

ON QUADRUPLER OF INFINITE-DIMENSIONAL MANIFOLDS IN THE SPACE OF PROBABILITY MEASURES

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ABSTRACT. By investigating the geometric and topological properties of the space $P(X)$ of all probability measures defined on an infinite compactum X , we identify several subspaces of $P(X)$ that are homeomorphic to triples and quadruples of infinite-dimensional manifolds. Furthermore, we distinguish pairs of subspaces in the space of probability measures that are homeomorphic to pairs of infinite-dimensional manifolds.

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Introduction

In this paper, we study topological properties such as manifolds, connectivity, infinite-dimensionality, homogeneity, and homeomorphy, as well as $A(N)$ -geometric properties such as convexity, local convexity, homotopy density, contractibility, and boundary sets of the space of all probability measures $P(X)$, defined on a given infinite compactum X . We also investigate its everywhere dense subsets $B \subset P(X)$ and the set $P_\omega(X)$ of all probability measures with finite supports.

We study how geometric properties of a topological space influence the properties of probability measures on this space. The construction of a general theory of infinite-dimensional manifolds significantly increased interest in infinite-dimensional objects occurring in nature. It is known that the space $P(X)$ of all probability measures on the considered compactum X is a convex subset of a locally convex infinite-dimensional space. The study of the above-listed properties of the space $P(X)$ of all probability measures and its dense subsets is conducted at the intersection of at least the following (main) mathematical disciplines: topology, functional analysis, geometry, and measure theory.

It turns out that probability measures are not only objects of study but also tools for research, through which A. Pelczynski laid the foundations of the theory of Dugundji spaces. [1-2]. We consider both geometric and topological properties of compact and σ -compact subsets of the space $P(X)$ related to their nesting in each other [3-4], geometric and topological properties of boundary, homotopically dense subsets of the space $P(X)$ [5], for subsets E, M, N , $P(X) \supset E \supset M \supset N$ of the space $P(X)$ of topological equivalence of the triple $(P(X), E, N)$, (E, M, N) to the triples (Q, S, Σ) , (Q, Σ, σ) , (S, Σ, σ) .

For an infinite ANR compactum X , the topological equivalence of pairs of manifolds $(P_\infty(X), P_\omega(X))$ to the pairs of manifolds (R^∞, σ) and (Q^∞, Σ) is studied.

Recall that a topological space Y is called an absolute (neighborhood) retract in the class K (written $Y \in A(N)R(K)$ [6], if $Y \in K$ and for every homeomorphism h mapping Y onto a closed subset $h(Y)$ of the space X from the class K , the set $h(Y)$ is a (neighborhood) retract of the space X .

Recall that a topological space X is called a manifold modeled on the space Y , or a Y -manifold [6], if every point of the space X has a neighborhood homeomorphic to an open subset of the space X .

Usually, a Q -manifold is called a separable metric space locally homeomorphic to the Hilbert cube Q , where $Q = \prod_{i=1}^\infty [-1, 1]_i$ is the Hilbert cube [7], $W_i^\pm = \{(g_j) \in Q | g_i = \pm 1\}$ is the i -th face of the Hilbert cube Q , $BdQ = \bigcup_{i=1}^\infty W_i^\pm$ is called the pseudo-boundary of the cube Q , and $S = Q \setminus BdQ$ is the pseudo-interior of the cube Q . It is known that $S \cong \prod_{i=1}^\infty (-1, 1)_i$ [8].

In the theory of infinite-dimensional manifolds, the following objects play an important role: the Hilbert cube Q , the separable Hilbert space ℓ_2 , Σ – the linear span of the standard brick $Q' = \prod_{n=1}^\infty [0, \frac{1}{2^n}]$ in the Hilbert space ℓ_2 . By ℓ_2^f or σ is denoted the linear subspace of the Hilbert space ℓ_2 , consisting of all points only a finite number of coordinates of

which are different from zero, and Q^f — the subspace of the Hilbert cube Q , consisting of all points only a finite number of coordinates of which are different from zero.

By the Anderson-Kadec theorem [8], the Hilbert space ℓ_2 is homeomorphic to S . From the results of Bessaga-Pelczynski, it follows that $\sum$ is homeomorphic to $\text{rint}Q$ and $\sigma \simeq \ell_2^f \simeq Q^f$ [9]. Here, by $\text{rint}Q$ is denoted the set $\{x = (x_n) \in Q \mid |x_n| < t < 1 \text{ for all } n \in N\}$. Further, it is obvious that $\text{rint}Q \approx BdQ$, hence it is true: $BdQ \approx \sum$.

It is known that the spaces Q , $\sum$ and ℓ_2 are strongly infinite-dimensional, while the spaces ℓ_2^f , σ and Q^f are weakly infinite-dimensional and all these spaces are homogeneous.

Definition. [8] A closed set A of a space X is called a Z -set in X , if the identity mapping id_X of the space X can be approximated arbitrarily closely by mappings $f : X \rightarrow X \setminus A$.

A countable union of Z -sets in X is called a σ - Z -set in X .

Following [7], a σ - Z -set, B of the Hilbert cube Q is called a boundary set in Q (denoted by $B(Q)$), if $Q \setminus B \approx \ell_2$. More generally, a boundary set in a Q -manifold is called a σ - Z -set, the complement of which is an ℓ_2 -manifold.

