

A CLASS OF ONE-DIMENSIONAL SOLVABLE EXTENSIONS OF THE NATURALLY GRADED FILIFORM 3-LIE ALGEBRA $NGF_{5,3}$

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ABSTRACT. In this paper, we study one-dimensional solvable extensions of the naturally graded filiform 3-Lie algebra $NGF_{5,3}$. Using the description of derivations of the algebra $NGF_{m,3}$, we determine the general form of the multiplication table of such extensions and derive the restrictions on its parameters from the fundamental identity. The analysis naturally leads to two cases according to the coefficients occurring in the product $[x, e_1, e_2]$. In the present paper, we consider in detail the case $\alpha = \beta = 0$. By applying suitable adapted changes of basis, the corresponding multiplication tables are reduced to simpler forms and the remaining parameters are analyzed. The case $(\alpha, \beta) \neq (0, 0)$ is left for a separate investigation.

MSC (2020): 17B05; 17B30; 17B40.

Key words: 3-Lie algebra, naturally graded filiform algebra, solvable extension, hypo-nilpotent ideal, derivation.

Introduction

The concept of an n -Lie algebra was introduced by Filippov as a natural multilinear generalization of a Lie algebra [1]. In contrast to the binary case, the structure theory of nilpotent and solvable n -Lie algebras contains a number of phenomena which have no direct analogue in the theory of Lie algebras. In particular, the notions of nilpotence as a subalgebra and nilpotence as an ideal need not coincide. This leads naturally to the notion of a hypo-nilpotent ideal, which plays an important role in the description of solvable n -Lie algebras; see [2-3].

Naturally graded filiform n -Lie algebras and their solvable extensions were studied in [4]. Among the naturally graded filiform algebras, the authors distinguished a model algebra with the simplest multiplication table, denoted by $NGF_{m,n}$. It is an n -ary analogue of the model filiform Lie algebra and is defined, in a suitable basis $\{e_1, \dots, e_m\}$, by

$$[e_1, e_2, \dots, e_{n-1}, e_j] = e_{j+1}, \quad n \leq j \leq m-1, \quad (6)$$

all other products being zero or obtained from skew-symmetry. This model is the starting point of the present paper.

The specialization $n = 3, m = 5$ gives the five-dimensional algebra $NGF_{5,3}$ used below. Its low dimension allows us to display the derivations and compatibility relations of a codimension-one extension explicitly.

The model used here is not a new nilpotent algebra. In [3, p. 44], the authors recall the simplest filiform model of [2], with products $[f_1, f_2, f_j] = f_{j-1}$ for $4 \leq j \leq m$, and the earlier description of its codimension-one solvable extensions. For $m = 5$, the identification

$$f_1 = e_1, \quad f_2 = e_2, \quad f_3 = e_5, \quad f_4 = e_4, \quad f_5 = e_3$$

Received: 07.08.2026, **Accepted:** 24.09.2026

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identifies that model with $NGF_{5,3}$. Thus the extension problem below lies within the scope of [2]. The calculations are presented as an explicit low-dimensional treatment of this known problem, rather than as a new classification. The second model in [3, equation (3.1)] has additional products for general m and should be distinguished from the simplest model of [2].

We give a direct derivation calculation, a normalized parameter table, and explicit adapted basis reductions for $\alpha = \beta = 0$. We establish reduction to three displayed normal forms, without asserting that they represent three distinct abstract isomorphism classes. The embedded ideal N is preserved in the reductions; no assertion that every abstract isomorphism preserves this ideal is needed. A precise family-by-family comparison with [2] is required before any independent originality claim can be made.

Throughout the paper the ground field is $\mathbb{C}$.

Preliminaries

We recall the notions used below. For subspaces A, B, C of a 3-Lie algebra L , by $[A, B, C]$ we mean the linear span of all products $[a, b, c]$, where $a \in A$, $b \in B$, and $c \in C$.

Definition 1. A complex vector space L endowed with a skew-symmetric trilinear operation $[\cdot, \cdot, \cdot]$ is called a 3-Lie algebra if

$$[a, b, [u, v, w]] = [[a, b, u], v, w] + [u, [a, b, v], w] + [u, v, [a, b, w]] \quad (7)$$

for all $a, b, u, v, w \in L$. Identity (7) is called the *fundamental identity* [1].

