

DYNAMICS OF RANDOM PRODUCTS OF POSITIVE CONTRACTIONS ON L_1

JALILOV OLIMJON*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN
olimjonjalilovmath@mail.ru

KHURSANOV SHOKHRUKH

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN
shohruhmath@gmail.com

ABSTRACT. We study a trajectorywise analogue of the classical zero–two phenomenon for products of positive contractions on L^1 -spaces. Motivated by the strong zero–two law for a single positive contraction, we ask whether a uniform bound strictly below 2 for the limiting successive differences along a fixed realization forces these differences to converge to zero. We show that this is false in general. More precisely, for every $c \in (0, 2)$ we construct two rank-one positive contractions on $L^1([0, 1])$ and an alternating realization ω such that

$$\|T_\omega^{n+1}f - T_\omega^n f\|_1 = c$$

for every $n \geq 1$ and every nonnegative f with $\|f\|_1 = 1$. Hence the hypothesis is satisfied for a suitable $\varepsilon > 0$, while the conclusion fails. In fact, the same equality holds in operator norm. Thus a direct pathwise extension of the deterministic zero–two criterion to non-autonomous, and hence random, products is impossible without additional assumptions.

Key words: positive contractions, zero–two law, random products, L^1 -spaces, rank-one operators, counterexample.

Introduction

The asymptotic behaviour of iterates of linear operators on Banach spaces is a classical topic in functional analysis, with important connections to ergodic theory and Markov processes (see, for example, [4–6]). For positive contractions on L^1 -spaces, a fundamental question concerns the decay of the successive differences

$$\|T^{n+1}f - T^n f\|_1.$$

A classical result of Ornstein and Sucheston is the strong zero–two law; related forms and refinements were subsequently obtained by several authors (see [8–10, 12]). One useful formulation is that if the limiting successive difference for every positive unit vector is bounded uniformly away from 2, then the limiting successive difference must in fact be zero for every vector in L^1 .

The purpose of the present note is to examine whether an analogous statement remains valid for products of different positive contractions. Let $\{T_\alpha\}_{\alpha \in A}$ be a family of positive contractions and, for a realization $\omega = (\alpha_1, \alpha_2, \dots)$, define

$$T_\omega^n = T_{\alpha_n} \circ \dots \circ T_{\alpha_1}.$$

The question considered here is trajectorywise: for a fixed realization ω , does a uniform strict bound below 2 for

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1}f - T_\omega^n f\|_1$$

over all positive unit vectors force this limit to be zero?

We show that the answer is negative, even in a particularly simple situation. For every prescribed $c \in (0, 2)$ we construct two rank-one Markov contractions on the standard probability space $[0, 1]$ and an

Received: 19.08.2026, **Accepted:** 19.09.2026

*Corresponding author

alternating realization for which the successive difference is identically equal to c for every positive unit vector. Thus the obstruction is not caused by a specially chosen initial function; it holds uniformly over the whole positive unit sphere. Moreover, the operator-norm successive difference is also equal to c .

The paper is organized as follows. In the main part we recall the deterministic zero-two criterion, formulate its pathwise analogue for products of positive contractions, and then give the counterexample.

Main part

Let $(X, \mathcal{A}, \mu)$ be a measure space and let $L^1 = L^1(X, \mathcal{A}, \mu)$ be the real Banach space of integrable functions with norm $\|\cdot\|_1$. Denote by L^1_+ the cone of nonnegative elements of L^1 . A linear operator $T : L^1 \rightarrow L^1$ is called *positive* if $T(L^1_+) \subset L^1_+$, and it is called a *contraction* if $\|T\| \leq 1$. As usual, equalities between L^1 -functions are understood modulo sets of measure zero.

For a contraction T and $f \in L^1$, the sequence $\|T^n(f - Tf)\|_1 = \|T^n f - T^{n+1} f\|_1$ is non-increasing and therefore has a limit.