From the above, it follows that the pseudo-boundary BdQ of the Hilbert cube Q is its boundary set.

Let X be a topological space.

A set $A \subset X$ is called homotopically dense in X [5], if there exists a homotopy $h(x, t) : X \times [0, 1] \rightarrow X$ such that $h(x, 0) = id_X$ and $h(X \times (0, 1]) \subset A$.

A set $A \subset X$ is homotopically negligible in X , if $X \setminus A$ is homotopically dense in X .

An embedding $e : Y \rightarrow X$ is homotopically dense (correspondingly, homotopically negligible), if $e(Y)$ is a homotopically dense set (correspondingly homotopically negligible) in X [5].

For a compactum X , there is a simple topological classification of the spaces $P(X)$ of all probability measures. In the case of a finite n -point space $X = \{n\}$ the points μ of the space $P(n) = P_n(n)$ are convex linear combinations of Dirac measures:

$$\mu = m_0\delta(0) + m_1\delta(1) + \dots + m_{n-1}\delta(n-1)$$

Therefore, they are naturally identified with the points of the $(n-1)$ dimensional simplex σ^{n-1} . Here the Dirac $\delta(i)$ form the vertices of the simplex, and the masses m_i , placed at the points i , are the barycentric coordinates of the measure μ . Thus, the compactum $P(n)$ is affinely homeomorphic to the simplex σ^{n-1} .

In the case of an infinite compactum X , the space $P(X)$ is also a compactum [1]. Further, containing simplices of arbitrarily large number of dimensions, they are infinite-dimensional. By Cayley's theorem, a convex compactum $P(X) \subset R^{C(X)}$ affinely embeds into ℓ_2 . Consequently, by Keller's theorem, the compactum $P(X)$, as an infinite-dimensional convex compactum lying in ℓ_2 , is homeomorphic to the Hilbert cube $Q = I^{\aleph_0}$. On the other hand, the space $P(X)$ of all probability measures on a compactum X is called the set of all regular Borel probability measures on X , equipped with the weakest of the topologies for which each functional $f_u : C(X) \rightarrow R$ is continuous, mapping the measure μ to $\mu(U)$ (U is an open set in X).

It is also known that for any χ_1 -degree of a non-degenerate compactum K the space of all probability measures $P(K^{\chi_1})$ is homeomorphic to the Tychonoff cube I^{χ_1} , $P(K^{\chi_1}) = I^{\chi_1}$, I is the interval $[0, 1]$. Note, in particular, that all these spaces are topologically homogeneous. But for spaces $P(K^{\chi_1})$ at $\tau > \chi_1$ the situation is different [10].

For an arbitrary compactum X and measure $\mu \in P(X)$, its support $\text{supp}(\mu)$ is defined — it is the smallest of the closed sets $F \subset X$ for which $\mu(F) = \mu(X)$ i.e. $\text{supp}(\mu) = \bigcap \{A : A = \bar{A}, \mu \in P(A)\}$;

$P_n(X) = \{\mu \in P(X) : |\text{supp} \mu| \leq n\}$ — the set of all measures μ with no more than n supports, $P_\omega(X) = \bigcup_{n=1}^\infty P_n(X)$ — the set of all probability measures μ with finite supports. Recall that the space $P_f(X) \subset P(X)$ consists of all probability measures [11]

$$\mu = m_1\delta(x_1) + m_2\delta(x_2) + \dots + m_n\delta(x_n)$$

with finite supports, for each of which $m_i \geq \frac{n}{n+1}$ for some i ;

$$P_{f,n}(X) = \{\mu \in P_f(X) : |\text{supp} \mu| \leq n\}.$$

From here, the compact subspace $P_f(X)$ is also a union of compacta $P_{f,n}(X)$, i.e., $P_f(X) = \bigcup_{n=1}^\infty P_{f,n}(X)$. It is obvious that $P_{f,n}(X) \subseteq P_n(X)$ and $P_f(X) \subseteq P_\omega(X)$.

It is obvious that for a metric compactum X and any $n \in \mathbb{N}$, the sets $P_n(X)$, $P_f(X)$, $P_{f,n}(X)$ are closed in $P(X)$.

Consequently, the subspace $P_\omega(X) \subset P(X)$ is σ -compact and everywhere dense in $P(X)$.

Note that the subspaces $P(X) \setminus P_n(X)$, $P(X) \setminus P_{f,n}(X)$ and $P(X) \setminus P_f(X)$ are Q -manifolds for any infinite compactum X and any $n \in \mathbb{N}$. In the particular case, $P(Q) \setminus P_n(Q)$, $P(Q) \setminus P_{f,n}(Q)$, $P(Q) \setminus P_f(Q)$ are Q -manifolds. And the subspace $P(X) \setminus P_\omega(X)$ is a G_δ -set in $P(X)$.

For subsets $F \subset E$ and $G \subset F \subset E$, the pair (E, F) or triple (E, F, G) of spaces is called an (E, F) -pair of manifolds or an (E, F, G) -triple (X, M, N) of spaces, if there is an open cover U of the space X and an open embedding $\Phi_u : U \rightarrow E, u \in E$ such that $\Phi_u(M \cap U) = F \cap \Phi_u(U)$ and, accordingly, $\Phi_u(N \cap U) = G \cap \Phi_u(U)$ [12].

