

Regular parameters and local pluriregularity

Narzillaev Nurbek

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER MIRZO ULUGBEK,
TASHKENT, UZBEKISTAN
n.narzillaev@nuu.uz

Abstract. We study weighted extremal functions on a compact subset of $\mathbb{C}^n$ when the weight is fixed and the logarithmic growth parameter varies. Concavity and two-sided estimates show that the regular parameters at a point form either the empty set, the smallest contact parameter alone, or the whole contact ray. At a regular parameter, we estimate the right derivative and characterize both persistence of regularity and equality of this derivative with the unweighted Green function. A local estimate involving dd^c shows that a global admissible extension that is strictly plurisubharmonic near the point makes weighted regularity equivalent to local pluriregularity. We also describe failure of local pluriregularity by removing a negligible subset. This gives a characterization of the points at which a compact set is non-plurithin but not locally pluriregular, and identifies its largest relatively plurifine open negligible subset. The paper presents a focused account of these results from the author's longer preprint.

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1. Introduction

Weighted extremal functions connect polynomial approximation with pluripotential theory. Their regularity depends on both the compact set and the weight. A further parameter is the allowed logarithmic growth at infinity. We keep the compact set and the weight fixed and ask how regularity changes when this parameter increases.

The notion of weighted regularity used here was studied by Sadullaev [6], Alan [1], and Narzillaev [4]. It requires equality between the regularized extremal function and the weight at a contact point. This is stronger than continuity of the extremal function alone; see [1, Example 2.3]. Alan asked whether regularity is preserved as the growth parameter increases when the weight has a global plurisubharmonic extension of the prescribed growth [1, Open Problem 3.7].

The present paper gives a short, self-contained account of selected results from [5]. We first show that regularity can be lost only at the smallest contact parameter. The proof uses concavity in the growth parameter. We then study the right derivative at a regular parameter and obtain a local criterion for persistence under strict plurisubharmonicity. Finally, we use negligible sets and the plurifine topology to distinguish local pluriregularity from non-plurithinness. The measure representation of the derivative and the detailed counterexamples in [5] are outside the scope of this short account.

The results have two complementary parts. Parameter estimates describe the possible changes in the weighted envelope. Local regularity and negligible sets explain the geometry of the points at which those changes can occur. In particular, strict plurisubharmonicity is needed only near the point under consideration, provided that a global extension with the required growth is available.

2. Preliminaries

Let $\text{PSH}(\Omega)$ denote the plurisubharmonic functions on an open set $\Omega \subset \mathbb{C}^n$. We use the Euclidean norm and write $\log^+ t = \max\{\log t, 0\}$, with $\log^+ 0 = 0$. For $\delta > 0$, put

$$\mathcal{L}_\delta = \{u \in \text{PSH}(\mathbb{C}^n) : u(z) \leq \delta \log^+ \|z\| + C_u\}.$$

The constant C_u may depend on u . The subclass $\mathcal{L}_\delta^+$ consists of the functions that also satisfy $u(z) \geq \delta \log^+ \|z\| - C$ for some constant C . Thus $\mathcal{L}_\delta = \delta \mathcal{L}_1$.

Throughout the paper, $K \subset \mathbb{C}^n$ is a non-pluripolar compact set and ψ is a bounded real-valued function on K , unless stronger assumptions are stated. The weighted extremal function and its regularization are

$$V_{\delta,K,\psi}(z) = \sup\{u(z) : u \in \mathcal{L}_\delta, u \leq \psi \text{ on } K\}, \quad F_\delta = V_{\delta,K,\psi}^*. \quad (1)$$

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Here $f^*(z) = \limsup_{\zeta \rightarrow z} f(\zeta)$ denotes upper semicontinuous regularization. For $\delta = 1$ and $\psi = 0$, (1) gives the classical Siciak–Zahariuta extremal function [7],

$$V_K := V_{1,K,0}.$$

Its regularization V_K^* is the pluricomplex Green function with pole at infinity. It is finite and belongs to $\mathcal{L}_1^+$. We have $V_K = 0$ on K , whereas V_K^* may be positive at some points of K .

