

DENSITY-TYPE PROPERTIES OF TOPOLOGICAL GROUPS

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ABSTRACT. In this paper, we study the density-type properties of topological groups. In particular, proved that for a topological group G , which the identity element 1 is closed in G , and there is a countable basis of neighborhoods for 1 in G the following relations hold: 1) $d(G) = wd(G)$; 2) $ld(G) = lwd(G)$. Besides, shown that for a topological group G , which G is a second countable locally compact (Hausdorff) space, the following relations hold: 1) $d(G) = wd(G)$; 2) $ld(G) = lwd(G)$. Finally, shown that for a topological group G , which G is Lie groups and locally Euclidean, the following relations hold: 1) $d(G) = wd(G)$; 2) $ld(G) = lwd(G)$.

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Introduction

A fundamental chapter of topological spaces that has attracted the interest of researchers are undoubtedly the density-type properties of topological spaces.

Topological groups arise when a topological space carries binary operations with the elements of the topological space satisfying all conditions of groups. In recent years, such structures, especially with Lie groups, have attracted considerable attention in mathematics and mathematical physics.

One of the central problems in the field is the cardinal invariants of these topological groups. A major step in this direction has been the density-type invariants of the topological groups. This approach gives us the opportunity to evaluate geometric properties of topological groups.

In recent years many authors have investigated influence of different functors to various density-type properties. In particular, such studies have been done for the functor of permutation degree, finite hyperspace, and finite product (see, for instance, [3, 4, 5, 6, 7]). For example, most important density-type properties of the space of permutation degree of a topological space were considered in [5].

The papers [4] and [6] contain several similar interesting results on various density-type properties, and the paper [7] presents several results about cardinal invariants in the space of permutation degree, finite hyperspace, and finite product.

Throughout this paper, all spaces are assumed to be an infinite T_1 -spaces. Topological terminology and notation follow [1, 2].

Preliminaries

In this section we recall the notions and elementary facts that will be used below.

A topological group G is a topological space equipped with a group structure such that the multiplication

$$\cdot : G \times G \longrightarrow G, \quad (x, y) \longmapsto xy,$$

and the inversion

$$^{-1} : G \longrightarrow G, \quad x \longmapsto x^{-1},$$

are continuous. The product $G \times G$ is endowed with the product topology. A topology satisfying these requirements is called a group topology [1].

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Continuity of multiplication can equivalently be expressed as follows: for arbitrary $x, y \in G$ and every neighborhood W of xy , there exist neighborhoods U of x and V of y such that $UV \subseteq W$, where $UV = \{uv : u \in U, v \in V\}$. Similarly, continuity of inversion means that for every $x \in G$ and every neighborhood V of x^{-1} there is a neighborhood U of x with $U^{-1} \subseteq V$, where $U^{-1} = \{u^{-1} : u \in U\}$.

It is enough to require continuity of the single map

$$G \times G \longrightarrow G, \quad (x, y) \longmapsto xy^{-1}.$$

In neighborhood form, for every $x, y \in G$ and every neighborhood W of xy^{-1} , there are neighborhoods U of x and V of y such that $UV^{-1} \subseteq W$.

A subset $A \subset X$ is called dense in X when $\overline{A} = X$. The density of X , denoted by $d(X)$, is the least cardinality of a dense subset of X . Thus

$$d(X) = \min\{|A| : A \subset X, \overline{A} = X\}.$$

A space with $d(X) \leq \aleph_0$ is called separable [2].

The local density of X at x is the least cardinal τ for which x possesses a neighborhood of density τ in X . We denote this cardinal by $ld(x)$. The global local density is then

$$ld(X) = \sup\{ld(x) : x \in X\}.$$

It is standard that

$$ld(X) \leq d(X)$$

for every topological space X .

Example. Consider the real line R endowed with the discrete topology τ_d . For every $x \in R$, the singleton $\{x\}$ is an open neighborhood and has density one. Hence

$$ld(R, \tau_d) = 1.$$

On the other hand, a discrete space has no proper dense subset, so its density equals its cardinality. Therefore

$$d(R, \tau_d) = |R| = c,$$

and consequently

$$1 = ld(R, \tau_d) < d(R, \tau_d) = c.$$

The weak density of a topological space is the least cardinal $\tau \geq \aleph_0$ for which there exists a π -base B that can be represented as

$$B = \bigcup_{\alpha \in A} B_\alpha, \quad |A| = \tau,$$

where each B_α is a centered family of open sets. We denote this cardinal invariant by $wd(X)$ [4].

