

CONSISTENCY OF WILD BOOTSTRAP FOR RANDOM VARIABLES WITH VALUES IN $L_p[0; 1]$

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ABSTRACT. This paper investigates the consistency of the wild bootstrap for independent identically distributed random elements in the separable Banach space $L_p[0; 1]$, $p > 2$. It is shown that, conditionally on the original sample, the normalized bootstrap mean converges weakly in probability to the same limiting distribution as the normalized sample mean. The proof is based on a central limit theorem for triangular arrays of random elements in $L_p[0; 1]$.

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Key words: wild bootstrap, L_p -space, bootstrap consistency, central limit theorem, triangular arrays.

Introduction

The bootstrap, introduced in [3], is one of the most widely used resampling methods in modern statistics. Its main idea is to approximate the unknown distribution of a statistic by the conditional distribution of the corresponding statistic computed from a bootstrap sample. One of the fundamental theoretical problems in this context is to establish the consistency of the bootstrap procedure.

Fundamental results on bootstrap consistency were obtained in [1], as well as in [4]. For dependent observations, classical bootstrap procedures generally fail to preserve the dependence structure of the original data, which has led to the development of block bootstrap methods and other resampling techniques for dependent sequences; see, for example, [6, 12].

The wild bootstrap originates from the ideas of [14] and was further developed in [8] and [9]. Consistency of wild bootstrap for dependent random variables with values in Hilbert space were considered in [13]. In [2] the authors demonstrate the validity of the wild bootstrap in $C[0; 1]$.

In the present paper, we establish the consistency of the Gaussian wild bootstrap for independent and identically distributed random elements taking values in a $L_p[0; 1]$ space. Using independent standard Gaussian multipliers and a central limit theorem for triangular arrays [10], we verify a Lindeberg-type condition, and convergence of the covariance structure. As a result, we prove the conditional weak convergence of the normalized bootstrap mean to the same centered Gaussian limit as that of the normalized mean of the original observations.

Main part

In this section, we establish the conditional weak convergence of the Gaussian multiplier wild bootstrap in the separable Banach space $L_p[0, 1]$, $p > 2$. The space $L_p[0, 1]$ consists of all equivalence classes of measurable functions $f : [0, 1] \rightarrow \mathbb{R}$ such that $\int_0^1 |f(t)|^p \mu(dt) < \infty$ with the norm $\|f\|_p = \left(\int_0^1 |f(t)|^p \mu(dt) \right)^{1/p}$. In what follows we will use $\|\cdot\|$ for the norm of all spaces under consideration, hopefully that this will not lead to confusion.

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Definition.[7] A separable Banach space B is called type 2 Banach space if for any finite collection of B -valued independent random variables with $\mathbb{E}X_i = 0$, $\mathbb{E}\|X_i\|^2 < \infty$, $i = \overline{1, n}$ the following inequality holds

$$\mathbb{E} \left\| \sum_{i=1}^n X_i \right\|^2 \leq C \sum_{i=1}^n \mathbb{E}\|X_i\|,$$

where a constant C depends on B only.

Theorem 1.[7] Let $\{X_i(t)\}_{i \geq 1}$ be a sequence of independent identically distributed mean-zero random elements taking values in a separable Banach space B of type 2, and suppose that $\mathbb{E}\|X_1\|^{2+\delta} < \infty$. Then the following weak convergence holds as $n \rightarrow \infty$

$$\frac{1}{\sqrt{n}} \sum_{i=1}^n X_i(t) \Rightarrow Y(t) \quad \text{in } B \text{ and } K(s, t) = \mathbb{E}Y(s)Y(t) = \mathbb{E}X_1(s)X_1(t) \quad s, t \in [0, 1],$$

where $Y(t)$ is a centered Gaussian random element. Conversely, if every sequence of independent identically distributed mean-zero B -valued random elements satisfying $\mathbb{E}\|X_1\|^2 < \infty$ satisfies the central limit theorem, then B is of type 2.