The contact parameter set and the global regular parameter set are

$$\begin{aligned}\Lambda(K, \psi) &= \{\delta > 0 : V_{\delta,K,\psi}|_K = \psi\}, \\ \Lambda_{\text{reg}}(K, \psi) &= \{\delta \in \Lambda(K, \psi) : F_\delta|_K = \psi\}.\end{aligned}$$

These sets were introduced in [4]. If $\Lambda(K, \psi)$ is nonempty, then it is $(0, \infty)$ or $[\lambda, \infty)$ for some $\lambda > 0$ [4, Section 4, Proposition 1]. We also use the pointwise notation from [5],

$$\Lambda_{\text{reg}}(a; K, \psi) = \{\delta \in \Lambda(K, \psi) : F_\delta(a) = \psi(a)\}, \quad a \in K.$$

Thus $\Lambda_{\text{reg}}(K, \psi)$ is the intersection of these pointwise sets. At a contact parameter δ , we say that K is (δ, ψ) -regular at a if $F_\delta(a) = \psi(a)$. If $\psi = \Psi|_K$ for some $\Psi \in \mathcal{L}_\tau$, then $[\tau, \infty) \subset \Lambda(K, \psi)$, since Ψ itself is admissible at every such parameter.

Put $m = \inf_K \psi$ and $M = \sup_K \psi$. Scaling and comparison of the obstacles in (1) give

$$V_{\delta,K,\psi} = \delta V_{1,K,\psi/\delta}, \quad V_{\delta,K,\psi+c} = V_{\delta,K,\psi} + c,$$

and

$$\delta V_K^* + m \leq F_\delta \leq \delta V_K^* + M. \quad (2)$$

These inequalities hold on all of $\mathbb{C}^n$.

A set is pluripolar if locally it is contained in the $-\infty$ locus of a plurisubharmonic function that is not identically $-\infty$. A set is negligible if locally it is contained in a set $\{v < v^*\}$, where v is the supremum of a locally uniformly bounded above family of plurisubharmonic functions. Negligible sets and pluripolar sets coincide [2; 3, Section 4.7]. In particular, countable unions of negligible sets are negligible. An extremal envelope agrees with its regularization outside such a set.

For a bounded set E , not necessarily compact, define V_E by the same unweighted supremum. Pluripolar removal gives

$$V_{E \cup P}^* = V_E^* \quad \text{if } P \text{ is pluripolar and } E \cup P \text{ is bounded.} \quad (3)$$

We will use this identity for non-pluripolar E ; see [3, Theorem 5.2.4 and Corollary 5.2.5]. The regularization in (3) is essential.

We need one more consequence of negligibility. If U and W are envelopes of locally uniformly bounded above families of plurisubharmonic functions and have finite regularizations, then, for $s, t \geq 0$,

$$(sU + tW)^* = sU^* + tW^*. \quad (4)$$

Indeed, the two sums agree outside a pluripolar set. A plurisubharmonic function is recovered by upper regularization of its values off a pluripolar set. Applying this to $sU^* + tW^*$ proves (4). Terms with zero coefficient are omitted.

We write $B(a, r)$ for the open Euclidean ball and $\overline{B}(a, r)$ for the closed ball. The compact set K is locally pluriregular at $a \in K$ if

$$V_{K \cap \overline{B}(a,r)}^*(a) = 0 \quad \text{for every } r > 0.$$

It is non-pluripolar at every point if $K \cap B(a, r)$ is non-pluripolar for every $a \in K$ and $r > 0$. The latter assumption will be imposed in Section 5. Global pluriregularity, namely $V_K^*|_K = 0$, is a weaker condition than local pluriregularity at every point.

For an open ball B and a compact set $E \subset B$, the relative extremal function is

$$h_{E,B} = \sup\{u \in \text{PSH}(B) : -1 \leq u \leq 0, u|_E \leq -1\}. \quad (5)$$

The standard local characterization says that K is locally pluriregular at a if and only if

$$h_{K \cap \overline{B}(a,r), B(a,R)}^*(a) = -1 \quad (0 < r < R)$$

for sufficiently small concentric balls [3, Chapter 4].

