

ON THE SOLVABILITY OF A FORWARD NONLOCAL PROBLEM FOR A NONHOMOGENEOUS BOUSSINESQ-TYPE EQUATION

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ABSTRACT. In this paper, we study a direct nonlocal problem for a nonhomogeneous Boussinesq-type equation in a bounded multidimensional domain. The problem is considered under homogeneous boundary conditions and a weighted integral condition involving the time variable. Using the Fourier method and spectral expansion, we derive an explicit form of the solution. Sufficient conditions for the existence and uniqueness of the solution are obtained, and the convergence of the corresponding series is established.

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Introduction

Let $\Omega \subset \mathbb{R}^n$ be an arbitrary bounded domain with piecewise smooth boundary $\partial\Omega$ and $T > 0$. Consider in the cylinder $\Omega \times (0, T)$ the following equation

$$u_{tt} - \Delta u_{tt} + \nu^2 \Delta u = f(x), \quad x \in \Omega, \quad 0 < t < T, \quad (1)$$

where parameter ν is a positive number.

Let A be a positive self-adjoint extension of Laplace operator $-\Delta$: $Au = -\Delta u$, $u \in C_0^\infty(\Omega)$.

$$(Au, u) \geq \mu(u, u), \quad \mu > 0, \quad u \in D(A)$$

We assume that A has a compact inverse operator A^{-1} . In particular, this self-adjoint extension may be produced by the boundary conditions like

$$\alpha(x) \frac{\partial u}{\partial n} + \beta(x) u = 0, \quad x \in \partial\Omega$$

with some appropriate $\alpha(x)$ and $\beta(x)$.

We consider non-local conditions

$$u(x, 0) = u(x, T), \quad (2)$$

$$\int_0^T t u(x, t) dt = \varphi(x) \quad (3)$$

Equations of the form (1) are known as linearized Boussinesq-type equations (see [1, Eq. (26)]). Such equations arise in the mathematical modeling of various physical processes and therefore the study of their solvability is of particular interest. Initial and boundary value problems for Boussinesq-type equations, including nonlinear and nonhomogeneous cases, have been investigated in many works (see, for example, [2-9]).

A number of papers are devoted to the study of nonlocal problems for Boussinesq-type equations, where conditions for the existence and uniqueness of solutions were established by means of spectral and Fourier methods.

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In [3], a nonlocal problem for a Boussinesq-type differential equation was studied, and the existence and uniqueness of the solution were proved by the Fourier method. The approach used in our paper is close to that work. However, the nonlocal condition considered here is different, since it is given as integral condition with respect to time and contains the factor t . Because of this, the relations for the Fourier coefficients take a different form, which in turn changes the solvability conditions of the problem.

Similar problems were considered in [7] and [10] by Khalmukhamedov A. R. and Kuchkorov E. I. Our study is also methodologically related to these works, especially in the use of spectral decomposition and the Fourier method. The main difference is that the term containing ν^2 appears with the opposite sign in the equation considered here. This change leads to a different form of the corresponding scalar equations and affects the spectral conditions that arise in the analysis of solvability.

In the present paper, we study a direct nonlocal problem for a nonhomogeneous Boussinesq-type equation subject to time-integral condition. The presence of the factor t in the integral condition, together with the changed sign of the ν^2 -term, distinguishes the present problem from those studied earlier. Due to the operator taken with the opposite sign, the form of the solution and the solution method changes. Using the Fourier method, we obtain an explicit representation of the solution and establish sufficient conditions for its existence and uniqueness.

Definition 1. We say that a function $u(x, t)$ is a solution to the non-local problem (1), (2) and (3) if the following conditions are satisfied:

- 1) $u(\cdot, t) \in D(A)$ for all $t \in (0, T)$, and the function $u(x, t)$ is continuous with respect to t in the norm of $L_2(\Omega)$;
- 2) the function $u(x, t)$ and $Au(x, t)$ are twice continuously differentiable with respect to t on the open interval $(0, T)$ in the $L_2(\Omega)$ -norm;
- 3) the function $u(x, t)$ satisfies equation (1) and the non-local conditions (2) and (3).