Definition 2. A linear mapping $d : L \rightarrow L$ is called a *derivation* if

$$d([u, v, w]) = [d(u), v, w] + [u, d(v), w] + [u, v, d(w)] \quad (8)$$

for all $u, v, w \in L$. The set of all derivations is denoted by $\text{Der}(L)$. By the fundamental identity, for arbitrary fixed $a, b \in L$, the mapping

$$\text{ad}(a, b)(z) = [a, b, z]$$

is a derivation.

Definition 3. Let I be an ideal of a 3-Lie algebra L . Put

$$I^0 = I, \quad I^{r+1} = [I^r, I, L], \quad r \geq 0.$$

The ideal I is nilpotent if $I^s = 0$ for some s . On the other hand, regarding I as a 3-Lie subalgebra, put

$$J^0 = I, \quad J^{r+1} = [J^r, I, I].$$

The ideal I is called *hypo-nilpotent* if the latter sequence reaches zero while the former does not. A hypo-nilpotent ideal is called maximal if it is not properly contained in another hypo-nilpotent ideal [3].

Definition 4. We use the derived series $L^{(0)} = L$ and $L^{(r+1)} = [L^{(r)}, L^{(r)}, L]$. The algebra is solvable when this series terminates at zero.

Following [4], consider the model naturally graded filiform n -Lie algebra $NGF_{m,n}$ defined by (6). Specializing first to $n = 3$ and then to $m = 5$, we obtain the five-dimensional 3-Lie algebra

$$N = NGF_{5,3} = \text{Span}\{e_1, e_2, e_3, e_4, e_5\}, \quad (9)$$

whose multiplication table is

$$\begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5. \end{cases} \quad (10)$$

All products not displayed in (10) are zero or follow from skew-symmetry. In particular, if

$$U = \text{Span}\{e_3, e_4, e_5\},$$

then

$$[U, U, N] = 0. \quad (11)$$

Relation (11) will be used repeatedly below.

We now describe the derivation algebra of N . The statement is written independently of the extension problem and will subsequently be applied to the inner derivations arising from the complementary element.

Proposition 1. Relative to the ordered basis $(e_1, \dots, e_5)$, with columns representing images of basis vectors, all derivations of N have the form

$$D = \begin{pmatrix} a_{11} & a_{21} & 0 & 0 & 0 \\ a_{12} & a_{22} & 0 & 0 & 0 \\ a_{13} & a_{23} & a_{33} & 0 & 0 \\ a_{14} & a_{24} & a_{34} & a_{11} + a_{22} + a_{33} & 0 \\ a_{15} & a_{25} & a_{35} & a_{34} & 2a_{11} + 2a_{22} + a_{33} \end{pmatrix},$$

where the thirteen displayed independent coefficients are arbitrary. This is the five-dimensional specialization of the derivation calculation for the model algebra; see [4] and the related conference contribution [5]. We verify it directly to make the present calculation self-contained.

Proof. Write $D(e_i) = \sum_k a_{ik}e_k$. Applying the derivation identity to $[e_1, e_2, e_3] = e_4$ gives

$$D(e_4) = (a_{11} + a_{22} + a_{33})e_4 + a_{34}e_5.$$

Applying it to $[e_1, e_2, e_4] = e_5$ gives $D(e_5) = (2a_{11} + 2a_{22} + a_{33})e_5$. The zero products $[e_1, e_3, e_4] = 0$ and $[e_2, e_3, e_4] = 0$ imply $a_{32} = a_{31} = 0$, respectively. The images of e_1, e_2 are unrestricted. Conversely, the displayed matrix satisfies the identity on both nonzero triples. On each of the other eight increasing triples, substitution gives zero on both sides. Skew-symmetry and multilinearity complete the verification. $\square$

One-dimensional solvable extensions

Let $R = N \dot{+} \text{Span}\{x\}$ be a six-dimensional solvable 3-Lie algebra in which $N = NGF_{5,3}$ is a maximal hypo-nilpotent ideal. Since N is an ideal, every product $[x, e_i, e_j]$ belongs to N .

