cyclic operator

$$x'_k = x_k(1 + a_{k-1}x_{k-1} - a_kx_{k+1}), \quad k = 1, \dots, m, \quad 0 < a_k \leq 1, \quad (7)$$

with indices modulo m and skew-symmetric matrix A with entries $A_{k,k-1} = a_{k-1}$, $A_{k,k+1} = -a_k$.

Proposition 1. For odd m , $\det A = 0$ for all $a_k > 0$.

Proof. Since $A^\top = -A$, $\det A = \det(A^\top) = \det(-A) = (-1)^m \det A = -\det A$, so $\det A = 0$. $\square$

Remark. For even m , $\det A = (\text{Pf } A)^2 \geq 0$ $\text{Pf } A = 0 \iff$ interior fixed point exists.

The unique interior fixed point. The equation $(AM)_k = 0$ reads

$$a_{k-1}M_{k-1} - a_kM_{k+1} = 0, \quad \text{i.e.} \quad M_{k+1} = \frac{a_{k-1}}{a_k}M_{k-1}. \quad (8)$$

This is a two-step recurrence. For odd m , $\gcd(2, m) = 1$, so stepping by 2 modulo m visits every index in a single orbit. Thus every M_k is determined by $M_1 > 0$, and normalising $\sum_k M_k = 1$ gives a unique $M \in \text{ri}S^{m-1}$.

Proposition 2. For m odd and all $a_k > 0$, $\dim \text{Ker } A = 1$ and $\text{Ker } A \cap S^{m-1} = \{M\}$.

Proof. Any solution of $Av = 0$ is determined by v_1 alone (single orbit under step-2 shifts for odd m), so $\dim \text{Ker } A = 1$. The unique positive solution, after normalisation, gives $M \in \text{ri}S^{m-1}$. $\square$

Example 1 ($m = 5$). From recurrence (8): $M_3 = (a_1/a_2)M_1$, $M_5 = (a_1a_3)/(a_2a_4) \cdot M_1$, and completing the cycle yields

$$M_1 = \frac{a_2a_4}{\Delta}, \quad M_2 = \frac{a_3a_5}{\Delta}, \quad M_3 = \frac{a_1a_4}{\Delta}, \quad M_4 = \frac{a_2a_5}{\Delta}, \quad M_5 = \frac{a_1a_3}{\Delta},$$

where $\Delta = a_2a_4 + a_3a_5 + a_1a_4 + a_2a_5 + a_1a_3$.

Example 2 ($m = 7$). The same procedure gives

$$\begin{aligned} M_1 &= \frac{a_2a_4a_6}{\Delta}, & M_2 &= \frac{a_3a_5a_7}{\Delta}, & M_3 &= \frac{a_1a_4a_6}{\Delta}, & M_4 &= \frac{a_2a_5a_7}{\Delta}, \\ M_5 &= \frac{a_1a_3a_6}{\Delta}, & M_6 &= \frac{a_2a_4a_7}{\Delta}, & M_7 &= \frac{a_1a_3a_5}{\Delta}, \end{aligned}$$

where Δ is the sum of all seven numerators.

Lemma 1. The function $\varphi_M(x) = \prod_{k=1}^m x_k^{M_k}$ satisfies $\varphi_M(Vx) \leq \varphi_M(x)$ on $\text{ri}S^{m-1}$, with equality if and only if $x = M$.

Proof. Write $u_k = (Ax)_k$, so $x'_k = x_k(1 + u_k)$. Then

$$\frac{\varphi_M(Vx)}{\varphi_M(x)} = \prod_{k=1}^m (1 + u_k)^{M_k}.$$

By skew-symmetry and $AM = 0$: $\sum_k M_k u_k = \langle M, Ax \rangle = -\langle AM, x \rangle = 0$. The weighted AM–GM inequality with weights $M_k > 0$, $\sum M_k = 1$ gives

$$\prod_{k=1}^m (1 + u_k)^{M_k} \leq \sum_{k=1}^m M_k(1 + u_k) = 1 + \sum_{k=1}^m M_k u_k = 1.$$

Equality requires all $1 + u_k$ equal; combined with $\sum M_k u_k = 0$ this forces each $u_k = 0$, i.e. $Ax = 0$, hence $x = M$ by Proposition 2. $\square$

Theorem 6. Let $m \geq 3$ be odd and let M be the unique interior fixed point. For every $x^{(0)} \in \text{ri}S^{m-1} \setminus \{M\}$: (i) $\alpha(x^{(0)}) = \{M\}$; (ii) $\omega(x^{(0)}) \subset \partial S^{m-1}$.

Proof.

Forward direction. For $x^{(0)} \neq M$ Lemma 1 gives strict decrease: $\varphi_M(x^{(n+1)}) < \varphi_M(x^{(n)})$. Being bounded below by 0, the sequence converges to some $L \geq 0$ with $L < \varphi_M(M)$. Suppose $y \in \omega(x^{(0)}) \cap \text{ri}S^{m-1}$. Then φ_M is constant on the forward orbit of y (since $y \in \omega$), yet Lemma 1 gives strict decrease unless $y = M$. Since $L < \varphi_M(M)$ we have $y \neq M$, a contradiction. Hence $\omega(x^{(0)}) \cap \text{ri}S^{m-1} = \emptyset$, proving (ii).