In order to formulate the results on existence and uniqueness we introduce some important parameters. Let λ_k and $v_k(x)$ be the eigenvalues and corresponding eigenfunctions of the self-adjoint operator A :

$$Av_k(x) = \lambda_k v_k(x), \quad x \in \Omega.$$

The spectral decomposition of any function $\varphi \in L_2(\Omega)$, $f \in L_2(\Omega)$ has the form

$$\varphi(x) = \sum_{k=1}^{\infty} \varphi_k v_k(x), \quad \varphi_k = (\varphi, v_k).$$

$$f(x) = \sum_{k=1}^{\infty} f_k v_k(x), \quad f_k = (f, v_k).$$

According to our assumptions, $\mu \leq \lambda_1 \leq \lambda_2 \leq \dots \leq \lambda_k \rightarrow \infty$. Set

$$\nu_k = \nu \sqrt{\frac{\lambda_k}{1 + \lambda_k}}. \quad (4)$$

Now, we introduce the following set:

$$\Lambda_0(\nu) := \{\lambda_k \in SpA : \varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k = 0\}.$$

Consider a linear space $L(\nu)$ formed by eigenfunctions corresponding to the eigenvalues from the $\Lambda_0(\nu)$:

$$L(\nu) = \{h | h(x) = \sum_{k=1}^{\infty} h_k v_k(x)\}.$$

In this paper, the validity of the following statement is proved.

Theorem 1. Let $\varphi \in D(A)$ and $f \in L_2(\Omega)$. Then the solution of problem (1)-(3) exists and is unique.

Preliminaries

In this section, we determine the solution to the problem using the spectral method and derive the general form of the solution.

Firstly, we can rewrite equation (1):

$$u_{tt} + Au_{tt} - \nu^2 Au = f(x), \quad x \in \Omega, \quad 0 < t < T. \quad (5)$$

Assume that $u(x, t)$ is solution of problem (5), (2) and (3). Then, we reduce problem (1)-(3) to a homogeneous problem:

$$u(x, t) = \omega(x, t) - \frac{1}{\nu^2} A^{-1} f(x).$$

In addition, the spectral decomposition of solution has the form

$$u(x, t) = \sum_{k=1}^{\infty} u_k(t) v_k(x), \quad u_k(t) = (u, v_k);$$

$$\omega(x, t) = \sum_{k=1}^{\infty} \omega_k(t) v_k(x), \quad \omega_k = (\omega, v_k).$$

The function $\omega(x, t)$ is a solution to the following equation

$$\omega_{tt} + A\omega_{tt} - \nu^2 A\omega = 0, \quad x \in \Omega, \quad 0 < t < T. \quad (6)$$

satisfying the nonlocal conditions.

Applying the $(I + A)^{-1}$ operator to both sides of equation (6) and we get

$$\omega_{tt} - \nu^2 (I + A)^{-1} A\omega = 0.$$

Recall that the operator A is self-adjoint and we can write

$$\left((I + A)^{-1} A\omega, v_k \right) = \left(\omega, A(I + A)^{-1} v_k \right) = \left(\omega, \frac{\lambda_k}{1 + \lambda_k} v_k \right) = \frac{\nu_k^2}{\nu^2} \omega_k.$$

Therefore, it follows from the above equations that

$$\begin{cases} \omega_k''(t) - \nu_k^2 \omega_k(t) = 0, & 0 < t < T, \\ \omega_k(0) = \omega_k(T), \\ \int_0^T t \omega_k(t) dt = \varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k, \end{cases} \quad (7)$$

is valid, where $k \in N$. Hence, $\omega_k(t) = a_k \operatorname{ch} \nu_k t + b_k \operatorname{sh} \nu_k t$.