For the differential operator we use the normalization

$$d^c = \frac{\partial - \bar{\partial}}{4i}, \quad dd^c = \frac{i}{2} \partial \bar{\partial}, \quad \omega_0 = dd^c \|z\|^2.$$

For a real-valued C^2 function u , plurisubharmonicity means $dd^c u \geq 0$, and strict plurisubharmonicity means $dd^c u > 0$, as $(1, 1)$ -forms. For continuous functions, these inequalities are understood in the sense of currents. In particular, $dd^c \Psi \geq c\omega_0$ near a means that $\Psi - c\|z - a\|^2$ is plurisubharmonic there.

3. Regular parameters

We begin with estimates that hold without any continuity assumption on the weight. They imply both the classification of regular parameters and the bounds for the right derivative.

Lemma 1. For $\alpha, \beta > 0$ and $0 \leq t \leq 1$,

$$F_{(1-t)\alpha+t\beta} \geq (1-t)F_\alpha + tF_\beta.$$

Moreover, if $0 < \alpha < \beta$, then on $\mathbb{C}^n$,

$$(\beta - \alpha)V_K^* \leq F_\beta - F_\alpha \leq \frac{\beta - \alpha}{\alpha}(F_\alpha - m). \quad (6)$$

Proof. If $u \in \mathcal{L}_\alpha$ and $v \in \mathcal{L}_\beta$ are at most ψ on K , then $(1-t)u + tv$ is admissible at parameter $(1-t)\alpha + t\beta$. Taking independent suprema first proves concavity for the unregularized envelopes. Formula (4) then proves the stated inequality.

For the lower bound in (6), take $u \in \mathcal{L}_\alpha$ with $u \leq \psi$ on K , and $v \in \mathcal{L}_1$ with $v \leq 0$ on K . The function $u + (\beta - \alpha)v$ is admissible at parameter β . Hence

$$V_{\beta, K, \psi} \geq V_{\alpha, K, \psi} + (\beta - \alpha)V_K.$$

Regularization and (4) give the lower bound. For the upper bound, set $q = \psi - m \geq 0$ on K . Scaling and monotonicity in the obstacle yield

$$V_{\beta, K, q} = \beta V_{1, K, q/\beta} \leq \beta V_{1, K, q/\alpha} = \frac{\beta}{\alpha} V_{\alpha, K, q}.$$

Adding back m and regularizing proves the upper bound in (6). $\square$

Since $V_K^* \geq 0$, Lemma 1 shows that $F_\delta(z)$ is nondecreasing in δ . For each fixed z , it is also finite and concave, hence continuous for $\delta > 0$. The signs in (6) are useful: increasing the growth parameter always increases the admissible envelope, and the unweighted Green function gives a lower bound for the increase.

Theorem 1. Suppose that $\Lambda = \Lambda(K, \psi)$ is nonempty. If $\Lambda = [\lambda, \infty)$ with $\lambda > 0$, then, for every $a \in K$,

$$\Lambda_{\text{reg}}(a; K, \psi) \in \{\emptyset, \{\lambda\}, [\lambda, \infty)\}.$$

The same alternatives hold for $\Lambda_{\text{reg}}(K, \psi)$. If $\Lambda = (0, \infty)$, each of these regular parameter sets is either empty or equal to $(0, \infty)$.

Proof. For $a \in K$ and $\delta \in \Lambda$, put

$$d_\delta(a) = F_\delta(a) - \psi(a).$$

Contact makes this function nonnegative. By Lemma 1, it is concave and nondecreasing in δ . Suppose $d_\alpha(a) = 0$ and that some $\eta \in \Lambda$ satisfies $\eta < \alpha$. For any $\beta > \alpha$, write

$$\alpha = (1-t)\eta + t\beta, \quad t = \frac{\alpha - \eta}{\beta - \eta} \in (0, 1).$$

Then

$$0 = d_\alpha(a) \geq (1-t)d_\eta(a) + td_\beta(a) \geq 0.$$

Thus $d_\beta(a) = 0$. For smaller parameters, monotonicity gives $0 \leq d_\delta(a) \leq d_\alpha(a) = 0$. Therefore, a zero at any parameter other than the smallest one forces vanishing on the whole contact ray.

