

**ANALYTICAL INVESTIGATION OF PURSUIT AND EVASION GAMES FOR SECOND-ORDER
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ABSTRACT. This paper considers a pursuit-evasion differential game where the dynamics of players are governed by second-order parameterized differential equations. This model substantially generalizes the problem formulation by extending the classical inertial motion to oscillating and exponentially unstable dynamic regimes. A family of parallel pursuit strategies (Π -strategy) for the pursuer is constructed, and exact necessary and sufficient conditions for guaranteed capture in a finite time interval are established alongside explicit analytical formulas for optimal capture time. Furthermore, an explicit evasion strategy for the evader is proposed, lower bounds for the distance between players are derived, and conditions for guaranteed evasion over an infinite time horizon are proven. The results obtain generalize and extend classical research on second-order controlled systems by Isaacs, Petrosyan, Pshenichnyi, Azamov, Samatov, and others.

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Key words: Differential game, Pursuit problem, Evasion problem, Parameterized dynamics, Parallel pursuit strategy, Π -strategy, Guaranteed capture time, Oscillatory motion, Unstable system.

1. Introduction

Differential games theory investigates conflict processes and control problems modeled by systems of differential equations. Following the development of L. S. Pontryagin's Maximum Principle, an intrinsic connection was established between optimal control theory and differential games. Essentially, differential game problems serve as a natural generalization of optimal control problems to multi-agent settings.

The foundation of differential games theory was laid by the American mathematician R. Isaacs [1], whose famous monograph provided significant momentum for the rapid development of the field. Fundamental and modern contributions in this direction have been obtained by L. S. Pontryagin [2], N. N. Krasovskii [3], L. A. Petrosyan [4], B. N. Pshenichnyi, A. A. Chikrii, A. I. Subbotin, and numerous other scholars.

In his monograph, R. Isaacs [1] formulated several special game problems that proved vital for subsequent research. For scenarios with geometric constraints on players' controls, the parallel approach strategy method was extensively analyzed by L. A. Petrosyan [4]. This approach served as a foundational step in establishing norms for multi-player differential games under complex geometric conditions. Subsequently, parallel approach strategies under geometric, integral, and mixed constraints on controls were successfully studied by A. Azamov, B. T. Samatov [5–7], and their students.

At the same time, significant progress has been made in recent years regarding pursuit-evasion and optimal control problems for systems governed by differential equations. Specifically, pursuit-evasion differential games described by first- and second-order differential equations with geometric constraints on controls, as well as optimal control problems for distributed parameter systems, have been thoroughly investigated by N. A. Mamadaliyev and Kh. Ya. Mustapokulov [8–12]. Their works deeply explore the construction of parallel

approach strategies in impulse-controlled and second-order systems, the application of the resolving functions method, and the invariance of set-valued mappings in infinite-dimensional systems.

In the works of B. T. Samatov, U. B. Soyibboev [7], and other aforementioned researchers, the dynamics of the objects were restricted to simple inertial motion regimes.

This paper addresses the general case where players' controls are subject to geometric constraints, and the dynamics of motion are governed by second-order parametric differential equations. The introduction of a dynamic parameter enables the classical inertial motion results studied by B. T. Samatov and U. B. Soyibboev [7] to be fully extended and generalized to oscillatory and exponentially unstable dynamic regimes. A parallel approach strategy that guarantees a guaranteed pursuit time for the pursuer is constructed, and conditions for guaranteed evasion over an infinite time interval along with distance estimates are established for the evader.

2. Statement of problem

Let us assume that in the space $\mathbb{R}^n$, a controlled object called the pursuer P is pursuing another object called the evader E . We denote the position of the pursuer in $\mathbb{R}^n$ by x , and the position of the evader by y .