We add to this equation conditions of (7) that follow this determinant

$$\begin{cases} (1 - \operatorname{ch} \nu_k T) a_k + b_k \operatorname{sh} \nu_k T = 0 \\ a_k \operatorname{sh} \nu_k T + b_k \operatorname{ch} \nu_k T = \frac{\nu_k}{T} \phi_k \end{cases} \quad (8)$$

We note that $\begin{vmatrix} 1 - \operatorname{ch} \nu_k T & -\operatorname{sh} \nu_k T \\ \operatorname{sh} \nu_k T & \operatorname{ch} \nu_k T \end{vmatrix} = 2 \operatorname{sh}^2 \frac{\nu_k T}{2} \neq 0$, there we can determine $\phi_k = \varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k$.

Finding the coefficients a_k and b_k from system (8), we write the form of each function $\omega_k(t)$ as follows:

$$\begin{aligned} \omega_k(t) &= a_k \operatorname{ch} \nu_k t + b_k \operatorname{sh} \nu_k t = \frac{\nu_k}{T} \phi_k \left[\frac{\operatorname{ch} \nu_k T / 2}{\operatorname{sh} \nu_k T / 2} \operatorname{ch} \nu_k t - \operatorname{sh} \nu_k t \right] = \\ &= \frac{\nu_k \phi_k}{T \operatorname{sh} \nu_k T / 2} \operatorname{ch} [\nu_k (t - T/2)]. \end{aligned}$$

Thus,

$$\omega(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \frac{\nu_k}{T \operatorname{sh} \nu_k T/2} \operatorname{ch}[\nu_k(t - T/2)] \left(\varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k \right)$$

Hence, if a solution to the problem (1)-(3) exists, then it has the following form:

$$u(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{\nu_k}{T \operatorname{sh} \nu_k T/2} \operatorname{ch}[\nu_k(t - T/2)] \left(\varphi_k + \frac{T^2}{2\nu^2 \lambda_k} f_k \right) \right] v_k(x) - \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x) \quad (9)$$

or

$$u(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{\nu_k \operatorname{ch}[\nu_k(t - T/2)]}{T \operatorname{sh} \nu_k T/2} \varphi_k + \left(\frac{T \nu_k \operatorname{ch}[\nu_k(t - T/2)]}{2 \operatorname{sh} \nu_k T/2} \right) \frac{1}{\nu^2 \lambda_k} f_k \right] v_k(x) - \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x). \quad (10)$$

Proposition 1. Let the function $u(x, t)$ be the solution of problem (1)-(3) for $\varphi, f \in L_2(\Omega)$. Then its spectral expansion has the form (9).

Auxiliary results and proof of the main Theorem

In this section, we present auxiliary lemmas and propositions which are needed for the proof of the main theorem. After establishing these auxiliary results, we proceed to the proof of the theorem itself.

For convenience, let us introduce the function

$$B(\nu_k, t) = \frac{\nu_k}{\operatorname{sh}(\nu_k T/2)} \operatorname{ch}[\nu_k(t - T/2)]. \quad (11)$$

In this regard, we can rewrite the form of the solution as follows.

$$u(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} B(\nu_k, t) \varphi_k + \frac{T}{2\nu^2 \lambda_k} B(\nu_k, t) f_k \right] v_k(x) - \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x). \quad (12)$$

Lemma 1. There exists a constant $C > 0$ such that

$$|B(\nu_k, t)| \leq C$$

for all $k \in \mathbb{N}$ and $t \in [0, T]$.

