

LOTKA-VOLTERRA DYNAMICS ON S^3 WITH A PARTIALLY ORIENTED INTERACTION GRAPH

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ABSTRACT. We study a five-parameter Lotka–Volterra quadratic stochastic operator on the three-dimensional simplex S^3 in the degenerate case $ce = ad$. The corresponding skew-symmetric matrix has exactly one zero interaction coefficient, and its partially oriented interaction graph contains two strong three-vertex subtournaments. On the fixed labelled vertex set $\{1, 2, 3, 4\}$, the isomorphism class of this graph contains exactly twelve labelled realizations. The degeneracy reduces the four coordinate growth forms to two independent linear forms. We determine the complete fixed-point set, prove that the kernel intersects the simplex in a segment of fixed points, construct the six non-empty signature polytopes and their adjacency cycle, and describe the admissible forward and backward crossing directions. We also prove convergence of global negative trajectories contained in the relative interior to unique points of the fixed segment and compute the Jacobian spectrum along that segment, including explicit eigenvalues and eigenvectors at its endpoints.

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Key words: Lotka–Volterra operator, simplex, partially oriented graph, strong tournament, fixed point, signature, convex polytope, trajectory, Jacobian spectrum.

Introduction

Quadratic stochastic operators and their Lotka–Volterra subclasses provide discrete models in population genetics, ecology, epidemiology, and related nonlinear dynamical systems. Their connection with Lyapunov functions, tournaments, and positive/negative trajectories is classical in the theory of Volterra operators [1, 2, 10, 11]. A useful structural feature of a Lotka–Volterra operator is its connection with a skew-symmetric interaction matrix and, in the non-degenerate case, with a tournament. When some interaction coefficients vanish, the associated object becomes a partially oriented graph, as in recent studies of degenerate Lotka–Volterra systems [3]; this is precisely the setting considered here.

For broader background on quadratic stochastic operators, trajectory and fixed-point problems, and their population-dynamical interpretation, see [4–8]. A recent graph-based study of discrete Lotka–Volterra operators associated with weighted oriented bigraphs is [9].

The operator studied in this article has one vanishing interaction, namely $a_{14} = 0$, while the remaining five interaction strengths are positive. Its associated partially oriented graph contains the two directed 3-cycles on the vertex sets $\{1, 2, 3\}$ and $\{2, 3, 4\}$. Thus, it has exactly two strong three-vertex subtournaments. Up to graph isomorphism this is one partially oriented graph type; however, on a fixed set of four labels its isomorphism class contains exactly twelve labelled graphs. Our principal degeneracy condition is $ce = ad$. It is equivalent to $\det A = 0$ for the skew-symmetric interaction matrix. Under this condition, the four growth forms are determined by only two linear forms. This reduction produces a common fixed segment, six strict signature domains, six closed signature polytopes meeting face-to-face, and a cyclic admissible order of transitions between neighbouring regions.

The paper is organized as follows. We first recall the basic notions used throughout, including fixed points, positive and negative trajectories, ω - and α -limit sets, signatures, convex hulls, and adjacent polytopes. After establishing the operator and its degeneracy, we determine the kernel segment and the entire fixed-point set. The six signature regions and their convex-hull descriptions are derived next, followed by their adjacency structure and trajectory directions. Finally, we analyze the Jacobian spectrum.

Preliminaries

Let $S^3 = \{x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 : x_i \geq 0, x_1 + x_2 + x_3 + x_4 = 1\}$ be the standard three-dimensional simplex, with vertices

$$e_1 = (1, 0, 0, 0), \quad e_2 = (0, 1, 0, 0), \quad e_3 = (0, 0, 1, 0), \quad e_4 = (0, 0, 0, 1).$$

For $I \subset \{1, 2, 3, 4\}$, we write $\Gamma_I = \{x \in S^3 : x_j = 0 \text{ for every } j \notin I\}$ for the corresponding coordinate face.

Definition 1. Let $V : S^3 \rightarrow S^3$. A point $x \in S^3$ is called a fixed point of V if $V(x) = x$. The set of all fixed points is denoted by $\text{Fix}(V) = \{x \in S^3 : V(x) = x\}$.