The dynamics of the players' motion are governed by the following differential equations and initial conditions, respectively:

$$P : \ddot{x} + bx = u, \quad x(0) = x_0, \quad \dot{x}(0) = x_1, \quad (2.1)$$

$$E : \ddot{y} + by = v, \quad y(0) = y_0, \quad \dot{y}(0) = y_1, \quad (2.2)$$

where $x, y, x_0, y_0, x_1, y_1, u, v \in \mathbb{R}^n$, $b \in \mathbb{R}$, $n \geq 2$; x_0 and y_0 are the initial positions of the players, while x_1 and y_1 are their initial velocity vectors. We assume that $x_0 \neq y_0$ and $x_1 = y_1$.

Here, the time-dependent variation of u , represented as $u(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n$, must be a measurable function. We impose a geometric constraint (in short, a G -constraint) on this vector-function in the following form:

$$|u(t)| \leq \alpha \quad \text{for almost all } t \geq 0, \quad (2.3)$$

where α is a given positive parameter representing the maximum acceleration value of the pursuer.

Similarly, the time-dependent variation of v , represented as $v(\cdot) : [0, \infty) \rightarrow \mathbb{R}^n$, must be a measurable function, and we impose the following G -constraint on this vector-function:

$$|v(t)| \leq \beta \quad \text{for almost all } t \geq 0, \quad (2.4)$$

where β is a given non-negative parameter representing the maximum acceleration value of the evader.

The control parameters of the players are the acceleration vectors u and v , which depend on time ($t \geq 0$). We denote the sets of all control parameters u and v satisfying conditions (2.3) and (2.4), respectively, by U and V .

Definition 1. A measurable function $u(\cdot)$ ($v(\cdot)$) satisfying condition (2.3) (condition (2.4)) is called an admissible control of the pursuer (evader) in the class U (V), where the introduced pair of admissible control classes (U, V) defines a differential game.

Integrating the differential equations (2.1) and (2.2) under the corresponding initial conditions yields the trajectories of the pursuer P and the evader E , expressed analytically depending on the value of the parameter b :

Case 1 (for $b > 0$ – harmonic oscillatory motion):

$$x(t) = x_0 \cos(\sqrt{b}t) + \frac{x_1}{\sqrt{b}} \sin(\sqrt{b}t) + \frac{1}{\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))u(s) ds, \quad (2.5a)$$

$$y(t) = y_0 \cos(\sqrt{b}t) + \frac{y_1}{\sqrt{b}} \sin(\sqrt{b}t) + \frac{1}{\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))v(s) ds. \quad (2.6a)$$

Case 2 (for $b = 0$ – classical inertial motion):

$$x(t) = x_0 + x_1 t + \int_0^t (t-s)u(s) ds, \quad (2.5b)$$

$$y(t) = y_0 + y_1 t + \int_0^t (t-s)v(s) ds. \quad (2.6b)$$

Case 3 (for $b < 0$ – exponentially unstable motion):

$$x(t) = x_0 \cosh(\sqrt{-b}t) + \frac{x_1}{\sqrt{-b}} \sinh(\sqrt{-b}t) + \frac{1}{\sqrt{-b}} \int_0^t \sinh(\sqrt{-b}(t-s))u(s) ds, \quad (2.5c)$$

$$y(t) = y_0 \cosh(\sqrt{-b}t) + \frac{y_1}{\sqrt{-b}} \sinh(\sqrt{-b}t) + \frac{1}{\sqrt{-b}} \int_0^t \sinh(\sqrt{-b}(t-s))v(s) ds. \quad (2.6c)$$

It is well known that control functions depending solely on the time parameter $t \geq 0$ are insufficient for the pursuer to solve the pursuit problem; appropriate forms of control must be formulated as strategies. Several methods exist for defining such concepts. Below, we introduce some notions to formalize these definitions.