A topological space X is said to be locally weakly τ -dense at x if τ is the least cardinal for which x has a neighborhood whose weak density is τ . The corresponding pointwise invariant is denoted by $lwd(x)$, and the global local weak density is

$$lwd(X) = \sup\{lwd(x) : x \in X\}.$$

Main part

A real Lie group is a group that is also a finite-dimensional real smooth manifold, in which the group operations of multiplication and inversion are smooth maps. Smoothness of the group multiplication:

$$\mu : G \times G \rightarrow G, \quad \mu(x, y) = xy$$

means that μ is a smooth mapping of the product manifold $G \times G$ into G . The two requirements can be combined to the single requirement that the mapping:

$$(x, y) \mapsto x^{-1}y$$

be a smooth mapping of the product manifold into G [1].

Theorem 1. Let a topological group G , the identity element 1 is closed in G , and there is a countable basis of neighborhoods for 1 in G . Then the following relations hold:

- 1) $d(G) = wd(G)$;
- 2) $ld(G) = lwd(G)$.

Proof. We first compare the global invariants and subsequently establish the corresponding local equality.

Because $\{1\}$ is closed and left translations are homeomorphisms, every singleton is closed. Indeed, for each $x \in G$ the map

$$L_x : G \longrightarrow G, \quad L_x(y) = xy,$$

is a homeomorphism and therefore

$$\{x\} = L_x(\{1\})$$

is closed. Thus G is a T_1 -space. A T_1 topological group is Hausdorff. To see this directly, take distinct $x, y \in G$. Then $x^{-1}y \neq 1$. Since $\{x^{-1}y\}$ is closed,

$$G \setminus \{x^{-1}y\}$$

is a neighborhood of 1. By continuity of the group operations, one can choose a symmetric neighborhood V of 1 such that

$$V^{-1}V = V^2 \subseteq G \setminus \{x^{-1}y\}.$$

If xV and yV had a common point, then $xu = yv$ for some $u, v \in V$, which would imply

$$x^{-1}y = uv^{-1} \in V^2,$$

a contradiction. Hence xV and yV are disjoint, and G is Hausdorff.

The countable local base at 1 can be transported by translations. Hence every point of G has a countable neighborhood base. Thus G is first countable and Hausdorff. By the Birkhoff–Kakutani metrization theorem, G admits a compatible metric.

We use the standard fact that for every metrizable T_1 -space X ,

$$d(X) = wd(X).$$

For completeness, one direction follows immediately from the definitions. If D is dense and $|D| = d(X)$, then, using countable local bases at the points of D , the union of these bases is a π -base that is a union of $d(X)$ centered families. Consequently,

$$wd(X) \leq d(X).$$

The reverse inequality for metrizable spaces follows from the density–weak-density theorem for stratifiable spaces; metrizable spaces are stratifiable. Hence the two invariants agree.

Applying this result to the metrizable group G gives

$$\boxed{d(G) = wd(G)}.$$

It remains to prove the local equality. Fix $x \in G$ and let $\mathcal{N}(x)$ denote the collection of neighborhoods of x . Every neighborhood $U \in \mathcal{N}(x)$, with the subspace topology, is metrizable. Therefore

$$d(U) = wd(U)$$

for each such U . By the definitions,

$$ld(x) = \min\{d(U) : U \in \mathcal{N}(x)\}, \quad lwd(x) = \min\{wd(U) : U \in \mathcal{N}(x)\}.$$

Since the two quantities agree for every neighborhood involved, we obtain

$$ld(x) = lwd(x).$$

As x was arbitrary,

$$\begin{aligned} ld(G) &= \sup\{ld(x) : x \in G\} \\ &= \sup\{lwd(x) : x \in G\} \\ &= lwd(G). \end{aligned}$$

Therefore

$$\boxed{ld(G) = lwd(G)}.$$

Both assertions follow.

Theorem 2. Let a topological group G is a second countable locally compact (Hausdorff) space. Then the following relations hold:

- 1) $d(G) = wd(G)$;
- 2) $ld(G) = lwd(G)$.

Proof. We again treat the global and local invariants separately.

Part 1: Proof that $d(G) = wd(G)$.

Choose a countable base

$$\mathcal{B} = \{B_n : n \in \mathbb{N}\}$$

for G . For each n select $x_n \in B_n$ and define

$$D = \{x_n : n \in \mathbb{N}\}.$$

If O is a nonempty open subset of G , some basic set B_n satisfies

$$\emptyset \neq B_n \subseteq O.$$

The selected point x_n therefore belongs to O , and so $D \cap O \neq \emptyset$. Thus D is dense and

$$d(G) \leq \aleph_0.$$

Since G is infinite,

$$d(G) = \aleph_0.$$

The same countable base is a π -base. For every $n \in \mathbb{N}$ put

$$\mathcal{B}_n = \{B_n\}.$$

Each $\mathcal{B}_n$ is centered, and

$$\mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n.$$

Hence $\mathcal{B}$ witnesses

$$wd(G) \leq \aleph_0.$$

By the definition of weak density, $wd(G) \geq \aleph_0$, so

$$wd(G) = \aleph_0.$$

Consequently,

$$\boxed{d(G) = wd(G)}.$$

Alternatively, second countability makes G first countable; since G is Hausdorff, the Birkhoff–Kakutani theorem shows that it is metrizable, and the standard metrizable-space equality $d(G) = wd(G)$ gives the same conclusion. The local compactness assumption is retained as part of the class specified in the theorem.