Definition 2. Let $x^{(0)} \in S^3$. The positive trajectory of $x^{(0)}$ is $O^+(x^{(0)}) = \{x^{(n)} : n \geq 0\}$, $x^{(n)} = V^n(x^{(0)})$. A negative trajectory of $x^{(0)}$ is a sequence $O^-(x^{(0)}) = \{x^{(-n)} : n \geq 0\}$ with $x^{(0)}$ as its terminal point and $V(x^{(-n-1)}) = x^{(-n)}$ for $n \geq 0$. If V is invertible, then the negative trajectory is unique and $x^{(-n)} = V^{-n}(x^{(0)})$. Without invertibility, a negative trajectory can be non-unique or may fail to exist globally.

Definition 3. The ω -limit set of the positive trajectory is

$$\omega(x^{(0)}) = \{y \in S^3 : \exists n_k \rightarrow \infty \text{ with } x^{(n_k)} \rightarrow y\}.$$

Whenever a negative trajectory is fixed, its α -limit set is

$$\alpha(x^{(0)}) = \{y \in S^3 : \exists n_k \rightarrow \infty \text{ with } x^{(-n_k)} \rightarrow y\}.$$

Thus $O^+(x^{(0)})$ and $O^-(x^{(0)})$ denote the trajectories themselves, whereas $\omega(x^{(0)})$ and $\alpha(x^{(0)})$ denote their sets of accumulation points.

Definition 4. Let $B \subset \mathbb{R}^n$. The convex hull of B , denoted by $\text{co}(B)$ or $\text{conv}(B)$, is the smallest convex set containing B . Equivalently,

$$\text{co}(B) = \left\{ \sum_{i=1}^m \lambda_i x_i : x_i \in B, \lambda_i \geq 0, \sum_{i=1}^m \lambda_i = 1, m \in \mathbb{N} \right\}.$$

Definition 5. A collection of signs $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4)$, where $\sigma_i \in \{+, -\}$, is called a signature. For a given signature σ , we associate the set of all $x \in S^3$ satisfying

$$x'_k > x_k \text{ if } \sigma_k = +, \quad x'_k < x_k \text{ if } \sigma_k = -, \quad k = 1, 2, 3, 4.$$

The set of points corresponding to a given signature is relatively open and convex in S^3 (possibly empty). We denote by F_σ its closure in S^3 . Thus, F_σ is a convex polyhedron determined by a system of linear inequalities corresponding to the signature σ .

Definition 6. (Adjacent polytopes) Let P and Q be full-dimensional convex polytopes in the same d -dimensional affine space. They are called adjacent if $P \cap Q$ is a common facet of both polytopes, that is, $\dim(P \cap Q) = d - 1$. In the present three-dimensional simplex, two full-dimensional signature polytopes are adjacent precisely when they share a two-dimensional face.

For a skew-symmetric matrix $A = (a_{ij})$, we use the following convention. If $a_{ij} > 0$, the edge between i and j is oriented $j \rightarrow i$; if $a_{ij} < 0$, it is oriented $i \rightarrow j$; and if $a_{ij} = 0$, the edge is left unoriented. A tournament is called strong if every vertex can be reached from every other vertex by a directed path; see [12, 13] for standard graph-theoretic background.

Definition 7. Let d be the Euclidean distance on S^3 . An invariant set $Y \subset S^3$ is called attracting if there exists a neighbourhood U of Y such that $\text{dist}(V^n(x), Y) \rightarrow 0$ ($n \rightarrow \infty$) for every $x \in U$. Its basin of attraction is $B(Y) = \{x \in S^3 : \text{dist}(V^n(x), Y) \rightarrow 0\}$. A fixed point p is attracting when $Y = \{p\}$. A fixed point is a local repeller if every sufficiently close point different from it eventually leaves a prescribed neighbourhood of it.