Let R^∞ and Q^∞ be the direct limit of spaces R^n and Q^n respectively, i.e., $R^\infty = \lim_{\rightarrow} R^n$ and $Q^\infty = \lim_{\rightarrow} Q^n$, where Q is the Hilbert cube and R is the real line; $R^n = \mathbb{R}^n \times \{0\} \subset R^{n+1}$, $R^\infty = \bigcup_{n=1}^\infty R^n$, $Q^\infty = \bigcup_{n=1}^\infty Q^n$, $Q^n = Q^n \times \{0\} \subset Q^{n+1}$ [13].

On the families $\{R^n : n \in \mathbb{N}\}$ and $\{Q^n : n \in \mathbb{N}\}$, the topology is weak with respect to the hereditary topology in R^ω and Q^ω [13], where $R^\omega = \prod_{n=1}^\infty R^n$ and $Q^\omega = \prod_{n=1}^\infty Q^n$.

Definition.[13] A bitopological space is a pair (X_1, X_2) defined on a single set X ($X_1 = X_2$) or a space (X, τ_1, τ_2) with two topologies τ_1 and τ_2 , denoted as $X_i = (X, \tau_i)$, $i = \overline{1, 2}$.

Let X be a metric compactum and $P_\infty(X) = \varprojlim P_n(X)$ be the direct upper limit of compacta (tower) $P_1(X) \subset P_2(X) \subset \dots$. In this case, there is a bitopological space $(P_\infty(X), P_\omega(X))$ of all probability measures with finite supports. Note that the space $P_\infty(X)$ is a free convex set over the compactum X possessing the property: every continuous mapping $f : X \rightarrow C$ has a unique affine extension $\tilde{f} : P_\infty(X) \rightarrow C$, where C is a convex set in a linear topological space. The subspace $P_\omega(X)$ is a locally convex free convex set over the compactum X possessing the following property: every continuous mapping $f : X \rightarrow C$ has a unique continuous extension $\tilde{f} : P_\omega(X) \rightarrow C$, where C is a convex set of a locally convex linear topological space.

A bitopological space $M = (M, \omega, m)$ is called an (R^∞, σ) -manifold if the topology m is metrizable, $m \subset \omega$, and for every point of the space M there is an m -open neighborhood U and a bitopological open embedding $h : (U, \omega, m) \rightarrow (R^\infty, \sigma)$ such that h is not open for all open embeddings U_m in σ , but is an open embedding for U_ω in R^∞ .

A triple $M = (M, \omega, m)$ is called bilocally equiconnected (denoted as *bi-LEC*) if the diagonal $\Delta_M \subset M \times M$ has an m -neighborhood U (i.e., a neighborhood in the product $M_m \times M_m$) such that there is a bilocally equiconnecting function $\lambda : U \times [0, 1] \rightarrow M$, where $\lambda(x, y, 0) = x$, $\lambda(x, y, 1) = y$, and $\lambda(x, x, t) = x$ for every $(x, y) \in U$ and $t \in [0, 1]$.

Theorem 5. [13] A bitopological space M is an (R^∞, σ) -manifold (or a (Q^∞, Σ) -manifold) if and only if M is *bi-LEC*, M_ω is an R^∞ -manifold (or a Q^∞ -manifold), and every compactum in M_m is a strong Z -set.

Main part

The results of works [15-17] show that for an infinite compactum X , the bitopological pair $(P_\infty(X), P_\omega(X))$ is a *bi-LEC* pair (i.e., satisfies the condition of Theorem 5 [13]) in the following cases:

- a) X is finite-dimensional;
- b) $A \subset X$, A is an F_σ -set or A is σ -compact and $\bar{A} = X$;
- v) $U \subset X$, U is open and $\bar{U} = X$;
- g) A is countable and $\bar{A} = X$;
- i.e., a) $(P_\infty(X), P_\omega(A)) \simeq (Q^\infty, \sigma)$;
- b) $(P_\infty(X), P_\omega(A)) \simeq (R^\infty, \sigma)$;
- v) $(P_\infty(X), P_\omega(U)) \simeq (R^\infty, \sigma)$;
- g) $(P_\infty(X), P_\omega(A)) \simeq (R^\infty, \sigma)$.

In the following cases, the pair $(P_\infty(X), P_\omega(A))$ is homeomorphic to the pair (Q^∞, Σ) if X is an *ANR* compactum:

- a) $P_\omega(X)$ contains a Hilbert cube;
 - b) X contains a Hilbert cube Q ;
- In particular, $(P_\infty(Q), P_\omega(Q)) \simeq (Q^\infty, \Sigma)$.

We have noticed that in many cases for subsets $A \subset X$ of an infinite compactum X , the subspace $P_\omega(A)$ is homeomorphic to σ . There exist different infinite compacta X and Y for which $P_\omega(X)$ is homeomorphic to $P_\omega(Y)$, but $P_\infty(X) \neq P_\infty(Y)$ [13]. That is, the following holds:

Proposition 2.8.[13] For different compacta X and Y , from $P_\infty(X) \simeq P_\infty(Y)$ follows $P_\omega(X) \simeq P_\omega(Y)$.