When $\Lambda = [\lambda, \infty)$, the only remaining possibilities are no zero or a zero only at λ . When $\Lambda = (0, \infty)$, each parameter has a smaller parameter in Λ , so any zero forces vanishing everywhere. Intersecting the pointwise possibilities over $a \in K$ gives the global assertion. $\square$

Theorem 1 rules out nondegenerate bounded intervals of regular parameters among the possibilities left by the interval description in [4]. It also locates any loss of regularity in Alan's question. If regularity holds at τ and fails at some larger parameter, then necessarily

$$\Lambda(K, \psi) = [\tau, \infty), \quad \Lambda_{\text{reg}}(a; K, \psi) = \{\tau\}.$$

The constructions in [5] realize this alternative even with a smooth extending weight and a polynomially convex compact set that is non-pluripolar and non-plurithin at every point.

There are two immediate special cases. If ψ is constant, then $F_\delta = \delta V_K^* + \psi$, and regularity at a either holds for every $\delta > 0$ or fails for every $\delta > 0$, according as $V_K^*(a) = 0$ or $V_K^*(a) > 0$. More generally, (6) shows that $V_K^*(a) > 0$ permits at most one regular parameter, since $F_\delta(a)$ is then strictly increasing.

We next measure the change immediately after a regular parameter. The following proposition also separates the initial slope from the slope for large parameters.

Proposition 1. Let $\tau \in \Lambda_{\text{reg}}(a; K, \psi)$. The right derivative

$$D_\tau(a) = \lim_{h \downarrow 0} \frac{F_{\tau+h}(a) - \psi(a)}{h} \quad (7)$$

exists and is finite. The quotient is nonincreasing as a function of $h > 0$, and

$$V_K^*(a) \leq \frac{F_{\tau+h}(a) - \psi(a)}{h} \leq D_\tau(a) \leq \frac{\psi(a) - m}{\tau}. \quad (8)$$

Furthermore,

$$0 \leq \frac{F_{\tau+h}(a) - \psi(a)}{h} - V_K^*(a) \leq \frac{M - m}{h}. \quad (9)$$

Equality $D_\tau(a) = V_K^*(a)$ holds if and only if

$$F_\delta(a) = \psi(a) + (\delta - \tau)V_K^*(a) \quad \text{for every } \delta \geq \tau.$$

Regularity persists for every $\delta > \tau$ if and only if $D_\tau(a) = 0$. If $D_\tau(a) > 0$, regularity fails at every $\delta > \tau$.

Proof. Since $d_\tau(a) = 0$, concavity gives, for $0 < h_1 < h_2$,

$$d_{\tau+h_1}(a) \geq \frac{h_1}{h_2} d_{\tau+h_2}(a).$$

This proves monotonicity of the quotient. Inequality (6), applied with $\alpha = \tau$ and $\beta = \tau + h$, bounds it between $V_K^*(a)$ and $(\psi(a) - m)/\tau$. Thus the limit in (7) exists, is finite, and satisfies (8).

For (9), combine (2) and (6) to obtain

$$\begin{aligned} 0 &\leq F_{\tau+h}(a) - \psi(a) - hV_K^*(a) \\ &\leq M + \tau V_K^*(a) - \psi(a) \leq M - m. \end{aligned}$$

The last inequality uses $\psi(a) = F_\tau(a) \geq \tau V_K^*(a) + m$. Dividing by h proves (9).

If $D_\tau(a) = V_K^*(a)$, the lower and upper bounds for every quotient in (8) agree. This gives the affine formula. Conversely, that formula immediately gives equality of the derivatives. If $D_\tau(a) = 0$, (8) forces every defect to vanish. If $D_\tau(a) > 0$, the defect is positive for all sufficiently small $h > 0$, and its monotonicity makes it positive for every $h > 0$. $\square$

It follows from (9) that $V_K^*(a)$ is the limiting slope as $h \rightarrow \infty$, whereas $D_\tau(a)$ is the initial slope. Vanishing of the former does not by itself force vanishing of the latter. In fact, [5] contains a globally pluriregular example with $D_\tau(a) > 0$. A condition that does force both slopes to vanish is given in the next section.