Theorem 1. (see [10].) Let T be a positive linear contraction on L^1 . Assume that there exists $\varepsilon > 0$ such that, for every $f \in L^1_+$ with $\|f\|_1 = 1$,

$$\lim_{n \rightarrow \infty} \|T^n f - T^{n+1} f\|_1 < 2(1 - \varepsilon).$$

Then, for every $f \in L^1$,

$$\lim_{n \rightarrow \infty} \|T^n f - T^{n+1} f\|_1 = 0.$$

We now formulate the corresponding question for products of different positive contractions. Let $(A, \mathcal{P}, \lambda)$ be a probability space, so that $\lambda(A) = 1$, and let $\{T_\alpha : L^1(X) \rightarrow L^1(X)\}_{\alpha \in A}$ be a family of positive linear contractions. Set $\Omega = A^{\mathbb{N}}$, $\omega = (\alpha_1, \alpha_2, \dots) \in \Omega$, and, for $n \geq 1$, define $T_\omega^n := T_{\alpha_n} \circ T_{\alpha_{n-1}} \circ \dots \circ T_{\alpha_1}$. When needed, Ω may be equipped with the product probability measure $\mathbb{P} = \lambda^{\mathbb{N}}$. The question below, however, is pathwise and concerns a fixed realization ω .

Question 1. Fix $\omega \in \Omega$. Assume that there exists a constant $\varepsilon > 0$, independent of f , such that for every $f \in L^1_+$ with $\|f\|_1 = 1$ the limit exists and

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1} f - T_\omega^n f\|_1 < 2(1 - \varepsilon). \quad (27)$$

Does it follow that, for every $f \in L^1$,

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1} f - T_\omega^n f\|_1 = 0? \quad (28)$$

The answer is negative. In fact, the next construction shows that every value strictly between 0 and 2 can occur as a constant successive difference, uniformly over all positive unit vectors.

Theorem 2. For every $c \in (0, 2)$ there exist two positive linear contractions T_1, T_2 on $L^1([0, 1])$ and a realization $\omega = (1, 2, 1, 2, \dots)$ such that, for every $f \in L^1_+$ with $\|f\|_1 = 1$ and every $n \geq 1$,

$$\|T_\omega^{n+1} f - T_\omega^n f\|_1 = c. \quad (29)$$

Moreover,

$$\|T_\omega^{n+1} - T_\omega^n\| = c, \quad n \geq 1. \quad (30)$$

Consequently, Question 1 has a negative answer.

Proof. Fix $c \in (0, 2)$ and put $a = 1 - \frac{c}{2} \in (0, 1)$. On $X = [0, 1]$ with Lebesgue measure define

$$u(x) = 1, \quad v(x) = \frac{1}{a} \mathbf{1}_{[0, a]}(x).$$

Then $u, v \in L^1_+$ and $\|u\|_1 = \|v\|_1 = 1$. Furthermore,

$$\|u - v\|_1 = \int_0^a \left(\frac{1}{a} - 1 \right) dx + \int_a^1 1 dx = (1 - a) + (1 - a) = 2(1 - a) = c.$$

Define rank-one operators $T_1, T_2 : L^1([0, 1]) \rightarrow L^1([0, 1])$ by

$$T_1 h = \left(\int_0^1 h(t) dt \right) u, \quad T_2 h = \left(\int_0^1 h(t) dt \right) v. \quad (31)$$

Both operators are linear and positive. Moreover, for every $h \in L^1([0, 1])$,

$$\|T_i h\|_1 = \left| \int_0^1 h(t) dt \right| \|u_i\|_1 \leq \|h\|_1,$$

where $u_1 = u$ and $u_2 = v$. Hence $\|T_i\| \leq 1$. Since equality is attained on any nonnegative unit vector, we have $\|T_1\| = \|T_2\| = 1$. Because $\int_0^1 u dt = \int_0^1 v dt = 1$, for $i, j \in \{1, 2\}$ we obtain

$$T_i T_j h = \left(\int_0^1 T_j h(t) dt \right) u_i = \left(\int_0^1 h(t) dt \right) u_i = T_i h.$$

Thus

$$T_i T_j = T_i, \quad i, j \in \{1, 2\}. \quad (32)$$

For the alternating realization $\omega = (1, 2, 1, 2, \dots)$, formula (32) gives

$$T_\omega^n = \begin{cases} T_1, & n \text{ odd}, \\ T_2, & n \text{ even}. \end{cases}$$

Now let $f \in L_+^1$ with $\|f\|_1 = 1$. Since $f \geq 0$, $\int_0^1 f(t) dt = 1$, and therefore $T_1 f = u$, $T_2 f = v$. Hence for every $n \geq 1$,

$$\|T_\omega^{n+1} f - T_\omega^n f\|_1 = \|u - v\|_1 = c,$$

which proves (29). The same calculation gives the operator-norm statement. Indeed,

$$(T_2 - T_1)h = \left(\int_0^1 h(t) dt \right) (v - u),$$

so $\|(T_2 - T_1)h\|_1 \leq c\|h\|_1$. Taking a nonnegative h with $\|h\|_1 = 1$ gives equality. Thus $\|T_2 - T_1\| = c$, and (30) follows from the alternating form of T_ω^n .