Theorem 1. Let R be a six-dimensional solvable 3-Lie algebra with maximal hypo-nilpotent ideal $NGF_{5,3}$. Then there exists a basis $\{x, e_1, e_2, e_3, e_4, e_5\}$ of R such that the multiplication table of R with respect to this basis has the following form:

$$\begin{cases} [x, e_1, e_2] = \alpha e_1 + \beta e_2 + \gamma e_3, \\ [x, e_1, e_3] = p_3 e_3 + p_5 e_5, \\ [x, e_1, e_4] = (\beta + p_3) e_4, \\ [x, e_1, e_5] = (2\beta + p_3) e_5, \\ [x, e_2, e_3] = q_3 e_3 + q_5 e_5, \\ [x, e_2, e_4] = (-\alpha + q_3) e_4, \\ [x, e_2, e_5] = (-2\alpha + q_3) e_5, \\ [x, e_i, e_j] = 0, \quad 3 \leq i, j \leq 5. \end{cases} \quad (12)$$

Moreover, the parameters satisfy

$$\alpha p_3 + \beta q_3 = 0, \quad \alpha p_5 + \beta q_5 = 0. \quad (13)$$

Conversely, the tables (10) and (12) define a 3-Lie algebra precisely when (13) holds. Every such algebra is solvable. This converse alone does not assert that N is hypo-nilpotent: in the case $\alpha = \beta = 0$, that additional condition is equivalent to $(p_3, q_3) \neq (0, 0)$.

Proof. Let

$$R = NGF_{5,3} \dot{+} \langle x \rangle.$$

Since $NGF_{5,3}$ is an ideal of R , all products involving x and two elements of $NGF_{5,3}$ belong to $NGF_{5,3}$.

First, for each $i = 1, \dots, 5$, the mapping

$$\text{ad}(x, e_i) : NGF_{5,3} \longrightarrow NGF_{5,3}, \quad \text{ad}(x, e_i)(z) = [x, e_i, z],$$

is a derivation of $NGF_{5,3}$. Applying Proposition 1 to these derivations and taking into account the skew-symmetry of the bracket, we obtain

$$[x, e_1, e_2] = \alpha e_1 + \beta e_2 + \gamma_3 e_3 + \gamma_4 e_4 + \gamma_5 e_5,$$

and

$$\begin{cases} [x, e_1, e_3] = p_3 e_3 + p_4 e_4 + p_5 e_5, \\ [x, e_1, e_4] = (\beta + p_3) e_4 + p_4 e_5, \\ [x, e_1, e_5] = (2\beta + p_3) e_5. \end{cases}$$

whereas

$$\begin{cases} [x, e_2, e_3] = q_3 e_3 + q_4 e_4 + q_5 e_5, \\ [x, e_2, e_4] = (-\alpha + q_3) e_4 + q_4 e_5, \\ [x, e_2, e_5] = (-2\alpha + q_3) e_5. \end{cases}$$

Similarly, applying Proposition 1 to $\text{ad}(x, e_i)$, $3 \leq i \leq 5$, yields

$$[x, e_i, e_j] = 0, \quad 3 \leq i, j \leq 5.$$

Next, replacing x by

$$x' = x + p_4 e_2 - q_4 e_1 - \gamma_4 e_3 - \gamma_5 e_4,$$

and then omitting the prime, we may assume

$$p_4 = q_4 = \gamma_4 = \gamma_5 = 0.$$

Putting $\gamma = \gamma_3$, we obtain the table (12).

Finally, applying the fundamental identity to

$$[x, e_1, [x, e_2, e_3]]$$

and using the multiplication table obtained above, we have

$$[x, e_1, [x, e_2, e_3]] = [x, [x, e_1, e_2], e_3] + [x, e_2, [x, e_1, e_3]].$$

Comparing the coefficients of e_3 and e_5 on both sides gives, respectively,

$$\alpha p_3 + \beta q_3 = 0$$

and

$$\alpha p_5 + \beta q_5 = 0.$$

Thus all the asserted relations are obtained.

For the converse, define the residual

$$F(a, b; u, v, w) = [a, b, [u, v, w]] - [[a, b, u], v, w] - [u, [a, b, v], w] - [u, v, [a, b, w]].$$

It is alternating in a, b and separately in u, v, w . Hence it suffices to check the $\binom{6}{2} \binom{6}{3} = 300$ choices of an increasing basis pair and an increasing basis triple in the order $(x, e_1, \dots, e_5)$. Put $A = \alpha p_3 + \beta q_3$ and $B = \alpha p_5 + \beta q_5$. Substitution into the sparse table gives all potentially nonzero residuals as follows:

(a, b)	(u, v, w)	$F(a, b; u, v, w)$
(x, e_1)	(x, e_2, e_3)	$-Ae_3 + Be_5$
(x, e_1)	(x, e_2, e_4)	$-Ae_4$
(x, e_1)	(x, e_2, e_5)	$-Ae_5$
(x, e_2)	(x, e_1, e_3)	$Ae_3 - Be_5$
(x, e_2)	(x, e_1, e_4)	Ae_4
(x, e_2)	(x, e_1, e_5)	Ae_5
(x, e_3)	(x, e_1, e_2)	$-Ae_3 + Be_5$
(x, e_4)	(x, e_1, e_2)	$-Ae_4$
(x, e_5)	(x, e_1, e_2)	$-Ae_5$

The other 291 residuals vanish identically. This enumeration follows by using $[U, U, R] = 0$, the two internal nonzero products, and the displayed actions on U ; it does not assume any parameter constraints. It proves sufficiency as well as the absence of additional relations.