This theorem is applicable in our setting, as $L_p[0, 1]$, $p \geq 2$ is type 2 Banach Space.

Let $X_1(t), X_2(t), \dots$ be independent identically distributed $L_p[0, 1]$ -valued random elements. Denote

$$\bar{X}_n(t) = \frac{1}{n} \sum_{i=1}^n X_i(t).$$

Let $W_1, W_2, \dots$ be independent standard Gaussian random variables, independent of the original sample. Define the wild bootstrap observations by

$$X_i^*(t) = \bar{X}_n(t) + (X_i(t) - \bar{X}_n(t))W_i, \quad i = 1, \dots, n,$$

and define the corresponding bootstrap sample mean by

$$\bar{X}_n^*(t) = \frac{1}{n} \sum_{i=1}^n X_i^*(t).$$

Then

$$\sqrt{n}(\bar{X}_n^*(t) - \bar{X}_n(t)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i^*(t) - \bar{X}_n(t)) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (X_i(t) - \bar{X}_n(t))W_i.$$

Thus, the validity of the wild bootstrap is reduced to establishing the conditional weak convergence of the normalized centered bootstrap sample mean. The following theorem gives the corresponding result in $L_p[0, 1]$.

Theorem 2. Let $X_1(t), X_2(t), \dots$ be independent identically distributed $L_p[0, 1]$ -valued random elements such that

$$\mathbb{E}X_1(t) = 0, \quad \mathbb{E}\|X_1(t)\|^{2+\delta} < \infty \text{ for some } \delta > 0.$$

Then the following weak convergence holds as $n \rightarrow \infty$

$$\sqrt{n}(\bar{X}_n^*(t) - \bar{X}_n(t)) \Rightarrow Y(t) \text{ in Probability,}$$

where $Y(t)$ is a centered Gaussian random element in $L_p[0, 1]$ with mean zero and covariance function $K(s, t) = \mathbb{E}Y(s)Y(t) = \mathbb{E}X_1(s)X_1(t)$, $s, t \in [0, 1]$.

Thus theorems 1 and 2 imply consistency of the wild bootstrap.

Proof. The proof of our theorem is based on the central limit theorem for triangular arrays of independent random elements in $L_p[0, 1]$, $p > 2$, obtained in [10]. We first recall this result. For convenience, in what follows we omit the argument t and write X_i , $\bar{X}_n$, and X_i^* instead of $X_i(t)$, $\bar{X}_n(t)$, and $X_i^*(t)$, respectively. We set $S = [0, 1]$, let $\mathcal{S}$ be the Lebesgue σ -algebra on $[0, 1]$, and let μ be the Lebesgue measure.

Theorem 3.[10] Let $(X_{nk}, k = 1, \dots, r_n; n \in \mathbb{N})$ be a triangular array of mean zero independent random elements in the separable space $L_p(S, \mathcal{S}, \mu; \mathbb{R})$, $p > 2$. Assume that the following conditions are satisfied.

(a)

$$\sup_{n \in \mathbb{N}} \sum_{k=1}^{r_n} \mathbb{E} \|X_{nk}\|^2 < \infty.$$

(b) For every $\varepsilon > 0$,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{r_n} \mathbb{E} [\|X_{nk}\|^2 \mathbf{1}_{\{\|X_{nk}\| > \varepsilon\}}] = 0.$$

(c) There is a mean zero Gaussian random element $Y \in L_p(S, \mathcal{S}, \mu; \mathbb{R})$ such that

$$\sum_{k=1}^{r_n} \mathbb{E}(X_{nk} \otimes X_{nk}) \longrightarrow \Gamma := \mathbb{E}(Y \otimes Y) \text{ in } L_p(S^2, \mathcal{S} \otimes \mathcal{S}, \mu \otimes \mu; \mathbb{R}),$$

where $(X_{nk} \otimes X_{nk})(s, t) = X_{nk}(s)X_{nk}(t)$, $(s, t) \in S^2$. Since the random elements are centered, $\mathbb{E}(X_{nk} \otimes X_{nk})$ represents their covariance function.