Finally, choose any $0 < \varepsilon < 1 - \frac{c}{2}$. Then $c < 2(1 - \varepsilon)$, so the hypothesis (27) holds for every positive unit vector f , whereas by (29)

$$\lim_{n \rightarrow \infty} \|T_\omega^{n+1} f - T_\omega^n f\|_1 = c > 0.$$

Therefore (28) fails. This proves that Question 1 has a negative answer. $\square$

Remark 1. The operators in Theorem 2 preserve integrals:

$$\int_0^1 T_i h dx = \int_0^1 h dx, \quad i = 1, 2.$$

Thus the counterexample already occurs for rank-one Markov operators. It is also uniform in the initial positive unit vector and, by (30), persists at the operator-norm level.

In this note we have shown that the deterministic zero-two criterion for a single positive contraction does not admit a direct trajectorywise extension to products of positive contractions. The counterexample is elementary but strong: for any prescribed $c \in (0, 2)$ there exist two rank-one Markov contractions and an alternating realization for which

$$\|T_\omega^{n+1} f - T_\omega^n f\|_1 = c$$

for every n and every positive unit vector f . Hence the limiting successive differences may stay uniformly and strictly below 2 without converging to zero.

This shows that additional structural assumptions are indispensable if one seeks a zero-two type theorem for random or non-autonomous products. Possible assumptions may concern the admissible realizations, common invariant structures, mixing properties, or probabilistic hypotheses on the driving process. The present result isolates the basic obstruction: once the operator is allowed to vary with time, the deterministic dichotomy can fail even for very simple positive contractions on a standard L^1 -space.

REFERENCES

1. Bartoszek, W. On the asymptotic behaviour of iterates of positive contractions on L^1 . *Proceedings of the American Mathematical Society* **103** (1988), no. 4, 1115–1120.
2. Bogachev, V. I. *Measure Theory*, Vol. 1 & 2. Springer Science & Business Media, 2007.
3. Brunel, A. Sur les couples de fonctions de l'ergodisme dans les espaces L^1 . *Comptes Rendus de l'Académie des Sciences Paris* **261** (1965), 3266–3269.
4. Dunford, N.; Schwartz, J. T. *Linear Operators. Part I: General Theory*. Interscience Publishers, New York, 1958.
5. Foguel, S. R. *The Ergodic Theory of Markov Processes*. Van Nostrand Reinhold Co., New York, 1969.
6. Krengel, U. *Ergodic Theorems*. de Gruyter Studies in Mathematics, Vol. 6, Walter de Gruyter, Berlin, 1985.
7. Lin, M. Mixing for random operators. *Israel Journal of Mathematics* **28** (1977), no. 3, 215–226.
8. Lin, M.; Wittmann, R. The zero-two law for positive contractions and its converse. *Annals of Probability* **16** (1988), no. 4, 1725–1731.
9. Ornstein, D. The sums of iterates of a positive contraction. *Proceedings of the American Mathematical Society* **22** (1969), no. 2, 266–267.
10. Ornstein, D.; Sucheston, L. An operator theorem on L_1 convergence to zero with applications to Markov kernels. *Annals of Mathematical Statistics* **41** (1970), 1631–1639.
11. Pettis, B. J. On integration in vector spaces. *Transactions of the American Mathematical Society* **44** (1938), no. 2, 277–304.
12. Wittmann, R. A note on the zero-two law for positive contractions on L^1 . *Proceedings of the American Mathematical Society* **103** (1988), no. 3, 793–796.

[Cite this article](#)

Jalilov O., Khursanov Sh. *Dynamics of random products of positive contractions on L_1* , **Acta NUUZ**, 2026, 2/3, pp. 109-112.