Main part

Consider the Lotka–Volterra quadratic stochastic operator $V : S^3 \rightarrow S^3$ defined by

$$\begin{cases} x'_1 = x_1(1 + ax_2 - cx_3), \\ x'_2 = x_2(1 - ax_1 + bx_3 - ex_4), \\ x'_3 = x_3(1 + cx_1 - bx_2 + dx_4), \\ x'_4 = x_4(1 + ex_2 - dx_3), \end{cases} \quad (1)$$

or, equivalently, $V_i(x) = x_i(1 + (Ax)_i)$, $i = 1, 2, 3, 4$, where

$$A = \begin{pmatrix} 0 & a & -c & 0 \\ -a & 0 & b & -e \\ c & -b & 0 & d \\ 0 & e & -d & 0 \end{pmatrix}.$$

Throughout the article we assume

$$0 < a, b, c, d, e \leq 1, \quad ce = ad. \quad (2)$$

Define the coordinate growth forms

$$\begin{aligned} h_1(x) &= (Ax)_1 = ax_2 - cx_3, \\ h_2(x) &= (Ax)_2 = -ax_1 + bx_3 - ex_4, \\ h_3(x) &= (Ax)_3 = cx_1 - bx_2 + dx_4, \\ h_4(x) &= (Ax)_4 = ex_2 - dx_3. \end{aligned}$$

Then

$$x'_i - x_i = x_i h_i(x), \quad i = 1, 2, 3, 4. \quad (3)$$

Thus, in the relative interior of S^3 , the sign of $x'_i - x_i$ is exactly the sign of $h_i(x)$. In particular, the signature conditions in Definition 5 are equivalent to $h_k(x) > 0$ when $\sigma_k = +$ and $h_k(x) < 0$ when $\sigma_k = -$. For the above matrix A , the graph has edges $2 \rightarrow 1$, $1 \rightarrow 3$, $3 \rightarrow 2$, $2 \rightarrow 4$, $4 \rightarrow 3$, and the single unoriented edge $1 - 4$. Consequently, the induced subtournaments on $\{1, 2, 3\}$ and $\{2, 3, 4\}$ are directed 3-cycles and hence are strong.

The unique unoriented edge is $\{1, 4\}$. The only non-trivial relabelling preserving all five directed edges and this unoriented edge is the transposition $(1\ 4)$. Hence $|\text{Aut}(G)| = 2$, and the orbit-stabilizer theorem gives $|S_4|/|\text{Aut}(G)| = 12$ labelled realizations in this isomorphism class.

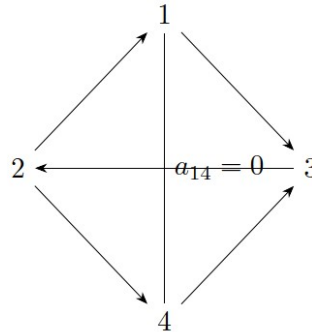


Figure 1. The partially oriented interaction graph.

For the skew-symmetric matrix A , $\det A = (ad - ce)^2$. Indeed, the Pfaffian of A is

$$\text{pf}(A) = a_{12}a_{34} - a_{13}a_{24} + a_{14}a_{23} = ad - (-c)(-e) + 0 = ad - ce,$$

and $\det A = \text{pf}(A)^2$. Thus $\det A = 0$ if and only if $ce = ad$. Put $u = h_1(x)$ and $v = h_2(x)$. Under $ce = ad$, we have $d = ce/a$. Hence

$$h_4 = \frac{e}{a}h_1, \quad h_3 = -\frac{b}{a}h_1 - \frac{c}{a}h_2 = -\frac{bu + cv}{a}. \quad (4)$$

Consequently, the full signature (h_1, h_2, h_3, h_4) is determined by the pair (u, v) . Set $S = a + b + c$, $T = b + d + e$, and define $C_1 = (\frac{b}{S}, \frac{c}{S}, \frac{a}{S}, 0)$, $C_2 = (0, \frac{d}{T}, \frac{e}{T}, \frac{b}{T})$. These points are obtained by solving $Ax = 0$ on the simplex. By (4), $h_1 = h_2 = 0$ implies $h_3 = h_4 = 0$; hence it suffices to solve $ax_2 = cx_3$, $-ax_1 + bx_3 - ex_4 = 0$. Thus, $x_2 = \frac{c}{a}x_3$, $x_1 = \frac{b}{a}x_3 - \frac{e}{a}x_4$. The endpoint $x_4 = 0$ gives C_1 , and the endpoint $x_1 = 0$ gives C_2 .

Theorem 1. (Kernel segment) Under condition (2), we have $\ker A \cap S^3 = [C_1, C_2]$.