Finally, put $v = \alpha e_1 + \beta e_2$. The table gives $R^{(1)} \subseteq \mathbb{C}v + U$ and $[\mathbb{C}v + U, \mathbb{C}v + U, R] \subseteq U$. Since $[U, U, R] = 0$, we have $R^{(3)} = 0$. Thus every parameter table satisfying the fundamental identity is solvable. If all parameters vanish, N is a nilpotent ideal, illustrating why the hypo-nilpotent condition must be imposed separately.

We now proceed to simplify the multiplication table obtained above. The further analysis essentially depends on the coefficients α and β occurring in the product $[x, e_1, e_2]$. In view of (13), there are two essentially different cases:

$$\alpha = \beta = 0 \quad \text{and} \quad (\alpha, \beta) \neq (0, 0).$$

In the present paper, we restrict our attention to the first case, that is, $\alpha = \beta = 0$, and investigate the corresponding one-dimensional solvable extensions of $NGF_{5,3}$. The case $(\alpha, \beta) \neq (0, 0)$ will be considered separately.

Case 1. $\alpha = \beta = 0$.

In this case, the multiplication table obtained above takes the form

$$\left\{ \begin{array}{l} [x, e_1, e_2] = \gamma e_3, \\ [x, e_1, e_3] = p_3 e_3 + p_5 e_5, \\ [x, e_1, e_4] = p_3 e_4, \\ [x, e_1, e_5] = p_3 e_5, \\ [x, e_2, e_3] = q_3 e_3 + q_5 e_5, \\ [x, e_2, e_4] = q_3 e_4, \\ [x, e_2, e_5] = q_3 e_5. \end{array} \right. \quad (14)$$

We first observe that

$$(p_3, q_3) \neq (0, 0).$$

Indeed, if $p_3 = q_3 = 0$, then the ideal series satisfies

$$[N, N, R] \subseteq \text{Span}\{e_3, e_4, e_5\}, \quad [[N, N, R], N, R] \subseteq \text{Span}\{e_4, e_5\},$$

and its next terms are contained in $\text{Span}\{e_5\}$ and then vanish. Thus N would be nilpotent as an ideal of R , contrary to the hypo-nilpotent assumption.

For completeness, if $(p_3, q_3) \neq (0, 0)$, one of $\text{ad}(x, e_1)|_U$ and $\text{ad}(x, e_2)|_U$ is invertible. Since $[N, N, R] \subseteq U$, this gives $[N, N, R] = U$ and $[U, N, R] = U$. Thus N is not nilpotent as an ideal, although its internal lower central series terminates. Maximality follows from codimension one, since R cannot be hypo-nilpotent as an ideal of itself.

We now specify the normalization completely. If $p_3 \neq 0$, put

$$x' = x, \quad e'_1 = p_3^{-1} e_1, \quad e'_2 = -q_3 e_1 + p_3 e_2, \quad e'_3 = e_3, \quad e'_4 = e_4, \quad e'_5 = e_5.$$

If $p_3 = 0$ and $q_3 \neq 0$, put

$$x' = x, \quad e'_1 = q_3^{-1} e_2, \quad e'_2 = -q_3 e_1, \quad e'_3 = e_3, \quad e'_4 = e_4, \quad e'_5 = e_5.$$

In both cases the determinant on $\text{Span}\{e_1, e_2\}$ is 1, so both defining products of N are preserved. The new coefficients satisfy $p'_3 = 1$, $q'_3 = 0$. The transformed products still have no e_4 component in $[x', e'_i, e'_j]$ and no e_4, e_5 component in $[x', e'_1, e'_2]$. Omitting primes gives the desired normalization.

Hence (14) reduces to

$$\begin{cases} [x, e_1, e_2] = \gamma e_3, \\ [x, e_1, e_3] = e_3 + p_5 e_5, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5, \\ [x, e_2, e_3] = q_5 e_5, \\ [x, e_2, e_4] = 0, \\ [x, e_2, e_5] = 0. \end{cases} \quad (15)$$

We next eliminate γ by the adapted change

$$e'_2 = e_2 - \gamma e_3 + \gamma p_5 e_5.$$

Indeed, $[x, e_1, e'_2] = 0$, while the defining products of N and the remaining products in (15) retain their form. Hence we may assume $[x, e_1, e_2] = 0$.