In the work [12], there is the following:

Theorem 3.1.[12] $\Sigma \setminus \sigma \simeq \Sigma \times S$.

We have noticed that for an infinite compactum X , various everywhere dense subspaces $Y \subset P(X)$ of the Hilbert cube space $P(X)$ are its boundary sets. Consequently, $P_\omega(X)$ is always a boundary set of the compactum $P(X)$. However, the subspace $P_\omega(X)$ can be homeomorphic to either Σ or σ . That is, for some infinite compacta X , if the subspace $P_\omega(X)$ preserves the Hilbert cube Q , then $P_\omega(X) \simeq \Sigma$. But there exists an everywhere dense subset $A \subset X$ different from X such that $P_\omega(A)$ is a boundary set. In this case, the subspace $P_\omega(X)$ is homeomorphic to σ . In these cases, the subspace $P_\omega(X) \setminus P_\omega(A)$ is a $\Sigma \times S$ manifold. That is, the following holds:

Theorem 1. Let X be such an infinite compactum that $P_\omega(X)$ contains the Hilbert cube Q , and $A \subset X$, $A \neq X$, $\bar{A} = X$ and $|A| = \aleph_0$. Then $P_\omega(X) \setminus P_\omega(A) \simeq \Sigma \times S$.

Corollary 1. Let X be an infinite compactum containing the Hilbert cube Q and $A \subset X$, $A \neq X$, $\bar{A} = X$ and $|A| = \aleph_0$. Then $P_\omega(X) \setminus P_\omega(A) \simeq \Sigma \times S$.

Corollary 2. If $A \subset Q$, A is countable and $\bar{A} = X$, then $P_\omega(Q) \setminus P_\omega(A) \simeq \Sigma \times S$.

In works [3,16], it was shown that for an everywhere dense subset $A \subset X$ of a compactum X , the subspaces $P(A)$ and $P_\omega(A)$ are boundary sets of the compactum $P(X)$, and $P(A) \cong \Sigma$ and $P_\omega(A) \simeq \sigma$. On the other hand, the following holds: $P_\omega(A) \subset P(A)$. Thus, the following is true:

Theorem 2. Let X be an infinite compactum and $A_1 \subseteq A_2 \subseteq \dots \subseteq A_n \subseteq \dots$ be closed subsets such that $A = \bigcup_{i=1}^\infty A_i$ is everywhere dense in X . a) $P(A) \simeq \Sigma$, if $P(A)$ contains the Hilbert cube Q ; Then b) $P(A) \simeq \ell_2^f$, if $P(A)$ does not contain the Hilbert cube Q .

Theorem 3. Let X be an infinite compactum and $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ be a sequence of closed sets such that $A \neq X$ and $A = \bigcup_{n=1}^\infty A_i$ is everywhere dense in X .

Then a) $P_\omega(A)$ is homeomorphic to ℓ_2^f , if $P_\omega(A)$ does not contain the Hilbert cube Q ;

b) $P_\omega(A) \simeq \Sigma$, if $P_\omega(A)$ contains the Hilbert cube Q .

Corollary 3. Under the conditions of Theorem 2, let N be a countable everywhere dense subset. Then $P(A) \setminus P(N) \simeq \Sigma \times S$, if $P(A)$ contains the Hilbert cube Q ;

Corollary 4. Under the conditions of Theorem 3, let N be a countable everywhere dense subset. Then $P_\omega(A) \setminus P_\omega(N) \simeq \Sigma \times S$, if $P_\omega(A)$ contains the Hilbert cube Q ;

Theorem 4. Let X be a finite-dimensional compactum, and $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ be a sequence of its closed subsets such that $A \neq X, \bar{A} = X$. Then $P(A) \setminus P(N) \simeq \Sigma \times S$, if there exists an infinite A_{i_0} , where N is a countable subset in X .

Corollary 5. Let X be a finite-dimensional compactum, and $A_1 \subset A_2 \subset \dots \subset A_n \subset \dots$ be its closed subsets such that $A = \bigcup_{n=1}^\infty A_i$ is everywhere dense and A is different from X .

Then $P(A) \setminus P_\omega(A) \simeq \Sigma \times S$ and $P(A) \setminus P_\omega(X) \simeq \Sigma \times S$, if there exists an index i_0 such that A_{i_0} is infinite.

In work [1], there is the following:

Proposition 3.27.[1] If for the triple (X, Y, Z) the pair (X, Y) is homeomorphic to the pair (Q, S) , and the pair (X, Z) is homeomorphic to the pair $(Q, rintQ)$, then the triple (X, Y, Z) is homeomorphic to the triple $(Q, S, rintQ)$.

In work [12], there is the following:

Theorem 3.6.[12] Let X be a Q -manifold (or S -manifold) and $N \subset M \subset X$. Then it is a (Q, Σ, σ) -triple (or (S, Σ, σ) -triple) of manifolds if and only if the pair (M, N) is a (Σ, σ) -manifold and M or N is embedded densely in X .

This theorem can be stated in the following way:

(X, M, N) is a (Q, Σ, σ) - (or (S, Σ, σ))- triple of manifolds, if and only if the pair (X, M) is a (Q, Σ) - (or, (S, Σ))-pair (or, (X, N) is a (Q, σ) - (or (S, σ))- manifold pair and (M, N) is a (Σ, σ) pair.