Proof. Recall that $\ker A \cap S^3 = \{x \in S^3 : Ax = 0\}$. First, we prove that $[C_1, C_2] \subseteq \ker A \cap S^3$. Let $x \in [C_1, C_2]$. Then $x = \lambda_1 C_1 + \lambda_2 C_2$, $\lambda_1, \lambda_2 \geq 0$, $\lambda_1 + \lambda_2 = 1$. Since $\lambda_2 = 1 - \lambda_1$, we have

$$x = \left(\frac{\lambda_1 b}{S}, \frac{\lambda_1 c}{S} + (1 - \lambda_1) \frac{d}{T}, \frac{\lambda_1 a}{S} + (1 - \lambda_1) \frac{e}{T}, (1 - \lambda_1) \frac{b}{T} \right).$$

For the second coordinate,

$$\begin{aligned} \lambda_1 cT + (1 - \lambda_1)dS &= \lambda_1(cT - dS) + dS = \lambda_1(c(b + d + e) - d(a + b + c)) + dS \\ &= \lambda_1(bc + ce - ad - bd) + dS = \lambda_1b(c - d) + dS, \end{aligned}$$

because $ce = ad$. Similarly,

$$\begin{aligned} \lambda_1 aT + (1 - \lambda_1)eS &= \lambda_1(aT - eS) + eS = \lambda_1(a(b + d + e) - e(a + b + c)) + eS \\ &= \lambda_1(ab + ad - be - ce) + eS = \lambda_1b(a - e) + eS. \end{aligned}$$

Therefore,

$$x = \left(\frac{\lambda_1 b}{S}, \frac{\lambda_1 b(c-d) + dS}{ST}, \frac{\lambda_1 b(a-e) + eS}{ST}, \frac{(1-\lambda_1)b}{T} \right).$$

By (4), it is enough to verify that $h_1(x) = h_2(x) = 0$. Indeed,

$$\begin{aligned} h_1(x) &= a \frac{\lambda_1 b(c-d) + dS}{ST} - c \frac{\lambda_1 b(a-e) + eS}{ST} = \frac{a\lambda_1 bc - a\lambda_1 bd + adS - c\lambda_1 ab + c\lambda_1 be - ceS}{ST} \\ &= \frac{\lambda_1 b(ce - ad) + (ad - ce)S}{ST} = 0. \end{aligned}$$

Also,

$$h_2(x) = -a \frac{\lambda_1 b}{S} + b \frac{\lambda_1 b(a-e) + eS}{ST} - e \frac{(1-\lambda_1)b}{T} = \frac{\lambda_1 b(-aT + b(a-e) + eS)}{ST} = \frac{\lambda_1 b(ce - ad)}{ST} = 0.$$

Hence, by (4), $h_1(x) = h_2(x) = h_3(x) = h_4(x) = 0$, and therefore $Ax = \mathbf{0}$. Thus, $[C_1, C_2] \subseteq \ker A \cap S^3$. Conversely, let $x = (x_1, x_2, x_3, x_4) \in \ker A \cap S^3$. Then $Ax = \mathbf{0}$, and in particular

$$h_1(x) = ax_2 - cx_3 = 0, \quad h_2(x) = -ax_1 + bx_3 - ex_4 = 0.$$

Hence $x_2 = \frac{c}{a}x_3$, $x_1 = \frac{b}{a}x_3 - \frac{e}{a}x_4$. Together with $x_1 + x_2 + x_3 + x_4 = 1$, these equations determine an affine line. Its intersection with the simplex S^3 is a closed line segment. Since $a, c > 0$, the equality $ax_2 = cx_3$ shows that $x_2 = 0$ if and only if $x_3 = 0$. If $x_2 = x_3 = 0$, then $h_2(x) = 0$ gives $ax_1 + ex_4 = 0$, which is impossible for $x \in S^3$. Therefore the endpoints of the segment occur when $x_4 = 0$ and when $x_1 = 0$. If $x_4 = 0$, then $ax_2 = cx_3$, $ax_1 = bx_3$, $x_1 + x_2 + x_3 = 1$, and hence $x = (\frac{b}{S}, \frac{c}{S}, \frac{a}{S}, 0) = C_1$. If $x_1 = 0$, then $ax_2 = cx_3$, $bx_3 = ex_4$, $x_2 + x_3 + x_4 = 1$. Using $ce = ad$, we obtain $x = (0, \frac{d}{T}, \frac{e}{T}, \frac{b}{T}) = C_2$. Thus the intersection has precisely the endpoints C_1 and C_2 . Consequently, $\ker A \cap S^3 = [C_1, C_2]$. Equivalently, the fixed segment can be parametrized by

$$X(t) = (1-t)C_1 + tC_2 = \left(\frac{(1-t)b}{S}, \frac{(1-t)c}{S} + \frac{td}{T}, \frac{(1-t)a}{S} + \frac{te}{T}, \frac{tb}{T} \right), \quad 0 \leq t \leq 1. \quad (5)$$

The proof is complete.