(d) Denoting by σ_n and σ the non-negative elements $L_p(S, \mathcal{S}, \mu; \mathbb{R})$ defined by $\sigma_n^2(s) := \sum_{k=1}^{r_n} \mathbb{E} X_{nk}(s)^2$ and $\sigma^2(s) := \mathbb{E} Y(s)^2$, μ -a.e. on S ,

$$\int_S \sigma_n^p, d\mu \xrightarrow{n \rightarrow \infty} \int_S \sigma^p, d\mu.$$

Then

$$\sum_{k=1}^{r_n} X_{nk} \xrightarrow{n \rightarrow \infty} Y \text{ in distribution.}$$

Theorem 4.[7] Let $X_1, X_2, \dots$ be independent identically distributed random elements with values in a separable Banach space B . Then $\frac{1}{n} \sum_{i=1}^n X_i \rightarrow 0$ a.s. if and only if $E|X_1| < \infty$ and $EX_1 = 0$.

A fixed realization of the original sample $X_1, \dots, X_n$, we denote by $x_1, \dots, x_n$, and set $\bar{x}_n = \frac{1}{n} \sum_{i=1}^n x_i$. Then $X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i$, $i = 1, \dots, n$.

Thus, it remains to verify that the wild bootstrap random elements satisfy the conditions of Theorem 2. We check these conditions successively.

a) By definition,

$$X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i, \quad i = 1, \dots, n.$$

Hence,

$$\begin{aligned} \sup_{n \in \mathbb{N}} \sum_{i=1}^n \mathbb{E} \left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\|^2 &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n \mathbb{E} \left[\frac{1}{n} W_i^2 \|x_i - \bar{x}_n\|^2 \right] \\ &= \sup_{n \in \mathbb{N}} \sum_{i=1}^n \frac{1}{n} \|x_i - \bar{x}_n\|^2. \end{aligned}$$

Let $M > 0$. Then, by Markov's inequality,

$$\begin{aligned} \mathbb{P} \left(\sum_{i=1}^n \frac{1}{n} \|X_i - \bar{X}_n\|^2 > M \right) &= \mathbb{P} \left(\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^2 > M \right) \\ &\leq \frac{\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^2 \right]}{M} \end{aligned}$$

$$\begin{aligned}
&\leq \frac{\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n (\|X_i\|^2 + \|\bar{X}_n\|^2) \right]}{M} \\
&\leq \frac{\mathbb{E} \left[\frac{1}{n} \sum_{i=1}^n \|X_i\|^2 + \frac{1}{n} \sum_{i=1}^n \|\bar{X}_n\|^2 \right]}{M} \\
&= \frac{\mathbb{E}\|X_i\|^2 + \mathbb{E}\|\bar{X}_n\|^2}{M}.
\end{aligned}$$

From the definition of type 2 Banach spaces we have

$$\begin{aligned}
&= \frac{\mathbb{E}\|X_i\|^2 + \mathbb{E}\|\bar{X}_n\|^2}{M} \leq \frac{\mathbb{E}\|X_i\|^2}{M} + \frac{\mathbb{E}\|\sum_{i=1}^n X_i\|^2}{n^2 M} \leq \\
&\leq \frac{\mathbb{E}\|X_i\|^2}{M} + \frac{C \sum_{i=1}^n \mathbb{E}\|X_i\|^2}{n^2 M} = \frac{\mathbb{E}\|X_i\|^2}{M} + \frac{C \mathbb{E}\|X_i\|^2}{nM} \rightarrow 0 \text{ as } M \rightarrow \infty.
\end{aligned}$$

Consequently,

$$\sup_{n \in \mathbb{N}} \sum_{i=1}^n \mathbb{E}\|X_i^*\|^2 < \infty$$

is bounded in Probability and thus condition (a) is satisfied.