Backward direction. Since V is a homeomorphism of S^{m-1} [1,2], the backward orbit $x^{(-n)}$ is well defined. By Lemma 1, the sequence $\varphi_M(x^{(-n)})$ is strictly increasing and bounded above, and therefore converges to some value L .

Let $z \in \alpha(x^{(0)})$. Then there exists a sequence $n_k \rightarrow \infty$ such that $x^{(-n_k)} \rightarrow z$, and hence, by continuity,

$$\varphi_M(z) = L.$$

Since the α -limit set is invariant under V , we also have $Vz \in \alpha(x^{(0)})$, and therefore

$$\varphi_M(Vz) = L = \varphi_M(z).$$

By the equality condition in Lemma 1, this is possible only if $z = M$. Hence every point of $\alpha(x^{(0)})$ is equal to M , and therefore

$$\alpha(x^{(0)}) = \{M\}.$$

Relation to previously known results. The results of [1] provide a general framework for the qualitative analysis of Volterra quadratic stochastic operators, including Lyapunov functions and estimates for limit sets. The present results develop this framework further for cyclic $R_2(m)$ operators by combining the graph structure with an explicit analysis of the global dynamics.

For $R_2(5)$ operators, Theorems 2–5 give a complete classification of all orientations into four D_5 -isomorphism classes and determine, for each class, the corresponding repeller–attractor structure, invariant sets P and Q , transverse stability properties, and global trajectory routes. Thus, these theorems provide a detailed dynamical classification that goes beyond the general qualitative statements available in [1].

For arbitrary odd m , Theorem 6 gives a further global result for the directed cyclic configuration. Namely, the unique interior fixed point is determined explicitly, and the associated Lyapunov function yields the precise asymptotic relations

$$\alpha(x^{(0)}) = \{M\}, \quad \omega(x^{(0)}) \subset \partial S^{m-1}$$

for every non-fixed interior trajectory. Hence, the results of Theorems 2–6 complement and refine the general theory in [1] by providing an explicit classification and a more detailed description of the global dynamics for cyclic Lotka–Volterra operators.

Proposition 3. *For the cyclic $\mathcal{R}_2(m)$ operator with m odd, $\text{Ker}A \cap \text{ri}S^{m-1} \neq \emptyset$ if and only if the orientation is a directed cycle; otherwise $\text{Ker}A \cap S^{m-1} \subset \partial S^{m-1}$.*

Proof. (If.) If the orientation is a directed cycle, every vertex has in-degree and out-degree 1, so the recurrence $(AM)_k = 0$ takes the form $a_{k-1}M_{k-1} = a_kM_{k+1}$ with all $a_k > 0$. For odd m , $\gcd(2, m) = 1$, so stepping by 2 visits every index; a unique positive solution exists and after normalisation gives $M \in \text{ri}S^{m-1}$.

(Only if.) Suppose $M \in \text{Ker}A \cap \text{ri}S^{m-1}$, so all $M_k > 0$. Then $(AM)_k = 0$ reads $a_{k-1}M_{k-1} = a_kM_{k+1}$ for each k . If vertex k had two outgoing edges (say to $k+1$ and $k-1$), then $(AM)_k = -a_{k,k+1}M_{k+1} - a_{k,k-1}M_{k-1} < 0$ since all terms are positive, contradicting $(AM)_k = 0$. Hence every vertex has exactly one outgoing and one incoming edge, i.e. the orientation is a directed cycle. $\square$

Corollary 1. *For $\mathcal{R}_2(5)$ on S^4 , an interior fixed point exists if and only if the orientation is class (I). In classes (II)–(IV), some vertex has degree imbalance, forcing the unique $\text{Ker}A$ -vector to lie on ∂S^4 .*

Proof. Apply Proposition 3 with $m = 5$. In class (II), vertex 1 has two outgoing edges, so $(AM)_1 = -a_1M_2 - a_5M_5 = 0$ with $a_1, a_5 > 0$ forces $M_2 = M_5 = 0$, hence $M \notin \text{ri}S^4$. The argument is analogous for classes (III) and (IV). $\square$

Conclusion

We have given a complete classification of $\mathcal{R}_2(5)$ Lotka–Volterra operators on S^4 into four D_5 -isomorphism classes (I)–(IV), and for each class proved the $P \rightarrow Q$ repeller–attractor scheme via coordinate signatures and Lyapunov functions. The general odd- m theorem shows that the directed-cycle operator always has a unique interior repeller M and that every other interior trajectory escapes to the boundary, with M as the sole α -limit point. These results connect combinatorial graph data (orientation type) directly to global qualitative dynamics on the simplex. For even m the situation is richer: the Pfaffian $\text{Pf}A$ may be nonzero, and the interior of S^{m-1} may carry no fixed point even for the directed-cycle orientation; a full classification for $m = 6$ into nine D_6 -isomorphism classes, together with the condition $a_1a_3a_5 = a_2a_4a_6$ distinguishing the degenerate regime, is the subject of ongoing work [3,4]. The coordinate-signature framework developed here extends naturally to $\mathcal{R}_r(m)$ operators with $r > 2$ interactions per vertex and to bigraph structures on higher-dimensional simplices.

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