Theorem 2. The fixed-point set is $\text{Fix}(V) = [e_1, e_4] \cup [C_1, C_2] \cup \{e_2, e_3\}$.

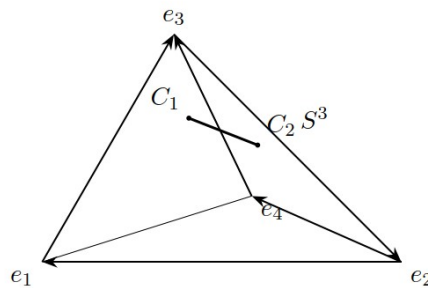


Figure 2. Schematic geometry of the two non-isolated fixed segments: the coordinate edge $[e_1, e_4]$ and the kernel segment $[C_1, C_2]$.

Proof. Every vertex is fixed. Since $a_{14} = 0$, every point of the edge $[e_1, e_4]$ is fixed. Theorem 1 shows that every point of $[C_1, C_2]$ is fixed. Conversely, let $x \in \text{Fix}(V)$. Then $x_i h_i(x) = 0$ for $i = 1, 2, 3, 4$. We distinguish the possibilities according to the support of x . If $|\text{supp } x| = 1$, then x is a vertex. If $|\text{supp } x| = 2$, an interior point of a coordinate edge can be fixed only when the interaction coefficient corresponding to that pair is zero. The only zero coefficient is a_{14} ; hence the only non-trivial fixed edge is $[e_1, e_4]$. Suppose $|\text{supp } x| = 3$. On relint Γ_{123} the fixed-point equations reduce to $ax_2 = cx_3$, $ax_1 = bx_3$, $x_1 + x_2 + x_3 = 1$, which have the unique solution C_1 . On relint Γ_{234} they reduce to $ax_2 = cx_3$, $bx_3 = ex_4$, $x_2 + x_3 + x_4 = 1$, and, using $ce = ad$, the unique solution is C_2 . The remaining two faces contain no fixed point in their relative interiors: on Γ_{124} one has $h_1 = ax_2 > 0$, while on Γ_{134} one has $h_1 = -cx_3 < 0$. Finally, if all four coordinates are positive, then $h_i(x) = 0$ for every i , so $x \in \ker A \cap S^3 = [C_1, C_2]$ by Theorem 1. This exhausts all possible supports and proves the assertion. The proof is completed.

Introduce the boundary points.

$$\begin{aligned} M_1 &= \left(0, \frac{c}{a+c}, \frac{a}{a+c}, 0 \right), & M_2 &= \left(\frac{b}{a+b}, 0, \frac{a}{a+b}, 0 \right), & M_3 &= \left(\frac{b}{b+c}, \frac{c}{b+c}, 0, 0 \right), \\ N_1 &= \left(0, 0, \frac{e}{b+e}, \frac{b}{b+e} \right), & N_2 &= \left(0, \frac{d}{b+d}, 0, \frac{b}{b+d} \right), & N_3 &= \left(0, \frac{d}{d+e}, \frac{e}{d+e}, 0 \right). \end{aligned} \quad (6)$$

Because $ce = ad$, one has $N_3 = M_1$. Using $u = h_1$ and $v = h_2$, we have $h_3 = -\frac{bu+cv}{a}$, $h_4 = \frac{e}{a}u$. Therefore the three lines

$$u = 0, \quad v = 0, \quad bu + cv = 0 \quad (7)$$

split the (u, v) -plane into exactly six open sectors.