Proposition 1. For any infinite compactum X , the spaces a) $CP_\omega(X)$, $S_P(A)$, $P(X) \setminus P(A)$ and $P(X \setminus A)$ are homotopically dense in $P(X)$, where $CP_\omega(X) = P(X) \setminus P_\omega(X)$, $A \subset X$, $A \neq \emptyset$, $A \neq X$;

b) For any $n \in \mathbb{N}$ the space $P_\omega(X) \setminus P_n(X)$ is homotopically dense in $P_\omega(X)$; v) For any $n \in \mathbb{N}$ the space $P_\omega(X \setminus A)$ is homotopically dense in $P_\omega(X)$ where $A \subset X$, $A \neq \emptyset$, $A \neq X$;

Proof. Let X be an infinite compact space; then $P(X) \simeq Q$. We shall show that the subspace $CP_\omega(X)$, consisting of the complement of the set of all probability measures with finite supports, is homotopically dense in $P(X)$.

Take a point $\mu_0 \in P(X) \setminus P_\omega(X)$. The support $supp \mu_0$ of this measure μ_0 is not finite, i.e., $|supp \mu_0| \geq \chi_0$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = t\mu_0 + (1 - t)\mu$.

At $t = 0$, $h(\mu, 0) = 0 \cdot \mu_0 + (1 - 0)\mu = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$, then $h(\mu, 1) = \mu_0$.

For $t > 0$, the measure $h(\mu, t) \notin P_\omega(X)$, since the support of the measure $h(\mu, t)$ consists of the set $supp \mu \cup supp \mu_0$. This set is infinite. Thus, the subspace $P_\omega(X)$ is homotopically dense in $P(X)$.

In the case of the homotopy density of the subsets $S_P(A)$, $P(X) \setminus P(A)$ and $P(X \setminus A)$, take a point μ_0 in the complement sets of these sets, i.e., $\mu_0 \in P(X) \setminus S_P(A)$, $\mu_0 \in P(A)$, $\mu_0 \in P(X) \setminus P(X \setminus A)$. The rest of the proof is identical. That is, the sets $S_P(A)$, $P(X) \setminus P(A)$ and $P(X \setminus A)$ are homotopically dense in $P(X)$.

b) It is known that $P_\omega(X)$ is a $\sigma - Z$ -set in $P(X)$ and $P_\omega(X)$ is a boundary set of the space $P(X)$. On the other hand, $P_\omega(X)$ is an AR -space and $P_\omega(X)$ is convex. Consequently, the space $P_\omega(X)$ is strongly universal [4]. From this it follows that $P_\omega(X)$ possesses the compact Z -property, i.e., any compactum in $P_\omega(X)$ is a Z -set in $P_\omega(X)$, including $P_n(X)$ which is also a Z -set. From the result [6] it follows that $P_\omega(X) \setminus P_n(X)$ is homeomorphic to $P_\omega(X)$.

v) Now we show that $P_\omega(X) \setminus P_n(X)$ is homotopically dense in $P_\omega(X)$. Take a measure $\mu_0 \in P(X)$ such that $|supp \mu_0| = \{x_1, x_2, \dots, x_{n+3}\}$. Construct the required homotopy $h(\mu, t) : P_\omega(X) \times [0, 1] \rightarrow P_\omega(X)$ by setting

$$h(\mu, t) = t\mu_0 + (1 - t)\mu$$

At $t = 0$, $h(\mu, 0) = \mu$, i.e., $h(\mu, 0) = id_{P_\omega(X)}$.

For $t \in (0, 1]$ the measure $h(\mu, t)$ has more than $n+3$ distinct points, i.e., $h(\mu, (0, 1]) \notin P_n(X)$. Hence, $h(\mu, (0, 1]) \in P_\omega(X) \setminus P_n(X)$. Thus, the subspace $P_\omega(X) \setminus P_n(X)$ is homotopically dense in $P_\omega(X)$.

g) It is obvious that $P_\omega(X \setminus A)$ is convex and an AR -space. Take a measure $\mu_0 \in P_\omega(X) \setminus P_\omega(X \setminus A)$ such that $\text{supp } \mu_0$ consists of $n+3$ points. Let the measure μ_0 have finite support, i.e., $\text{supp } \mu_0 = \{x_1, x_2, \dots, x_{n+3}\}$ and $x_i \notin A$. The required homotopy is $h(\mu, t) : P_\omega(X) \times [0, 1] \rightarrow P_\omega(X)$.

Construct it using the formula $h(\mu, t) = t\mu_0 + (1-t)\mu$. A simple verification shows that the subspace $P_\omega(X \setminus A)$ is homotopically dense in $P_\omega(X)$. $\square$

Theorem 5. For any infinite compactum X and its closed non-empty infinite subset A different from the compactum X itself, there exist such dense subsets $\Sigma_P(A)$ and $\sigma_P(A)$ in the space $S_P(A)$ that $\Sigma_P(A) \cong \Sigma$ and $\sigma_P(A) \cong \sigma$. The triple $(S_P(A), \Sigma_P(A), \sigma_P(A)) \simeq (S, \Sigma, \sigma)$.