First, we introduce the following notations:

$$z(t) = x(t) - y(t), \quad z_0 = x_0 - y_0, \quad \dot{z}(0) = x_1 - y_1.$$

Definition 2. A function $\mathbf{u} : \mathbb{R}^n \times S_\beta \rightarrow S_\alpha$ is called a strategy of the pursuer if the following conditions are satisfied:

- (a) The function $\mathbf{u}(z_0, v)$ is measurable with respect to $v \in V$ in the sense of Borel;
- (b) Admissibility: For every $v(\cdot) \in V$, the inclusion $\mathbf{u}(z_0, v(\cdot)) \in U$ holds on some time interval $[0, t]$. In this case, the function $\mathbf{u}(z_0, v(\cdot))$ is called a realization of the strategy $\mathbf{u}(z_0, v)$;
- (c) Volterra property: If $v_1(s) = v_2(s)$ for almost all $s \in [0, t]$ for any $v_1(\cdot), v_2(\cdot) \in V$ and $t \geq 0$, then $u_1(s) = u_2(s)$ for almost all $s \in [0, t]$, where $u_i(\cdot) = \mathbf{u}(z_0, v_i(\cdot))$, $i = 1, 2$; S_α and S_β are closed balls in $\mathbb{R}^n$ centered at the origin with radii α and β , respectively.

Definition 3. A strategy $\mathbf{u}(z_0, v)$ is called a parallel approach strategy (Π -strategy) if, for any $v(\cdot) \in V$, the solution $z(t)$ to the Cauchy problem

$$\ddot{z} + bz = \mathbf{u}(z_0, v(t)) - v(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0 \quad (2.7)$$

can be represented in the form:

$$z(t) = z_0 \Lambda(t, v(\cdot)), \quad \Lambda(0, v(\cdot)) = 1,$$

where $\Lambda(t, v(\cdot))$ is a scalar function of time $t \geq 0$. Typically, this function is referred to as the approach function in pursuit problems.

Definition 4. In the pursuit problem, a Π -strategy is said to guarantee capture of the evader within the time interval $[0, t_g]$ if, for every $v(\cdot) \in V$, the following conditions hold:

- (a) There exists some time $t^* \in [0, t_g]$ such that $z(t^*) = 0$;
- (b) The inclusion $\mathbf{u}(z_0, v(\cdot)) \in U$ holds on the time interval $[0, t^*]$. Here, t_g is called the guaranteed pursuit (or capture) time.

Definition 5. A function $\mathbf{v}^* : \mathbb{R}_+ \rightarrow \mathbb{R}^n$ is called a strategy of the evader if $\mathbf{v}^*(t)$ is measurable with respect to t in the sense of Lebesgue.

Now, let us consider the (U, V) game from the evader's perspective.

Definition 6. In the evasion problem, a control $\mathbf{v}^*(\cdot) \in V$ is said to guarantee evasion if, for any $u(\cdot) \in U$, the solution $z(t)$ to the Cauchy problem

$$\ddot{z} + bz = u(t) - \mathbf{v}^*(t), \quad z(0) = z_0, \quad \dot{z}(0) = 0 \quad (2.8)$$

remains non-zero, i.e., $z(t) \neq 0$ for all $t \geq 0$.

This paper is devoted to solving the following problems under the conditions where the controls of the players are subject to constraints (2.3) and (2.4), respectively:

Problem 1. Pursuit problem: To construct a Π -strategy for the pursuer and determine the guaranteed capture time in the (U, V) game.

Problem 2. Evasion problem: To construct a strategy for the evader and estimate how the distance between the players changes over time.

3. A solution of the pursuit problem

In parametric mathematical problems, an interesting property of the final analytical results is their explicit dependence on these parameters, which are treated as constants in the solution. However, these parameters can assist in determining the conditions for the solvability (feasibility) of these problems. In this section, we present necessary and sufficient conditions for the pursuit problem to be solvable in the (U, V) game.

If the pursuer and the evader choose admissible controls $u(\cdot) \in U$ and $v(\cdot) \in V$, respectively, then by integrating the differential equation of relative motion corresponding to equations (2.7) and (2.8), the relative trajectory $z(t) = x(t) - y(t)$ is expressed in the following analytical forms depending on the value of the parameter b :

Case 1 (for $b > 0$ – harmonic oscillatory motion):

$$z(t) = z_0 \cos(\sqrt{b}t) + \frac{1}{\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))(u(s) - v(s)) ds. \quad (3.1a)$$

Case 2 (for $b = 0$ – classical inertial motion):

$$z(t) = z_0 + \int_0^t (t-s)(u(s) - v(s)) ds. \quad (3.1b)$$

Case 3 (for $b < 0$ – exponentially unstable motion):

$$z(t) = z_0 \cosh(\sqrt{-b}t) + \frac{1}{\sqrt{-b}} \int_0^t \sinh(\sqrt{-b}(t-s))(u(s) - v(s)) ds. \quad (3.1c)$$

In terms of the newly introduced notations, the objective of the pursuer is to achieve the equality $z(t^*) = 0$ at some time $t^* > 0$. The objective of the evader is to maintain the relation $z(t) \neq 0$ for all $t \geq 0$.