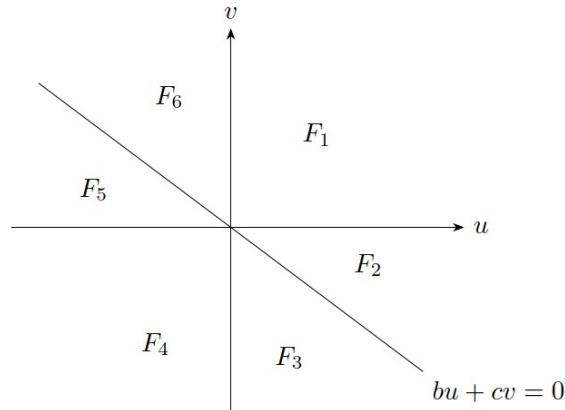


Figure 3. The six signature sectors in the (u, v) -plane.

Lemma 1. (Six strict signatures) Under $ce = ad$, there are exactly six non-empty strict signature domains. We denote their closures by $F_1, \dots, F_6$; the strict signatures on relint F_i are given in the following table.

Region	Conditions in the (u, v) -plane	Signature (h_1, h_2, h_3, h_4)
F_1	$u > 0, v > 0$	$(+, +, -, +)$
F_2	$u > 0, v < 0, bu + cv > 0$	$(+, -, -, +)$
F_3	$u > 0, v < 0, bu + cv < 0$	$(+, -, +, +)$
F_4	$u < 0, v < 0$	$(-, -, +, -)$
F_5	$u < 0, v > 0, bu + cv < 0$	$(-, +, +, -)$
F_6	$u < 0, v > 0, bu + cv > 0$	$(-, +, -, -)$

Proof. Since

$$h_4 = \frac{e}{a}h_1, \quad \frac{e}{a} > 0,$$

the signs of h_1 and h_4 always coincide. Thus, instead of the $2^4 = 16$ sign combinations that would be possible for four independent signs, there are at most $2^3 = 8$ possible sign combinations.

The two signatures $(+, +, +, +)$ and $(-, -, -, -)$ cannot occur. Indeed, in [9] given V maps S^3 into itself, $\sum_{i=1}^4 x'_i = \sum_{i=1}^4 x_i = 1$, and therefore $\sum_{i=1}^4 (x'_i - x_i) = 0$. Hence, in relint S^3 , the four differences $x'_i - x_i = x_i h_i(x)$, $i = 1, 2, 3, 4$, cannot all be strictly positive or all be strictly negative. Consequently, only six strict signatures are admissible. More explicitly, since $h_3 = -\frac{bu+cv}{a}$, the sign of h_3 is opposite to that of $bu + cv$. Hence the three lines in (7) $u = 0$, $v = 0$, $bu + cv = 0$ divide the (u, v) -plane into exactly six open sectors, which correspond precisely to the six signatures listed above.

For a strict sign vector $\sigma = (\sigma_1, \dots, \sigma_4) \in \{+, -\}^4$, its closed signature polytope is obtained by replacing each strict sign condition by the corresponding weak inequality. Thus, for example, $+$ means $h_i \geq 0$ and $-$ means $h_i \leq 0$.

Proposition 1. The six closed signature polytopes are

$$\begin{aligned} F_1 &= \text{co}\{e_2, M_1, C_1, C_2\}, & F_2 &= \text{co}\{e_2, N_2, M_3, C_1, C_2\}, \\ F_3 &= \text{co}\{e_1, e_4, N_2, M_3, C_1, C_2\}, & F_4 &= \text{co}\{e_1, e_4, N_1, M_2, C_1, C_2\}, \\ F_5 &= \text{co}\{e_3, N_1, M_2, C_1, C_2\}, & F_6 &= \text{co}\{e_3, M_1, C_1, C_2\}. \end{aligned}$$

Proof. Put $w = bu + cv$. Each F_i is the intersection of the three-dimensional simplex S^3 with three closed half-spaces bounded by the separating planes $u = 0, v = 0, w = 0$. Hence each F_i is a bounded convex polyhedron, and therefore a convex polytope.

We first determine all possible vertices of these intersections. We work in the affine hull

$x_1 + x_2 + x_3 + x_4 = 1$, which has dimension 3. At least three independent active boundary conditions must be active. These conditions are chosen from the coordinate planes $x_j = 0$ and the three separating planes above.