Proof. Let X be an infinite compactum, and $A \subset X$, $A = \bar{A}$ and $A \neq X$, A being infinite. Then $P(A) \simeq Q$ and $P(X) = Q$. On the other hand, the set $P(A) \subset S_P(A)$. It is known that $S_P(A) \subset P(X)$ and $S_P(A)$ is everywhere dense and homotopically dense in $P(X)$. From the homeomorphism of $P(A)$ to the Hilbert cube Q , we can consider $P(A)$ as a Keller cube. We may assume that $O \in P(A)$, where O is the zero point. Take a set $K = \{\mu_1, \mu_2, \dots, \mu_k\}$ in $S_P(A) \setminus P(A)$. Now for each $n \in \mathbb{N}$ set $S_P^n(A) = \text{conv}\{P(A), \mu_1, \mu_2, \dots, \mu_n\}$, where $S_P^n(A)$ is the convex hull of the set $\{P(A), \mu_1, \mu_2, \dots, \mu_n\}$. From Lemma 5.3.14 [5] it follows that $S_P^n(A)$ is a Keller cube and the zero point O belongs to $S_P^n(A)$.

Consequently, $S_P^n(A) \simeq Q$. Now denote $\bigcup_{n=1}^\infty S_P^n(A)$ by $\Sigma_P(A)$. It is obvious that $S_P^1(A), S_P^2(A), \dots, S_P^n(A), \dots$ is a skeletoid in $S_P(A)$ and $\Sigma_P(A)$ is a convex σ -compact AR space containing the Hilbert cube Q . In this case, from Theorem 5.3.13 [5] it follows that the subspace $\Sigma_P(A)$ is strongly μ_0 -universal. Consequently, from Corollary 5.3.15 [5] we have $\Sigma_P(A) \simeq \Sigma$.

Now let $\{\mu_1, \mu_2, \dots, \mu_k, \dots\}$ be a countable everywhere dense subset in $P(A)$. For each $n \in \mathbb{N}$ set $X_n = \text{conv}\{\mu_1, \mu_2, \dots, \mu_n\}$ – the convex hull of the points $\{\mu_1, \mu_2, \dots, \mu_n\}$. We have the sequence

$$X_1 \subset X_2 \subset \dots \subset X_n \subset \dots \quad (9)$$

The sequence (9) is a strongly universal space by Lemma 5.3.11 [5]. Denote $\bigcup_{i=1}^\infty X_i$ by $\sigma_P(A)$. In this case, for $n \in \mathbb{N}$, X_n is convex and X is σ -compact and A is a convex set. Note that it is an AR space. By Corollary 5.3.12 [5] it follows that $\sigma_P(A) \simeq \sigma$. Thus, the triple $(S_P(A), \Sigma_P(A), \sigma_P(A)) \simeq (Q, \Sigma, \sigma)$. $\square$

Theorem 6. Let X be an infinite compactum. In the space $P_{\nabla\omega}(X) = P(X) \setminus P_\omega(X)$ there exist such dense subsets $\Sigma(\omega)$ and $\ell_2^f(\omega)$ that $\Sigma(\omega) \simeq \Sigma$, $\ell_2(\omega) \simeq \ell_2$ and the triple $(P_{\nabla\omega}(X), \Sigma(\omega), \ell_2^f(\omega)) \simeq (S, \Sigma, \sigma)$.

Proposition 2. Let X be an infinite compactum. For any non-empty subset $A \subset X$ different from X , the subspace $P(X) \setminus P(A)$ is homotopically dense in $P(X)$.

Proof. Let X be an infinite compactum $A \subset X$ and $A \neq X$. In this case $P(A)$ is convex in $P(X)$ and $P(X) \simeq Q$. Take an arbitrary point $x_0 \in X \setminus A$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = (1-t)\mu + t\delta_{x_0}$.

At $t = 0$, $h(\mu, 0) = (1-0)\mu + 0 \cdot \delta_{x_0} = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$, $h(\mu, 1) = \delta_{x_0}$.

For $t \in (0, 1]$, i.e., $t > 0$, the measure $h(\mu, t) \notin P(A)$, since $\text{supp } h(\mu, t)$ consists of the set $\text{supp } \mu \cup \{x_0\}$. Thus, the subspace $P(X) \setminus P(A)$ is homotopically dense in $P(X)$. $\square$

Proposition 3. For any infinite compactum X and any $n \in \mathbb{N}$, the subspace $P(X) \setminus P_n(X)$ is homotopically dense in $P(X)$.

Proof. Let X be infinite and $n \in \mathbb{N}$. Take an arbitrary measure $\mu_0 \in P(X) \setminus P_n(X)$ such that $|\text{supp } \mu_0| \geq n+3$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = (1-t)\mu + t\mu_0$.

At $t = 0$ the measure $h(\mu, 0) = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$ the measure $h(\mu, 1) = \mu_0$.

For $t > 0$ the measure $h(\mu, t) \notin P_n(X)$ because the support $\text{supp } h(\mu, t)$ of the measure $h(\mu, t)$ consists of the set $\text{supp } \mu \cup \text{supp } \mu_0$. Thus, the subspace $P(X) \setminus P_n(X)$ is homotopically dense in $P(X)$. $\square$

Proposition 4. For any infinite compactum X , the subspace $P(X) \setminus P_f(X)$ is homotopically dense in $P(X)$.