For the pursuer, open-loop strategies (i.e., admissible controls depending solely on time t) are insufficient to achieve their goal. Therefore, similar to work [6], the pursuer's strategy in the present setting can be constructed based exclusively on the current values of the evader's acceleration function $v(t)$, $t \geq 0$, alongside the given constants z_0 and α .

To solve the pursuit problem, let us assume that at the current time t , the pursuer is aware of the initial parameters x_0, y_0, x_1, y_1 , the constants α, β , the current time t , and the value of the evader's control $v(t)$.

Definition 7. Suppose that $\alpha \geq \beta$. Then, in the (U, V) game, the function

$$\mathbf{u}(z_0, v) = v - \lambda(z_0, v)\xi_0 \quad (3.2)$$

is called the parallel pursuit strategy (in short, Π -strategy) of the pursuer, where

$$\lambda(z_0, v) = \langle v, \xi_0 \rangle + \sqrt{\langle v, \xi_0 \rangle^2 + \alpha^2 - |v|^2}, \quad \xi_0 = \frac{z_0}{|z_0|} \quad (3.3)$$

and $\langle v, \xi_0 \rangle$ denotes the inner product of the vectors v and ξ_0 in the space $\mathbb{R}^n$. The function $\lambda(z_0, v)$ is typically called the resolving function.

Now, we highlight several important properties of the strategy (3.2) and the resolving function (3.3).

Lemma 1. The strategy (3.2) is well-defined and continuous for all $v \in S_\beta$, and during the pursuit game, the equality $|\mathbf{u}(z_0, v)| = \alpha$ holds.

Lemma 2. The resolving function (3.3) is well-defined, continuous, and non-negative for all $v \in S_\beta$, and satisfies the following bounds:

$$\alpha - \beta \leq \lambda(z_0, v) \leq \alpha + \beta.$$

Definition 8. Depending on the value of the parameter $b \in \mathbb{R}$, let one of the following conditions hold: (i) for $b > 0$: $\alpha \geq \beta$; (ii) for $b = 0$: $\alpha > \beta$; (iii) for $b < 0$ ($k = \sqrt{-b} > 0$): $\alpha - \beta > k^2|z_0|$.

Then, in the (U, V) game, the approach function of the players $\Lambda(t, v(\cdot))$ is defined as the scalar function corresponding to the value of the parameter b as follows:

$$\Lambda(t, v(\cdot)) = \begin{cases} \cos(\sqrt{b}t) - \frac{1}{|z_0|\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))\lambda(z_0, v(s)) ds, & b > 0, \\ 1 - \frac{1}{|z_0|} \int_0^t (t-s)\lambda(z_0, v(s)) ds, & b = 0, \\ \cosh(kt) - \frac{1}{|z_0|k} \int_0^t \sinh(k(t-s))\lambda(z_0, v(s)) ds, & b < 0, (k = \sqrt{-b}). \end{cases} \quad (3.4)$$

Lemma 3. Under the correspondig parameter conditions specified in Definition 8, the following assertions hold for all $v(\cdot) \in V$:

- (a) The function $\Lambda(t, v(\cdot))$ is monotonically decreasing with respect to t on the corresponding time intervals;
- (b) The function $\Lambda(t, v(\cdot))$ satisfies the following two-sided estimate:

$$\Lambda_1(t) \leq \Lambda(t, v(\cdot)) \leq \Lambda_2(t), \quad (3.5)$$

where the functions $\Lambda_1(t)$ and $\Lambda_2(t)$ are defined depending on the value of the parameter b as follows:

Case 1 (for $b > 0$, on the interval $t \in [0, \frac{\pi}{2\sqrt{b}}]$):

$$\Lambda_1(t) = \cos(\sqrt{b}t) - \frac{\alpha + \beta}{b|z_0|} (1 - \cos(\sqrt{b}t)), \quad \Lambda_2(t) = \cos(\sqrt{b}t) - \frac{\alpha - \beta}{b|z_0|} (1 - \cos(\sqrt{b}t)).$$

Case 2 (for $b = 0$, for all $t \geq 0$):

$$\Lambda_1(t) = 1 - \frac{\alpha + \beta}{2|z_0|} t^2, \quad \Lambda_2(t) = 1 - \frac{\alpha - \beta}{2|z_0|} t^2.$$

Case 3 (for $b < 0$, $k = \sqrt{-b} > 0$, for all $t \geq 0$):

$$\Lambda_1(t) = \cosh(kt) - \frac{\alpha + \beta}{k^2|z_0|} (\cosh(kt) - 1), \quad \Lambda_2(t) = \cosh(kt) - \frac{\alpha - \beta}{k^2|z_0|} (\cosh(kt) - 1).$$

Proof. (a) *Proof of Monotonicity:* We apply Leibniz's rule for differentiation under the integral sign. We take into account the inequality $\lambda(z_0, v(s)) \geq \alpha - \beta > 0$ from Lemma 2 and the signs of the corresponding integrand functions:

$$\textbf{Case 1 } (b > 0): \Lambda(t, v(\cdot)) = \cos(\sqrt{b}t) - \frac{1}{|z_0|\sqrt{b}} \int_0^t \sin(\sqrt{b}(t-s))\lambda(z_0, v(s)) ds.$$

Taking the derivative with respect to t :

$$\frac{d\Lambda}{dt} = -\sqrt{b}\sin(\sqrt{b}t) - \frac{1}{|z_0|} \int_0^t \cos(\sqrt{b}(t-s))\lambda(z_0, v(s)) ds.$$

Since $\sin(\sqrt{b}t) > 0$ on the interval $t \in (0, \frac{\pi}{2\sqrt{b}}]$ and for all $s \in [0, t]$, $\cos(\sqrt{b}(t-s)) \geq 0$ as well as $\lambda(z_0, v(s)) > 0$, both terms of the derivative are negative. Thus, $\frac{d\Lambda}{dt} < 0$.

$$\textbf{Case 2 } (b = 0): \Lambda(t, v(\cdot)) = 1 - \frac{1}{|z_0|} \int_0^t (t-s)\lambda(z_0, v(s)) ds.$$

Taking the derivative with respect to t :

$$\frac{d\Lambda}{dt} = -\frac{1}{|z_0|} \int_0^t \lambda(z_0, v(s)) ds \leq -\frac{\alpha - \beta}{|z_0|} t < 0 \quad (\forall t > 0).$$

$$\textbf{Case 3 } (b < 0, k = \sqrt{-b} > 0): \Lambda(t, v(\cdot)) = \cosh(kt) - \frac{1}{|z_0|k} \int_0^t \sinh(k(t-s))\lambda(z_0, v(s)) ds.$$

Using the inequality $\lambda(z_0, v(s)) \geq \alpha - \beta$ and taking the derivative:

$$\frac{d\Lambda}{dt} = k \sinh(kt) - \frac{1}{|z_0|} \int_0^t \cosh(k(t-s)) \lambda(z_0, v(s)) ds \leq k \sinh(kt) - \frac{\alpha - \beta}{|z_0|k} \sinh(kt).$$

Since $\alpha - \beta > k^2|z_0|$ (i.e., $\frac{\alpha - \beta}{k^2|z_0|} > 1$) by condition:

$$\frac{d\Lambda}{dt} \leq k \sinh(kt) \left(1 - \frac{\alpha - \beta}{k^2|z_0|}\right) < 0 \quad (\forall t > 0).$$

Thus, in all cases, $\Lambda(t, v(\cdot))$ is strictly decreasing with respect to time t .