If three coordinate planes are active, we obtain one of the simplex vertices e_1, e_2, e_3, e_4 . If two coordinate planes and one separating plane are active, the point lies on a coordinate edge. The non-trivial interior edge intersections are exactly

edge	separating equation	intersection point
$[e_1, e_2]$	$w = 0$	M_3
$[e_1, e_3]$	$v = 0$	M_2
$[e_2, e_3]$	$u = 0$	M_1
$[e_2, e_4]$	$w = 0$	N_2
$[e_3, e_4]$	$v = 0$	N_1

These formulas follow immediately by solving the corresponding single linear equation on the indicated edge. The remaining edge $[e_1, e_4]$ is contained entirely in the plane $u = 0$, so this intersection produces no additional vertex. Also, the equation $h_4 = 0$ is equivalent to $u = 0$ by (4), and its intersection with $[e_2, e_3]$ gives $N_3 = M_1$ because $ce = ad$.

It remains to examine intersections involving two or more separating planes. Since $w = bu + cv$ with $b, c > 0$, any two equalities among $u = 0$, $v = 0$, $w = 0$ force $u = v = 0$. By Theorem 1, their intersection with S^3 is precisely the segment $[C_1, C_2]$. Intersecting this segment with a coordinate plane produces only its endpoints: $x_4 = 0$ gives C_1 and $x_1 = 0$ gives C_2 , while the planes $x_2 = 0$ and $x_3 = 0$ do not meet the kernel segment. If all three separating equations are imposed, one again obtains the whole segment $[C_1, C_2]$, not a new vertex.

This also shows that no extra vertices can arise from intersections of a separating plane with the relative interior of a two-dimensional face of the simplex. Such an intersection is a line segment; its endpoints occur on coordinate edges and have already been listed above. Thus the complete set of possible vertices is

$$\{e_1, e_2, e_3, e_4, M_1, M_2, M_3, N_1, N_2, C_1, C_2\}.$$

Finally, substituting these points into the weak sign conditions associated with the six sectors in Lemma 1 gives exactly the vertex sets displayed in the statement of the proposition. Since every bounded polyhedron is the convex hull of its vertices, the asserted convex-hull representations follow. The proof is complete.

Their union is S^3 , their strict interiors are pairwise disjoint, and different closed regions meet along common faces. In this precise sense they form a face-to-face polyhedral decomposition of the simplex. Given an initial point $x^{(0)}$, one computes $u_0 = ax_2^{(0)} - cx_3^{(0)}$, $v_0 = -ax_1^{(0)} + bx_3^{(0)} - ex_4^{(0)}$, and compares (u_0, v_0) with the three lines in (7) to determine its signature region. Each feasible point listed above is the unique solution of three independent active boundary equations; hence it is a vertex of the corresponding polytope.

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Proposition 2. All six closed polytopes contain the common edge $[C_1, C_2]$. Their two-dimensional common faces are

$$\begin{aligned} F_1 \cap F_2 &= \text{co}\{e_2, C_1, C_2\}, & F_2 \cap F_3 &= \text{co}\{M_3, N_2, C_1, C_2\}, \\ F_3 \cap F_4 &= \text{co}\{e_1, e_4, C_1, C_2\}, & F_4 \cap F_5 &= \text{co}\{M_2, N_1, C_1, C_2\}, \\ F_5 \cap F_6 &= \text{co}\{e_3, C_1, C_2\}, & F_6 \cap F_1 &= \text{co}\{M_1, C_1, C_2\}. \end{aligned} \quad (8)$$

Hence the adjacency graph is the six-cycle

$$F_1 - F_2 - F_3 - F_4 - F_5 - F_6 - F_1. \quad (9)$$

Proof. By Proposition 1, every F_i is a full-dimensional convex polytope in the affine hull of S^3 . Moreover, every point of $[C_1, C_2]$ satisfies $u = v = 0$, and hence also $bu + cv = 0$; consequently, the segment $[C_1, C_2]$ is contained in all six polytopes. We verify explicitly the dimension of one representative intersection. The regions F_1 and F_2 differ only in the sign of v . Hence $F_1 \cap F_2 = F_1 \cap \{v = 0\} = F_2 \cap \{v = 0\}$. By Proposition 1, the vertices of this common face are exactly e_2, C_1, C_2 , and therefore $F_1 \cap F_2 = \text{co}\{e_2, C_1, C_2\}$. The points C_1 and C_2 are distinct and span the kernel segment, whereas $e_2 \notin \ker A$ because $Ae_2 = (a, 0, -b, e)^T \neq 0$. Thus e_2, C_1, C_2 are affinely independent, so $\dim(F_1 \cap F_2) = 2$. Hence $F_1 \cap F_2$ is a common facet of F_1 and F_2 .