Consequently, the multiplication table can be written as

$$\begin{cases} [x, e_1, e_2] = 0, \\ [x, e_1, e_3] = e_3 + c e_5, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5, \\ [x, e_2, e_3] = d e_5, \\ [x, e_2, e_4] = 0, \\ [x, e_2, e_5] = 0, \end{cases} \quad (16)$$

where $c, d \in \mathbb{C}$.

The remaining parameters can be reduced as follows. If $d \neq 0$, the change $e'_1 = e_1 - (c/d)e_2$ eliminates c . The rescaling

$$x' = x, \quad e'_1 = e_1, \quad e'_2 = d e_2, \quad e'_3 = e_3, \quad e'_4 = d e_4, \quad e'_5 = d^2 e_5$$

then normalizes d to 1: $[x', e'_2, e'_3] = d^2 e_5 = e'_5$. The preceding shear has determinant 1 and leaves e_3, e_4, e_5, x fixed; the rescaling is applied to that sheared basis. If $d = 0$ and $c \neq 0$, choose $s \in \mathbb{C}$ with $s^2 = c$ and put

$$x' = s^{-1} x, \quad e'_1 = s e_1, \quad e'_2 = e_2, \quad e'_3 = e_3, \quad e'_4 = s e_4, \quad e'_5 = s^2 e_5.$$

This normalizes c to 1, since $[x', e'_1, e'_3] = e'_3 + (c/s^2)e'_5 = e'_3 + e'_5$. All previously vanishing products remain zero. Each change preserves the defining products of N . We therefore obtain the following three tables, corresponding respectively to $c = d = 0$, $d = 0, c \neq 0$, and $d \neq 0$.

$$R_{5,3}^{(1)} : \begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5, \\ [x, e_1, e_3] = e_3, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5. \end{cases} \quad (17)$$

$$R_{5,3}^{(2)} : \begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5, \\ [x, e_1, e_3] = e_3 + e_5, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5. \end{cases} \quad (18)$$

$$R_{5,3}^{(3)} : \begin{cases} [e_1, e_2, e_3] = e_4, \\ [e_1, e_2, e_4] = e_5, \\ [x, e_1, e_3] = e_3, \\ [x, e_1, e_4] = e_4, \\ [x, e_1, e_5] = e_5, \\ [x, e_2, e_3] = e_5. \end{cases} \quad (19)$$

All omitted products in the above multiplication tables are zero or follow from skew-symmetry.

It should be noted that the multiplication tables $R_{5,3}^{(1)}$, $R_{5,3}^{(2)}$, and $R_{5,3}^{(3)}$ define 3-Lie algebras. Indeed, the parameter relations obtained from the fundamental identity are satisfied in the case $\alpha = \beta = 0$, and all subsequent reductions are performed by changes of basis preserving the 3-Lie algebra structure.

We summarize the above considerations in the following statement.

Proposition 2. Let R be a six-dimensional solvable 3-Lie algebra with maximal hypo-nilpotent ideal $NGF_{5,3}$, and suppose that $\alpha = \beta = 0$ in the product

$$[x, e_1, e_2] = \alpha e_1 + \beta e_2 + \gamma e_3.$$

Then, by suitable adapted changes of basis, the multiplication table of R can be reduced to one of the forms

$$R_{5,3}^{(1)}, \quad R_{5,3}^{(2)}, \quad R_{5,3}^{(3)}$$

given in (17)–(19), respectively.

Proof. The preceding explicit reductions yield the three tables, whose parameters satisfy (13). In each case,

$$[R, R, R] = U, \quad [U, U, R] = 0, \quad [U, N, R] = U.$$

Thus R is solvable, while N is not nilpotent as an ideal. Internally, $[N, N, N] = \text{Span}\{e_4, e_5\}$, its next lower central term is $\mathbb{C}e_5$, and the following term is zero. Hence N is hypo-nilpotent. Codimension one proves maximality. These statements concern existence and reduction to normal forms, not pairwise non-isomorphism.

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Gaybullaev R., Nurmatova I., Tokhtabayeva O., *A class of one-dimensional solvable extensions of the naturally graded filiform 3-Lie algebra $NGF_{5,3}$* , *Acta NUUZ*, 2026, 2/3, pp. 85–91.