4. Local regularity

We first recall how local pluriregularity controls a weighted envelope. Suppose ψ is bounded and upper semicontinuous on K , and $\delta \in \Lambda(K, \psi)$. If a is locally pluriregular, set $K_r = K \cap \overline{B}(a, r)$ and $M_r = \sup_{K_r} \psi$. Monotonicity in the compact set and (2) give

$$\psi(a) \leq F_\delta(a) \leq V_{\delta, K_r, \psi}^*(a) \leq M_r + \delta V_{K_r}^*(a) = M_r.$$

As $r \downarrow 0$, upper semicontinuity gives $M_r \downarrow \psi(a)$. Hence $F_\delta(a) = \psi(a)$. The same argument applies on each smaller compact K_s : contact is inherited because the envelope on K_s lies between the envelope on K and the common obstacle. This also proves local weighted regularity at a . For continuous weights, equivalence of local weighted and unweighted regularity is established in [4, Theorem 4.1].

The converse for global weighted regularity needs more information about the weight. The following estimate uses a global admissible extension but requires strict plurisubharmonicity only near a .

Theorem 2. Let $a \in K$, let $\Psi \in \mathcal{L}_\tau \cap C(\mathbb{C}^n)$, and put $\psi = \Psi|_K$. Suppose that for some $R > 0$ and $c > 0$,

$$dd^c \Psi \geq c\omega_0 \quad \text{on } B(a, R).$$

Then, for $0 < r < R$ and every $\delta \geq \tau$,

$$F_\delta(a) - \Psi(a) \geq cr^2(1 + h_{K \cap \overline{B}(a, r), B(a, R)}^*(a)). \quad (10)$$

Consequently, (δ, ψ) -regularity at a for one $\delta \geq \tau$ is equivalent to local pluriregularity at a . If it holds at one such parameter, it holds at every $\delta \geq \tau$.

Proof. Put $B = B(a, R)$ and $E = K \cap \overline{B}(a, r)$. For a competitor u in (5), define on B

$$H_u(z) = \Psi(z) - c\|z - a\|^2 + cr^2(u(z) + 1).$$

This function is plurisubharmonic, since

$$dd^c H_u = dd^c \Psi - c\omega_0 + cr^2 dd^c u \geq 0.$$

The bound $u \leq 0$ gives $H_u \leq \Psi$ whenever $\|z - a\| \geq r$. Therefore the function

$$W_u(z) = \begin{cases} \max\{\Psi(z), H_u(z)\}, & z \in B, \\ \Psi(z), & z \notin B \end{cases}$$

is globally plurisubharmonic. It equals Ψ outside $B(a, r)$, so it belongs to $\mathcal{L}_\tau$. On E , we have $u = -1$ and $H_u \leq \Psi$. On $K \setminus E$, the latter inequality follows from $\|z - a\| \geq r$. Thus $W_u = \psi$ on K , and W_u is admissible in (1) for every $\delta \geq \tau$.

Taking the supremum over u near a gives

$$V_{\delta, K, \psi}(z) \geq \Psi(z) - c\|z - a\|^2 + cr^2(h_{E, B}(z) + 1), \quad z \in B.$$

Regularize at a . Continuity of Ψ proves (10). If $F_\delta(a) = \Psi(a)$, then (10) and $h_{E, B}^* \geq -1$ imply $h_{E, B}^*(a) = -1$ for every $0 < r < R$. The relative-extremal characterization in Section 2 gives local pluriregularity. The converse and persistence follow from the preceding observation, since $[\tau, \infty) \subset \Lambda(K, \psi)$. $\square$

Under the assumptions of Theorem 2, regularity at τ therefore implies

$$D_\tau(a) = V_K^*(a) = 0.$$

Indeed, persistence makes the derivative zero, and local pluriregularity makes the Green function zero. If a C^2 plurisubharmonic extension gives regularity at τ but this regularity is lost at a larger parameter, then necessarily $(dd^c \Psi)^n(a) = 0$. Otherwise, positivity and continuity of $dd^c \Psi$ would give the strict lower bound required in Theorem 2 on a small ball.