Proof. Let X be an infinite compactum. It is known that $P_f(X) \subset P(X)$ and $P(X) \setminus P_f(X)$ is convex. Take an arbitrary point $\mu_0 \in P(X) \setminus P_f(X)$ where the support $\text{supp } \mu_0$ is infinite; or the measure μ_0 has the form $\mu_0 = \frac{1}{n}\delta_{x_1} + \frac{1}{n}\delta_{x_2} + \dots + \frac{1}{n}\delta_{x_n}$, where $n > 2$, i.e., $\text{supp } \mu_0 = \{x_1, x_2, \dots, x_n\}$. Construct a homotopy $h(\mu, t) : P(X) \times [0, 1] \rightarrow P(X)$ by setting $h(\mu, t) = (1-t)\mu + t\mu_0$.

At $t = 0$, the measure $h(\mu, 0) = \mu$, i.e., $h(\mu, 0) = id_{P(X)}$.

At $t = 1$ the measure $h(\mu, 1) = \mu_0$.

For $t > 0$ the measure $h(\mu, t) \notin P_f(X)$, since the support of the measure $h(\mu, t)$ contains the set $\text{supp } \mu \cup \text{supp } \mu_0$.

Thus, the subspace $P(X) \setminus P_f(X)$ is homotopically dense in $P(X)$. $\square$

We have noticed that for an infinite compactum X and its everywhere dense subset A different from X , the subspaces $P_\omega(X)$ and $P_\omega(A)$ are boundary sets of the compactum $P(X)$, i.e., $P(X) \setminus P_\omega(X)$ and $P(X) \setminus P_\omega(A)$ are homeomorphic to ℓ_2 . In this case, the following can be asserted:

- a) for any $n \in \mathbb{N}$ the space $P_\omega(X) \setminus P_n(X)$ is homeomorphic to $P_\omega(X)$;
- b) for any $n \in \mathbb{N}$ the space $P_\omega(A) \setminus P_n(A)$ is homeomorphic to $P_\omega(A)$;

It is known that $P_f(X)$ is compact and is a countable union of compacta $P_{f,n}(X)$ which are subsets of the compactum $P_n(X)$. Consequently, the compacta $P_{f,n}(X)$ are Z -sets in $P_\omega(X)$. In this case, it can be asserted:

- v) for any infinite compactum X , the space $P_\omega(X) \setminus P_f(X)$ is homeomorphic to the space $P_\omega(X)$.

In work [3], we proved that for any infinite compactum X and any of its closed subsets A different from X , the space $S_P(A)$ is homeomorphic to the Hilbert space ℓ_2 .

If B is a compactum in $S_P(A)$, then B is a Z -set in $S_P(A)$. Then, by the theorem in [6], the space $S_P(A) \setminus B$ is homeomorphic to ℓ_2 . In particular, $S_P(A) \setminus P(A)$ is homeomorphic to ℓ_2 .

Theorem 7. For any infinite metric compactum X and any everywhere dense, non-open G_δ -subset Y different from X , the subspace $P(X) \setminus P(Y)$ is homeomorphic to the space Σ .

Proof. Let X be an infinite compactum and $Y \subset X$, where Y is a G_δ -set, $Y \neq X$, and Y is not open. In this case, we have: $Y = \bigcap_{i=1}^{\infty} U_i$, where U_i is open in X and $\bar{U}_i = X$. Note that $U_1 \supset U_2 \supset \dots \supset U_n \supset \dots$. Hence $X \setminus U_1 \subset X \setminus U_2 \subset \dots \subset X \setminus U_n \subset \dots$. For each $i \in \mathbb{N}$, the set $X \setminus U_i$ is closed and $X \setminus U_i$ is infinite. Consequently, $X \setminus U_i$ is compact.

By the equality $P(X \setminus A) = P(X) \setminus S_P(A)$, where $A \subset X, A \neq X$, we have:

$$P(X) \setminus P(Y) = \{\mu \in P(X) : \text{supp } \mu \cap (X \setminus Y) \neq \emptyset\}.$$

Thus, $P(X) \setminus P(Y) = S_P(X \setminus Y) = S_P(X \setminus \bigcap_{i=1}^{\infty} U_i) = S_P(\bigcup_{i=1}^{\infty} (X \setminus U_i)) = \bigcup_{i=1}^{\infty} S_P(X \setminus U_i) = \bigcup_{i=1}^{\infty} P(X \setminus U_i)$.

Hence, the space $P(X) \setminus P(Y) = \bigcup_{i=1}^{\infty} P(X \setminus U_i)$ is σ -compact. Note that $P(X \setminus U_i) \subset P(X \setminus U_{i+1})$. By virtue of the infinity of $X \setminus U_i$, the space $P(X \setminus U_i) \simeq Q$. From the results of work [4], the subspace $P(X) \setminus P(Y)$ is a σ - Z -set in $P(X)$.

It follows from the definition that $P(X) \setminus P(Y)$ is locally homotopically negligible in $P(X)$. Thus, the subspace $P(X) \setminus P(Y)$ is an increasing sequence of its infinite-dimensional convex compacta such that:

- 1) $P(X \setminus U_i)$ is a Z -set in $P(X \setminus U_{i+1})$ [4];
- 2) $\bigcup_{i=1}^{\infty} P(X \setminus U_i)$ is everywhere dense in $P(X)$.