Proof. Taking into account that $\nu_k < \nu$ and $t \in (0, T)$, we obtain

$$\operatorname{ch} \left[\nu_k \left| t - \frac{T}{2} \right| \right] \leq \operatorname{ch} \left(\frac{\nu_k T}{2} \right).$$

Therefore,

$$\begin{aligned} B(\nu_k, t) &\leq \frac{\nu_k \operatorname{ch}(\nu_k T/2)}{\operatorname{sh}(\nu_k T/2)} = \nu_k \operatorname{cth} \frac{\nu_k T}{2} = \left| \frac{\nu_k T}{2} \right| = a, \quad \operatorname{cth} = \frac{e^a + e^{-a}}{e^a - e^{-a}} = \\ &= \nu_k \left(1 + \frac{2e^{-a}}{e^a - e^{-a}} \right) = \nu_k \left(1 + \frac{2}{e^{2a} - 1} \right) \leq \nu_k \left(1 + \frac{1}{a} \right) \end{aligned}$$

Thus,

$$|B(\nu_k, t)| \leq C \quad (13)$$

where $C = \text{const}$. □

Recall that for $u(x, t)$ to be a solution of the considered problem, the function itself and its second-order derivatives with respect to t must satisfy the required belong to the domain of operator A . Therefore, we prove the following statements.

Proposition 2. Let the function $u(x, t)$ be defined by (9). Then $u(\cdot, t) \in D(A)$ uniformly with respect to $t \in (0, T)$, provided that

$$\sum_{\lambda_k \notin \Lambda_0(\nu)} (\lambda_k^2 |\varphi_k|^2 + |f_k|^2) < +\infty. \quad (14)$$

In particular, if $\varphi \in D(A)$ and $f \in L_2(\Omega)$, then condition (14) is satisfied.

Proof. Applying the operator A to equation (12), we obtain

$$Au(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} B(\nu_k, t) \lambda_k \varphi_k + \frac{T}{2\nu^2} B(\nu_k, t) f_k \right] v_k(x) + \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2} f_k v_k(x). \quad (15)$$

Since $f \in L_2(\Omega)$, the second sum in (15) is finite. It remains to estimate the first part of the expression.

Then, using (13) and Parseval's equality and the inequality $(a + b)^2 \leq 2a^2 + 2b^2$ we get

$$\begin{aligned} \|Au(\cdot, t)\|^2 &= \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} B(\nu_k, t) \lambda_k \varphi_k + \frac{T}{2\nu^2} B(\nu_k, t) f_k \right]^2 \leq \\ &\leq \sum_{\lambda_k \notin \Lambda_0(\nu)} \left(\frac{2}{T^2} B^2(\nu_k, t) \lambda_k^2 \varphi_k^2 + \frac{T^2}{2\nu^4} B^2(\nu_k, t) f_k^2 \right). \end{aligned}$$

By Lemma 1, it follows that the inequality

$$\|Au(\cdot, t)\|_{L_2(\Omega)}^2 \leq C_1 \sum_{\lambda_k \notin \Lambda_0(\nu)} (\lambda_k^2 |\varphi_k|^2 + |f_k|^2) < +\infty$$

Since condition (14) holds and $f \in L_2(\Omega)$, the right-hand side is finite.

Hence, $\|Au(\cdot, t)\|_{L_2(\Omega)} \leq \text{const.}$ uniformly with respect to $t \in (0, T)$. The proposition is proved. $\square$

Proposition 3. Let condition (14) be satisfied. Then the functions $u(x, t)$ and $Au(x, t)$ are twice continuously differentiable with respect to $t \in (0, T)$.

Proof. Differentiating with respect to t in equation (15), we obtain the following equality

$$\frac{\partial^2}{\partial t^2} Au(x, t) = \sum_{\lambda_k \notin \Lambda_0(\nu)} \left[\frac{1}{T} \nu_k^2 B(\nu_k, t) \lambda_k \varphi_k + \frac{T \nu_k^2}{2\nu^2} B(\nu_k, t) f_k \right] v_k(x).$$

By Parseval's equality, we get

$$\left\| \frac{\partial^2}{\partial t^2} Au(\cdot, t) \right\|^2 \leq \sum_{\lambda_k \notin \Lambda_0(\nu)} \left(\frac{2\nu_k^4}{T^2} B^2(\nu_k, t) \lambda_k^2 \varphi_k^2 + \frac{T^2 \nu_k^4}{2\nu^4} B^2(\nu_k, t) f_k^2 \right).$$