(b) *Proof of Two-Sided Estimate:* According to Lemma 2, the inequality $\alpha - \beta \leq \lambda(z_0, v(s)) \leq \alpha + \beta$ holds for all $s \geq 0$.

On the interval $s \in [0, t]$, the kernel terms satisfy $\sin(\sqrt{b}(t-s)) \geq 0$ for $b > 0$, $(t-s) \geq 0$ for $b = 0$, and $\sinh(k(t-s)) \geq 0$ for $b < 0$.

Integrating the inequalities and bounding the terms from above and below respectively:

$$\int_0^t K(t, s)(\alpha - \beta) ds \leq \int_0^t K(t, s)\lambda(z_0, v(s)) ds \leq \int_0^t K(t, s)(\alpha + \beta) ds$$

we obtain the corresponding relations, where $K(t, s)$ represents the respective kernel functions.

Due to the negative sign preceding the integral in $\Lambda(t, v(\cdot))$, applying the lower bound $\alpha - \beta$ yields the upper estimate $\Lambda_2(t)$, whereas applying the upper bound $\alpha + \beta$ yields the lower estimate $\Lambda_1(t)$.

Evaluating the integrals algebraically completes the proof of the two-sided estimate (3.5). $\square$

Theorem 1. Let the players' control resources satisfy $\alpha > \beta > 0$ with the initial state $z_0 \neq 0$, and in the case of $b < 0$, let the condition $\alpha - \beta > k^2|z_0|$ ($k = \sqrt{-b}$) hold. Then, the Π -strategy guarantees the capture of the evader within the time interval $[0, t_g]$, where the guaranteed capture time t_g is determined as follows:

$$t_g = \begin{cases} \frac{1}{\sqrt{b}} \arccos\left(\frac{\alpha - \beta}{\alpha - \beta + b|z_0|}\right), & b > 0, \\ \sqrt{\frac{2|z_0|}{\alpha - \beta}}, & b = 0, \\ \frac{1}{k} \ln\left(\frac{\alpha - \beta + k\sqrt{|z_0|(2(\alpha - \beta) - k^2|z_0|)}}{\alpha - \beta - k^2|z_0|}\right), & b < 0. \end{cases} \quad (3.6)$$

Proof. When the Π -strategy is applied for all admissible controls $v(\cdot) \in V$, the trajectory of relative motion has the form $z(t) = z_0\Lambda(t, v(\cdot))$.

According to Lemma 3, the upper estimate $\Lambda(t, v(\cdot)) \leq \Lambda_2(t)$ holds for all $v(\cdot) \in V$. To determine the capture time t_g , we solve the equation $\Lambda_2(t_g) = 0$ separately for each case of the parameter b :

Case 1 ($b > 0$):

$$\begin{aligned} \cos(\sqrt{b}t_g) - \frac{\alpha - \beta}{b|z_0|}(1 - \cos(\sqrt{b}t_g)) &= 0 \implies \cos(\sqrt{b}t_g) = \frac{\alpha - \beta}{\alpha - \beta + b|z_0|} \\ \implies t_g &= \frac{1}{\sqrt{b}} \arccos\left(\frac{\alpha - \beta}{\alpha - \beta + b|z_0|}\right). \end{aligned}$$

Case 2 ($b = 0$):

$$1 - \frac{\alpha - \beta}{2|z_0|}t_g^2 = 0 \implies t_g = \sqrt{\frac{2|z_0|}{\alpha - \beta}}.$$

Case 3 ($b < 0$, $k = \sqrt{-b}$):

$$\cosh(kt_g) - \frac{\alpha - \beta}{k^2|z_0|}(\cosh(kt_g) - 1) = 0 \implies \cosh(kt_g) = \frac{\alpha - \beta}{\alpha - \beta - k^2|z_0|}.$$

We apply the identity $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$ for the inverse hyperbolic function. Taking $x = \frac{\alpha - \beta}{\alpha - \beta - k^2|z_0|} > 1$:

$$x^2 - 1 = \frac{k^2|z_0|(2(\alpha - \beta) - k^2|z_0|)}{(\alpha - \beta - k^2|z_0|)^2}.$$