Then the pair $(P(X), P(Y))$ is homeomorphic to the pair $(Q, \text{rint } Q)$ by Lemma 2.10 [18]. By Theorem 2.2 [18], the pair $(P(X), P(Y))$ is homeomorphic to the pair (Q, S) .

Thus, the following holds:

Theorem 8. For any infinite metric compactum X and its proper non-open everywhere dense G_δ subset $Y \subset X$, the pair $(P(X), P(Y))$ is homeomorphic to the pair (Q, S) .

Corollary.[18] For space X , we have $P(X) \simeq S$ if and only if X is a complete separable non-compact metric space.

Using the functorial values of the probability measure functor $P : \text{Comp} \rightarrow \text{Comp}$ and the relative positions of compacta [16-17], Theorems 7-8, the Corollary in [18], and Corollary 16 in [18] is proved as follows:

Theorem 9. The quadruple $(P(X), P(Y), P(A), P_\omega(L))$ is homeomorphic to the quadruple (Q, s, Σ, σ) if and only if X is an infinite separable compactum, Y is a non-open dense G_δ -set with $Y \neq X$, A is open and dense, and L is a countable dense subspace in X .

Theorem 10. For any compactum X and any proper subset $A \subset X$, the subspace $S_{P_n}(A)$ ($n > 1$) is homotopically dense in $P_n(X)$.

Proof. Let X be an arbitrary compactum. It is known that $P_n : \text{Comp} \rightarrow \text{Comp}$. Consider the normal functor and the set $S_{P_n}(A) = \{\mu \in P_n(X) : \text{supp } \mu \cap A \neq \emptyset\}$, where A is a proper subset of the compactum X .

If X is a finite set, i.e., $|X| = n$, then $P_n(X) = \sigma^{n-1}$ – a simplex of dimension $n - 1$.

Note that the set $S_{P_n}(A)$ is everywhere dense in $P_n(X)$ for any proper subset $A \subset X$.

The required homotopy $h(\mu, t) : P_n(X) \times [0, 1] \rightarrow P_n(X)$ is constructed as $h(\mu, t) = (1 - t)\mu + t\mu_0$, where $\mu_0 \in P(A)$, i.e., $\text{supp } \mu_0 \subseteq A$.

If $\mu \in P_n(X)$, then $h(\mu, 0) = (1 - 0)\mu + 0 \cdot \mu_0 = \mu$, i.e., $h(\mu, 0) = \text{id}_{P_n(X)}$.

If $t > 0$, then the measure $h(\mu, t) = (1 - t)\mu + t\mu_0$ has a support $\text{supp } h(\mu, (0, 1])$ that intersects the set A , that is, $\text{supp } h(\mu, (0, 1]) \cap A \neq \emptyset$. Hence, $h(\mu, t) \in S_{P_n}(A)$ for $t > 0$. This means that $S_{P_n}(A)$ is homotopically dense in $P_n(X)$. $\square$

If X is a compactum and $A \subset X$ is its proper subset, then $S_{P_n}(A)$ is a convex subset in $P_n(X)$. Indeed, if $\mu_0 \in S_{P_n}(A)$ and $\mu_1 \in S_{P_n}(A)$, then the measure $(1-t)\mu + t \cdot \mu_1$ belongs to $S_{P_n}(A)$ because the support intersects with A .

On the other hand, the set $S_{P_n}(A)$ is locally convex. Thus, for a proper subset $A \subset X$ in the subspace $S_{P_n}(A)$ of the space $P_n(X)$, we have:

Theorem 11. For any compactum X and any of its proper subsets $A \subset X$, the subspace $S_{P_n}(A)$ is an absolute retract, i.e., $S_{P_n}(A) \in AR$.

It is known that for any compactum X , the set $P_{n,n-1}(X)$ is an open subspace in $P_n(X)$. If $\mu \in P_{n,n-1}(X)$, then its support consists of exactly n points, that is, $\text{supp } \mu = \{x_1, x_2, x_3, \dots, x_n\}$. The set of all measures μ whose supports consist of exactly n points forms an n -dimensional simplex σ^n , and its interior is $\text{int } \sigma^n$. It is known that $\text{int } \sigma^n$ is homeomorphic to the Euclidean space $\mathbb{R}^n$ of dimension n .

Thus, $\text{int } \sigma^n \subset P_{n,n-1}(X)$.

Take an arbitrary $\mu_0 \in P_n(X) \setminus P_{n-1}(X)$, then $\text{supp } \mu_0 = \{x_1, x_2, \dots, x_n\}$. Let $O(\mu_0)$ be an arbitrary neighborhood of the measure μ_0 in $P_n(X)$. Then $O(\mu_0) \cap (P_n(X) \setminus P_{n-1}(X)) = O'(\mu_0)$, where $O'(\mu_0)$ is open in $P_n(X) \setminus P_{n-1}(X)$. We may assume that $O'(\mu_0)$ is an open subset in $\text{int } \sigma^n \simeq \mathbb{R}^n$. Every open subset of $\text{int } \sigma^n$ is homeomorphic to $\mathbb{R}^n$.

The following holds:

Theorem 12. For any compactum X , the subspace $P_{n,n-1}(X)$ of the space $P_n(X)$ is an $\mathbb{R}^n$ -manifold.

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