b) We now verify the Lindeberg condition. Using $X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i$ and the inequality $\mathbf{1}_{\{x > \varepsilon\}} \leq \frac{x^\delta}{\varepsilon^\delta}$, for $x \geq 0$, we obtain

$$\begin{aligned}
&\sum_{i=1}^n \mathbb{E} \left[\left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\|^2 \mathbf{1}_{\left\{ \left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\| > \varepsilon \right\}} \right] \\
&\leq \frac{1}{\varepsilon^\delta} \sum_{i=1}^n \mathbb{E} \left[\left\| \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i \right\|^{2+\delta} \right] \\
&= \frac{\mathbb{E}|W_1|^{2+\delta}}{\varepsilon^\delta n^{1+\delta/2}} \sum_{i=1}^n \|x_i - \bar{x}_n\|^{2+\delta} \\
&= \frac{\mathbb{E}|W_1|^{2+\delta}}{\varepsilon^\delta n^{\delta/2}} \left(\frac{1}{n} \sum_{i=1}^n \|x_i - \bar{x}_n\|^{2+\delta} \right).
\end{aligned}$$

The convergence $\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^{2+\delta} \rightarrow \mathbb{E}\|X_1 - \mathbb{E}X_1\|^{2+\delta} = \mathbb{E}\|X_1\|^{2+\delta} < \infty$ a.s. implies $\frac{\mathbb{E}|W_1|^{2+\delta}}{\varepsilon^\delta n^{\delta/2}} \left(\frac{1}{n} \sum_{i=1}^n \|X_i - \bar{X}_n\|^{2+\delta} \right) \rightarrow 0$ a.s..

Therefore, condition **(b)** is satisfied.

c) Now we will verify condition (c).

It is sufficient to prove that $\sum_{i=1}^n \text{Cov}(X_i^*(s), X_i^*(t)) \rightarrow \text{Cov}(Y(s), Y(t))$ in probability in $L_p([0, 1]^2)$.

By the definition of covariance function

$$K(s, t) = \mathbb{E}[(X(s) - \mathbb{E}X(s))(X(t) - \mathbb{E}X(t))] = \mathbb{E}[X(s)X(t)] - \mathbb{E}[X(s)]\mathbb{E}[X(t)].$$

Now we calculate the covariance of the bootstrap variables. Since

$$\mathbb{E}[X_i^*(s)] = \mathbb{E}[X_i^*(t)] = 0,$$

we have

$$\begin{aligned} \sum_{i=1}^n \text{Cov}(X_i^*(s), X_i^*(t)) &= \sum_{i=1}^n (\mathbb{E}[X_i^*(s)X_i^*(t)] - \mathbb{E}[X_i^*(s)]\mathbb{E}[X_i^*(t)]) \\ &= \sum_{i=1}^n \mathbb{E}[X_i^*(s)X_i^*(t)]. \end{aligned}$$

Using the definition

$$X_i^* = \frac{1}{\sqrt{n}}(x_i - \bar{x}_n)W_i,$$

we obtain

$$\begin{aligned} \sum_{i=1}^n \mathbb{E}[X_i^*(s)X_i^*(t)] &= \sum_{i=1}^n \mathbb{E}\left[\frac{1}{\sqrt{n}}(x_i(s) - \bar{x}_n(s))W_i \frac{1}{\sqrt{n}}(x_i(t) - \bar{x}_n(t))W_i\right] \\ &= \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))(x_i(t) - \bar{x}_n(t))\mathbb{E}W_i^2. \end{aligned}$$

Since $\mathbb{E}W_i^2 = 1$, it follows that

$$\sum_{i=1}^n \text{Cov}(X_i^*(s), X_i^*(t)) = \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))(x_i(t) - \bar{x}_n(t)).$$