Substituting this expression into the logarithm and simplifying yields:

$$t_g = \frac{1}{k} \ln \left(\frac{\alpha - \beta + k\sqrt{|z_0|(2(\alpha - \beta) - k^2|z_0|)}}{\alpha - \beta - k^2|z_0|} \right).$$

Since initially $\Lambda(0, v(\cdot)) = 1 > 0$, and at the time instant t_g we have $\Lambda(t_g, v(\cdot)) \leq \Lambda_2(t_g) = 0$, by the continuity of the function $\Lambda(t, v(\cdot))$ with respect to time (Bolzano–Cauchy theorem), there exists some $t^* \in (0, t_g]$ such that $\Lambda(t^*, v(\cdot)) = 0$, which implies $z(t^*) = 0$ and capture is accomplished. $\square$

4. Solution of the Evasion Problem

In this section, we propose an admissible strategy for the evader that guarantees evasion (avoiding capture) in the evasion game. Using this strategy, we prove that strategy (3.2) is the optimal pursuit strategy and t_g is the optimal pursuit time.

Definition 9. In the (U, V) game, we call the following control function of the evader:

$$\mathbf{v}^*(t) = -\beta\xi_0 \quad (4.1)$$

his strategy.

Definition 10. A strategy $\mathbf{v}^*(t)$ is said to guarantee evasion on the time interval $[0, t_g]$ if for any control function $u(\cdot) \in U$ of the pursuer, the relation $z(t) \neq 0$ holds for all $t \in [0, t_g]$, where $z(t)$ is the solution of the Cauchy problem (2.8).

Theorem 2. (a) When $\alpha > \beta > 0$ (and $\alpha - \beta > k^2|z_0|$ in the case $b < 0$), strategy (4.1) guarantees evasion on the time interval $[0, t_g]$, where t_g is the optimal pursuit time given in Theorem 1.

(b) Evasion on the infinite time interval $[0, +\infty)$ ($z(t) \neq 0, \forall t \geq 0$) is guaranteed by strategy (4.1) if one of the following conditions holds: for $b = 0$ or $b < 0$: the condition $\alpha \leq \beta$ is satisfied; for $b > 0$: the condition $\beta - \alpha \geq b|z_0|$ is satisfied.

In these cases, the distance between the players is bounded from below as follows:

$$|z(t)| \geq \begin{cases} |z_0| \cos(\sqrt{b}t) + \frac{\beta - \alpha}{b}(1 - \cos(\sqrt{b}t)) \geq |z_0|, & b > 0, \\ |z_0| + \frac{\beta - \alpha}{2}t^2 \geq |z_0|, & b = 0, \\ |z_0| \cosh(kt) + \frac{\beta - \alpha}{k^2}(\cosh(kt) - 1) \geq |z_0|, & b < 0, (k = \sqrt{-b}). \end{cases} \quad (4.2)$$

Proof. (a) When the evader applies strategy (4.1) ($v^*(t) = -\beta\xi_0$), for an arbitrary control $u(\cdot) \in U$ ($|u(t)| \leq \alpha$) of the pursuer, the magnitude of the relative trajectory is bounded from below as follows:

Case 1 ($b > 0$):

$$|z(t)| \geq |z_0| \cos(\sqrt{b}t) - \frac{\alpha - \beta}{b}(1 - \cos(\sqrt{b}t)).$$

From the condition $t < t_g = \frac{1}{\sqrt{b}} \arccos\left(\frac{\alpha - \beta}{\alpha - \beta + b|z_0|}\right)$, we have $\cos(\sqrt{b}t) > \frac{\alpha - \beta}{\alpha - \beta + b|z_0|}$, which guarantees $|z(t)| > 0$.

Case 2 ($b = 0$):

$$|z(t)| \geq \left| \xi_0 \left(|z_0| + \frac{\beta t^2}{2} \right) \right| - \int_0^t (t-s) |u(s)| ds \geq |z_0| - \frac{(\alpha - \beta)t^2}{2}.$$

Since $t < t_g = \sqrt{\frac{2|z_0|}{\alpha - \beta}}$, it follows that $|z(t)| > 0$ for all $t \in [0, t_g]$.