Part 2: Proof that $ld(G) = lwd(G)$.

Let $x \in G$ and let $\mathcal{N}(x)$ be the neighborhood family at x . Every subspace of a second countable space is second countable. Thus, for each $U \in \mathcal{N}(x)$,

$$\mathcal{B}|_U = \{B_n \cap U : B_n \cap U \neq \emptyset\}$$

is a countable base for U . Since G is Hausdorff, U is Hausdorff as well.

Repeating the argument from Part 1 for the space U gives

$$d(U) = wd(U).$$

Therefore

$$ld(x) = \min\{d(U) : U \in \mathcal{N}(x)\}$$

and

$$lwd(x) = \min\{wd(U) : U \in \mathcal{N}(x)\}$$

have the same value. Hence

$$ld(x) = lwd(x)$$

for every $x \in G$. Taking suprema yields

$$\begin{aligned} ld(G) &= \sup\{ld(x) : x \in G\} \\ &= \sup\{lwd(x) : x \in G\} \\ &= lwd(G). \end{aligned}$$

Thus

$$\boxed{ld(G) = lwd(G)}.$$

Combining the two parts, we have

$$\boxed{d(G) = wd(G)} \quad \text{and} \quad \boxed{ld(G) = lwd(G)}.$$

This completes the proof.

Theorem 3. Let G is Lie groups, which are locally Euclidean. Then the following relations hold:

- 1) $d(G) = wd(G)$;
- 2) $ld(G) = lwd(G)$.

Proof. A Lie group is, in particular, a finite-dimensional smooth manifold. We exploit its local Euclidean structure together with the homogeneity supplied by translations.

Part 1: Proof that $d(G) = wd(G)$.

For each $x \in G$ there are an open neighborhood U_x of x , an integer $n \geq 0$, and a homeomorphism

$$\varphi_x : U_x \longrightarrow V_x,$$

where V_x is an open subset of $\mathbb{R}^n$. Since $\mathbb{R}^n$ is second countable and metrizable, the same is true of V_x and hence of U_x .

Choose such a coordinate neighborhood U at the identity element 1, together with a homeomorphism

$$\varphi : U \longrightarrow V \subseteq \mathbb{R}^n.$$

Let

$$\mathcal{B} = \{B_k : k \in \mathbb{N}\}$$

be a countable base for U . Select $u_k \in B_k$ and put

$$D_U = \{u_k : k \in \mathbb{N}\}.$$

Given any nonempty open subset O of U , some B_k satisfies

$$\emptyset \neq B_k \subseteq O.$$

Hence $u_k \in O$, showing that D_U is dense in U .

For each $g \in G$, left translation

$$L_g : G \longrightarrow G, \quad L_g(y) = gy,$$

is a homeomorphism. It follows that gU is a metrizable neighborhood of g and

$$gD_U = L_g(D_U)$$

is dense in gU . Thus the local Euclidean pieces of the group all have the same qualitative density behavior.

More importantly, a finite-dimensional manifold is first countable. Hence the Lie group G is first countable at 1 and, by translation, at every point. Since a Lie group is Hausdorff, the Birkhoff–Kakutani metrization theorem applies and shows that G is metrizable.

For metrizable T_1 -spaces the density and weak density coincide. Therefore

$$\boxed{d(G) = wd(G)}.$$

Part 2: Proof that $ld(G) = lwd(G)$.

Fix $x \in G$. A coordinate neighborhood U_x of x is metrizable. More generally, whenever V is a neighborhood of x contained in a coordinate neighborhood, V is metrizable and consequently

$$d(V) = wd(V).$$

Thus the two density notions agree on the neighborhood system generated by the local Euclidean structure.

The group structure gives an equivalent description at every point. Namely,

$$L_x : G \longrightarrow G, \quad L_x(y) = xy,$$

is a homeomorphism and maps neighborhoods of 1 bijectively onto neighborhoods of x . Therefore the local invariants are translation invariant:

$$ld(x) = ld(1), \quad lwd(x) = lwd(1).$$

At the identity, a Euclidean neighborhood is metrizable and has equal density and weak density. Hence

$$ld(1) = lwd(1),$$

and consequently

$$ld(x) = lwd(x)$$

for every $x \in G$.

Finally,

$$\begin{aligned} ld(G) &= \sup\{ld(x) : x \in G\} \\ &= \sup\{lwd(x) : x \in G\} \\ &= lwd(G). \end{aligned}$$

Thus

$$\boxed{ld(G) = lwd(G)}.$$

We have proved both required relations:

$$\boxed{d(G) = wd(G)} \quad \text{and} \quad \boxed{ld(G) = lwd(G)}.$$

This completes the proof of Theorem 3.

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