Sadullaev's global-to-local theorem uses a globally strictly plurisubharmonic extension [6, Theorem 2.4]. Alan assumes strictness on a ball containing the compact set [1, Theorem 3.14]. Estimate (10) isolates the local condition needed at the point. It does not dispense with the global extension in $\mathcal{L}_\tau$: that extension is what allows W_u to be a competitor with the prescribed growth.

5. Plurithinness and negligible sets

In this section, assume that K is non-pluripolar at every point. We describe the points that fail to be locally pluriregular, and then distinguish those at which K is plurithin from those at which it is not.

A set E is plurithin at a if a is not an accumulation point of $E \setminus \{a\}$, or if there is a plurisubharmonic function u near a such that

$$\limsup_{E \setminus \{a\} \ni z \rightarrow a} u(z) < u(a).$$

The puncture is part of the definition. Finite unions of sets plurithin at a are plurithin there. We also use the following form of the thinness theorem: if E is plurithin at an accumulation point a , there is $u \in \mathcal{L}_1$ such that

$$u(a) > -\infty, \quad \lim_{E \setminus \{a\} \ni z \rightarrow a} u(z) = -\infty. \quad (11)$$

See [3, Proposition 4.8.2 and Corollary 4.8.4].

Negligibility and plurithinness at a point are different notions. For example, let $\overline{\mathbb{D}} = \{\zeta \in \mathbb{C} : |\zeta| \leq 1\}$. The set $\{0\} \times \overline{\mathbb{D}} \subset \mathbb{C}^2$ is negligible because it is contained in $\{\log |z_1| = -\infty\}$. It is not plurithin at $(0, 0)$: restriction to the complex line $z_1 = 0$ and the submean inequality exclude a strict jump at the center of the disk. If that restriction is identically $-\infty$, a strict jump is also impossible. Conversely, being plurithin at one point does not make a whole set negligible. For instance, $\overline{\mathbb{D}} \cup \{2\}$ is thin at its isolated point 2, but contains a nonpolar disk.

The plurifine topology is the smallest topology in which all plurisubharmonic functions are continuous. Finite intersections of sets $B \cap \{u > c\}$, where B is a Euclidean ball and $u \in \text{PSH}(B)$, form a basis. A set E is plurithin at a if and only if some plurifine neighborhood of a misses $E \setminus \{a\}$ [3, Section 4.8]. We use this equivalence below; no general theory of plurifinely plurisubharmonic functions is needed.

Following [1], decompose K into

$$\begin{aligned} R &= \{a \in K : K \text{ is locally pluriregular at } a\}, \\ T &= \{a \in K : K \text{ is plurithin at } a\}, \\ S &= K \setminus (R \cup T). \end{aligned}$$

Local pluriregularity implies non-plurithinness, so these sets are disjoint. The set S is the part not detected by either local regularity or plurithinness. Alan asked for its characterization in [1, Open Problem 3.17].

Lemma 2. For $a \in K$, the following conditions are equivalent:

- (i) $a \notin R$;
- (ii) there is a negligible subset $N \subset K$ containing a such that $K \setminus N$ is plurithin at a ;
- (iii) there is a plurifine open neighborhood U of a such that $K \cap U$ is negligible.

Proof. Suppose $a \notin R$. Choose $r > 0$ with

$$g = V_{K \cap \overline{B}(a, r)}^*, \quad c = g(a) > 0.$$

Local non-pluripolarity makes g finite and plurisubharmonic. Set

$$U = B(a, r) \cap \{g > c/2\}, \quad N = K \cap U.$$

The set U is a plurifine open neighborhood of a . Since the unregularized extremal function is zero on $K \cap \overline{B}(a, r)$,

$$N \subset \{V_{K \cap \overline{B}(a, r)}^* < V_{K \cap \overline{B}(a, r)}^*\}.$$

Thus N is negligible. Near a , the function g is at most $c/2$ on $K \setminus N$, whereas $g(a) = c$. This proves (ii) and (iii).