Case 3 ($b < 0$, $k = \sqrt{-b}$):

$$|z(t)| \geq |z_0| \cosh(kt) - \frac{\alpha - \beta}{k^2} (\cosh(kt) - 1).$$

For $t < t_g$, since $\cosh(kt) < \frac{\alpha - \beta}{\alpha - \beta - k^2|z_0|}$, we obtain $|z(t)| > 0$.

(b) Next, we verify evasion on the infinite interval $[0, +\infty)$:

For $b = 0$ and $b < 0$: Since $\alpha \leq \beta \implies \beta - \alpha \geq 0$, the second terms in the estimates are non-negative. Consequently, $|z(t)| \geq |z_0| > 0$ holds for all $t \geq 0$.

For $b > 0$: We rewrite the estimate in the following form:

$$|z(t)| \geq \frac{\beta - \alpha}{b} + \left(|z_0| - \frac{\beta - \alpha}{b} \right) \cos(\sqrt{b}t).$$

From the given condition $\beta - \alpha \geq b|z_0|$, it follows that $|z_0| - \frac{\beta - \alpha}{b} \leq 0$. Taking into account that $\cos(\sqrt{b}t) \leq 1$, we get:

$$|z(t)| \geq \frac{\beta - \alpha}{b} + \left(|z_0| - \frac{\beta - \alpha}{b} \right) \cdot 1 = |z_0| > 0, \quad \forall t \in [0, +\infty).$$

Thus, under the specified conditions, $z(t) \neq 0$ holds for all $t \geq 0$, which guarantees evasion on the infinite time interval. $\square$

REFERENCES

1. Isaacs, R. Differential Games. *NY: John Wiley and Sons*, (1965), 385 p.
2. Pontryagin, L.S. Selected Works. *Moscow: MAKs Press*, (2004), 551 p. (in Russian)
3. Krasovskii, N.N.; Subbotin, A.I. Game-Theoretical Control Problems. *NY: Springer*, (2011), 517 p.
4. Petrosjan, L.A. Differential Games of Pursuit. *Ser. Optim, vol. 2. Singapore: World Scientific Publ.*, (1993), 340 p. DOI: 10.1142/1670
5. Azamov, A. On the quality problem for simple pursuit games with constraint. *Serdica Bulgariacae Math. Publ.*, **12** (1986), no. 1, 38–43. (in Russian)
6. Azamov, A.A.; Samatov, B.T. The P-strategy: analogies and applications. *In: Proc. The Fourth Int. Conf. Game Theory and Management (GTM 2010), June 28-30, 2010, St. Petersburg, Russia*, **4** (2010), 33–47.
7. Samatov, B.T.; Soyibboev, U.B. Differential game with a lifeline for the inertial movements of players. *Ural Math. J.* **7** (2021), no. 2, 94–109. DOI: 10.15826/umj.2021.2.007
8. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. Construction of P-strategies in a simple pursuit-evasion game with impulse control. *Acta NUUZ* **2/2.1.** (2024), 164–175. (in Russian)
9. Mustapokulov, Kh. Ya. Pursuit-Evasion Problems for Second-Order Differential Games with Impulse Control. *Acta NUUZ* **2/1.1.** (2025), 178–184. (in Russian)
10. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. A Linear Differential Pursuit Game with Impulse Controls. *Acta NUUZ* **2/2.** (2025), 171–178. (in Russian)

11. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. Pursuit Problem for Differential-Difference Equations of Neutral Type. *Journal of Mathematical Sciences* **277** (2023), no. 3, 438–448. DOI 10.1007/s10958-023-06846-8
12. Mamadaliyev, N. A.; Mustapokulov, Kh. Ya. On the Invariance of a Constant Multivalued Mapping in a Heat Conduction Problem with Impulse Control in the Presence of Time Delay. *Bulletin of the Institute of Mathematics* **6** (2023), no. 6, 90–98. (in Russian)

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