From definition (4), we have

$$\frac{\mu}{\mu + 1} \nu^2 \leq \nu_k^2 < \nu^2.$$

Thus ν_k^4 is uniformly bounded and together with Lemma 1, $B(\nu_k, t)$ is also uniformly bounded. Therefore the above series converges in $L_2(\Omega)$ under condition (13). Consequently,

$$\left\| \frac{\partial^2}{\partial t^2} Au(x, t) \right\|^2 \leq \text{const.} \quad 0 < t < T.$$

Moreover, $Au(x, t)$ is twice continuously differentiable with respect to t . The same argument applied directly to $u(x, t)$ shows that $u(x, t)$ is also twice continuously differentiable with respect to t . $\square$

Proof of the Main Theorem.

Let $\varphi \in D(A)$ and $f \in L_2(\Omega)$.

$$\sum_{k=1}^{\infty} \lambda_k^2 |\varphi_k|^2 < \infty, \quad \sum_{k=1}^{\infty} |f_k|^2 < \infty$$

Hence condition (14) is satisfied. Therefore by Proposition 2, the function defined by representation (12) belongs to $D(A)$ uniformly with respect to $t \in (0, T)$. By Proposition 3, the functions $u(x, t)$ and $Au(x, t)$ are twice

continuously differentiable with respect to $t \in (0, T)$. Hence all terms appearing in equation (5) are well defined in $L_2(\Omega)$.

From representation (12) we have

$$u_k(t) = \frac{1}{T} B(\nu_k, t) \varphi_k + \frac{T}{2\nu^2 \lambda_k} B(\nu_k, t) f_k - \frac{f_k}{\nu^2 \lambda_k}$$

By the definition of (11) $B(\nu_k, t)$, we have $B_{tt}(\nu_k, t) = \nu_k^2 B(\nu_k, t)$. Therefore,

$$u_k''(t) = \nu_k^2 \left[\frac{1}{T} B(\nu_k, t) \varphi_k + \frac{T}{2\nu^2 \lambda_k} B(\nu_k, t) f_k \right].$$

It follows from the above results lemma, propositions that the series defining $u(x, t)$, $Au(x, t)$, and $Au_{tt}(x, t)$ converge in $L_2(\Omega)$.

The remain series

$$- \sum_{\lambda_k \in \Lambda_0(\nu)} \frac{1}{\nu^2 \lambda_k} f_k v_k(x)$$

also converges in $L_2(\Omega)$. Indeed, since $\lambda_k \geq \lambda_1 > 0$, we have

$$\sum_{\lambda_k \in \Lambda_0(\nu)} \left| \frac{1}{\nu^2 \lambda_k} f_k \right|^2 \leq \frac{1}{\nu^4 \lambda_1^2} \sum_{\lambda_k \in \Lambda_0(\nu)} |f_k|^2 < +\infty.$$

Moreover, from representation (12), we obtain

$$|u_k(t)| \leq C_1 |\varphi_k| + \frac{C_2}{\nu^2 \lambda_k} |f_k|.$$

where C_1 and C_2 are positive constants independent of k and t . Hence,

$$|u_k(t)|^2 \leq 2C_1^2 |\varphi_k|^2 + \frac{2C_2^2}{\nu^4 \lambda_k^2} |f_k|^2.$$

Since $\lambda_k \geq \lambda_1 > 0$, we have

$$\sum_{k=1}^{\infty} |u_k(t)|^2 \leq 2C_1^2 \sum_{k=1}^{\infty} |\varphi_k|^2 + \frac{2C_2^2}{\nu^4 \lambda_1^2} \sum_{k=1}^{\infty} |f_k|^2 < \infty.$$

Therefore, the series (12) converges $L_2(\Omega)$. Consequently, the function $u(x, t)$ satisfies all the required conditions and is a solution of problem (1)-(3).

It remains to prove uniqueness and the uniqueness part of the theorem is straightforward to prove. Thus, the solution of problem (1)-(3) exists and is unique. The theorem is proved. $\square$

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