Assume (ii) and set $E = K \setminus N$. The point a is an accumulation point of E . Otherwise a small neighborhood of a would meet K only in the pluripolar set N , contrary to local non-pluripolarity. Apply (11) and choose $\rho > 0$ small enough that

$$u \leq u(a) - 1 \quad \text{on } E \cap \overline{B}(a, \rho).$$

Then $v = u - u(a) + 1 \in \mathcal{L}_1$ is nonpositive on $K \cap \overline{B}(a, \rho) \setminus N$, while $v(a) = 1$. By (3),

$$V_{K \cap \overline{B}(a, \rho)}^*(a) = V_{K \cap \overline{B}(a, \rho) \setminus N}^*(a) \geq 1.$$

Hence $a \notin R$. Finally, (iii) implies (ii) with $N = K \cap U$, because the complement of a plurifine neighborhood of a is plurithin at a . $\square$

Theorem 3. A point $a \in K$ belongs to S if and only if there is a negligible subset $N \subset K$ such that $a \in N$, N is non-plurithin at a , and $K \setminus N$ is plurithin at a .

Proof. If $a \in S$, Lemma 2 gives a negligible set N containing a for which $K \setminus N$ is plurithin at a . If N were plurithin there, their union K would also be plurithin, contrary to $a \notin T$. Conversely, if such an N exists, its non-plurithinness makes K non-plurithin at a . Lemma 2 gives $a \notin R$. Thus $a \in S$. $\square$

The theorem describes S without a local extremal function in its statement. Lemma 2 also gives a criterion for regular points: $a \in R$ if and only if $K \setminus N$ is non-plurithin at a for every negligible subset $N \subset K$. Indeed, replacing N by $N \cup \{a\}$ changes neither negligibility nor thinness at a .

There is a canonical choice of negligible subset. Define

$$N_K = \bigcup \{K \cap U : U \text{ is plurifine open and } K \cap U \text{ is negligible}\}. \quad (12)$$

For a set A , write $A'_{\mathcal{F}}$ for its plurifine derived set: $a \in A'_{\mathcal{F}}$ if every plurifine neighborhood of a meets $A \setminus \{a\}$.

Proposition 2. The set N_K is the largest relatively plurifine open negligible subset of K . Moreover,

$$\begin{aligned} R &= K \setminus N_K, \\ T &= \{a \in K : a \text{ is isolated in the relative plurifine topology}\}, \\ S &= N_K \cap (N_K)'_{\mathcal{F}}. \end{aligned}$$

Proof. Lemma 2 gives $N_K = K \setminus R$. We must still prove negligibility, since an arbitrary union of negligible sets need not be negligible. Let B_j run through the balls with rational centers and positive rational radii for which $K \cap B_j \neq \emptyset$. Put

$$A_j = K \cap \overline{B}_j, \quad N_j = K \cap B_j \cap \{V_{A_j}^* > 0\}.$$

Every A_j is non-pluripolar, and each N_j is negligible. We claim that $K \setminus R = \bigcup_{j \geq 1} N_j$.

If $a \notin R$, choose r with $V_{K \cap \overline{B}(a,r)}^*(a) > 0$ and a rational ball B_j such that $a \in B_j$ and $\overline{B}_j \subset B(a, r)$. Monotonicity in the set gives $V_{A_j}^*(a) > 0$, hence $a \in N_j$. Conversely, if $a \in N_j$, choose $\rho > 0$ with $\overline{B}(a, \rho) \subset B_j$. Then

$$V_{K \cap \overline{B}(a,\rho)}^*(a) \geq V_{A_j}^*(a) > 0,$$

so $a \notin R$. This proves the claim and shows that N_K is a countable union of negligible sets.

Relative plurifine openness and maximality follow from (12). The thinness criterion gives the description of T . Each point of T belongs to N_K , since its singleton is relatively plurifine open in K and is pluripolar. At a point of N_K , isolation in N_K is equivalent to isolation in K , because N_K is relatively plurifine open. Removing these isolated points from $N_K = K \setminus R$ gives $S = N_K \cap (N_K)'_{\mathcal{F}}$. $\square$

In one complex variable, polar sets are thin at each of their points. Theorem 3 therefore gives $S = \emptyset$. In several complex variables, the negligible disk considered above shows why this argument no longer applies. The product construction in [5] realizes an entire nonempty exceptional polydisc as S while the compact set remains non-pluripolar at every point. Thus the distinction captured by Theorem 3 is specific to the local geometry in several complex variables.

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