Expanding the product, we get

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))(x_i(t) - \bar{x}_n(t)) \\ &= \frac{1}{n} \sum_{i=1}^n (x_i(s)x_i(t) - x_i(s)\bar{x}_n(t) - \bar{x}_n(s)x_i(t) + \bar{x}_n(s)\bar{x}_n(t)). \end{aligned}$$

Using

$$\frac{1}{n} \sum_{i=1}^n x_i(s) = \bar{x}_n(s), \quad \frac{1}{n} \sum_{i=1}^n x_i(t) = \bar{x}_n(t),$$

we obtain

$$\begin{aligned} &\frac{1}{n} \sum_{i=1}^n x_i(s)x_i(t) - \bar{x}_n(s)\bar{x}_n(t) - \bar{x}_n(t)\bar{x}_n(s) + \bar{x}_n(s)\bar{x}_n(t) \\ &= \frac{1}{n} \sum_{i=1}^n x_i(s)x_i(t) - \bar{x}_n(s)\bar{x}_n(t). \end{aligned}$$

By Theorem 4, we have

$$\frac{1}{n} \sum_{i=1}^n X_i(s)X_i(t) - \bar{X}_n(s)\bar{X}_n(t) \rightarrow \mathbb{E}[X_1(s)X_1(t)] - \mathbb{E}[X_1(s)]\mathbb{E}[X_1(t)] = \text{Cov}(X_1(s), X_1(t)) = \text{Cov}(Y(s), Y(t)),$$

where the convergence holds almost surely.

Hence, the required covariance structure coincides with the covariance structure of the limiting Gaussian element, and condition (c) is satisfied.

d) Consider

$$\sigma_n^{*2}(s) = \sum_{i=1}^n \mathbb{E} [(X_i^*(s))^2] .$$

and obtain

$$\begin{aligned} \sigma_n^{*2}(s) &= \sum_{i=1}^n \mathbb{E} \left[\frac{1}{n} (x_i(s) - \bar{x}_n(s))^2 W_i^2 \right] \\ &= \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))^2 \mathbb{E} W_i^2 . \end{aligned}$$

Since $\mathbb{E} W_i^2 = 1$, it follows that

$$\sigma_n^{*2}(s) = \frac{1}{n} \sum_{i=1}^n (x_i(s) - \bar{x}_n(s))^2 .$$

Expanding the square, we get

$$\begin{aligned} \sigma_n^{*2}(s) &= \frac{1}{n} \sum_{i=1}^n (x_i^2(s) - 2x_i(s)\bar{x}_n(s) + \bar{x}_n^2(s)) \\ &= \frac{1}{n} \sum_{i=1}^n x_i^2(s) - \frac{2}{n} \sum_{i=1}^n x_i(s)\bar{x}_n(s) + \frac{1}{n} \sum_{i=1}^n \bar{x}_n^2(s) . \end{aligned}$$

Since

$$\frac{1}{n} \sum_{i=1}^n x_i(s) = \bar{x}_n(s) ,$$

we obtain

$$\begin{aligned} \sigma_n^{*2}(s) &= \frac{1}{n} \sum_{i=1}^n x_i^2(s) - 2\bar{x}_n^2(s) + \bar{x}_n^2(s) \\ &= \frac{1}{n} \sum_{i=1}^n x_i^2(s) - \bar{x}_n^2(s) . \end{aligned}$$

By Theorem 4

$$\frac{1}{n} \sum_{i=1}^n X_i^2(s) \longrightarrow \mathbb{E} X_1^2(s) \text{ a.s.}$$

and

$$\bar{X}_n(s) \longrightarrow \mathbb{E} X_1(s) = \mathbb{E} Y(s) \implies \bar{X}_n^2(s) \longrightarrow (\mathbb{E} X_1(s))^2 = \mathbb{E} Y(s)^2 \text{ a.s.},$$

we get $\sigma_n^{*2}(s) \rightarrow \mathbb{E} X_1^2(s) - (\mathbb{E} X_1(s))^2 = \sigma^2(s)$ a.s..

Thus, condition **(d)** is satisfied.

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