The remaining consecutive pairs are treated in exactly the same way. More precisely, the separating plane is $w = 0$ for $F_2 \cap F_3$, $u = 0$ for $F_3 \cap F_4$, $v = 0$ for $F_4 \cap F_5$, $w = 0$ for $F_5 \cap F_6$, and $u = 0$ for $F_6 \cap F_1$, where $w = bu + cv$. Each

set displayed in (8) lies in the corresponding separating plane, contains the line segment $[C_1, C_2]$, and contains at least one listed vertex outside that segment. Therefore each of these intersections has dimension exactly 2 and is a common facet.

If two regions are not consecutive in the cyclic order, their weak sign conditions force at least two of the separating equalities $u = 0$, $v = 0$, and $w = 0$. Any two of these imply $u = v = 0$, so the intersection is contained in $[C_1, C_2]$ and has dimension at most 1. Such regions are therefore not adjacent. Hence the adjacency graph is exactly the six-cycle (9).

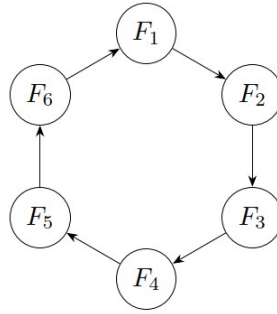


Figure 4. The underlying adjacency graph is a six-cycle. The arrows indicate the admissible first-crossing order for positive trajectories proved in Theorem 3.

Let $x^{(n+1)} = V(x^{(n)})$. From (3), we have $x_i^{(n+1)} - x_i^{(n)} = x_i^{(n)} h_i(x^{(n)})$, $i = 1, 2, 3, 4$. Therefore the signature determines which coordinates increase and which decrease at each iterate in a strict region.

Theorem 3. Suppose an orbit point lies in the relative interior of one of the six regions and its next iterate leaves that region by crossing a separating wall. Then the first possible neighbouring region follows the cyclic order (9).

Current region	Signs preserved after one step	Sign that may change	First possible next region
F_1	$h_1 > 0, h_3 < 0, h_4 > 0$	h_2	F_2
F_2	$h_1 > 0, h_2 < 0, h_4 > 0$	h_3	F_3
F_3	$h_2 < 0, h_3 > 0$	h_1, h_4	F_4
F_4	$h_1 < 0, h_3 > 0, h_4 < 0$	h_2	F_5
F_5	$h_1 < 0, h_2 > 0, h_4 < 0$	h_3	F_6
F_6	$h_2 > 0, h_3 < 0$	h_1, h_4	F_1

Proof. Put $y = V(x)$. From the definitions of the growth forms,

$$\begin{aligned} h_1(y) &= h_1(x) + ax_2h_2(x) - cx_3h_3(x), \\ h_2(y) &= h_2(x) - ax_1h_1(x) + bx_3h_3(x) - ex_4h_4(x), \\ h_3(y) &= h_3(x) + cx_1h_1(x) - bx_2h_2(x) + dx_4h_4(x), \end{aligned}$$

and, by (4), $h_4(y) = \frac{e}{a}h_1(y)$. We shall also use the skew-symmetry of A . Since $h(x) = Ax$ and $A^T = -A$,

$$x_1h_1(x) + x_2h_2(x) + x_3h_3(x) + x_4h_4(x) = x^T Ax = 0. \quad (10)$$

For brevity, in the following calculations we write $h_i = h_i(x)$.

Case F₁. Here $h_1 > 0$, $h_2 > 0$, $h_3 < 0$, $h_4 > 0$. Therefore $h_1(y) = h_1 + ax_2h_2 - cx_3h_3 > 0$. To control $h_3(y)$, use $0 < c, d \leq 1$. Since $x_1h_1 > 0$ and $x_4h_4 > 0$,

$$h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 \leq h_3 + x_1h_1 - bx_2h_2 + x_4h_4.$$

By (10), $x_1h_1 + x_4h_4 = -x_2h_2 - x_3h_3$, and hence $h_3(y) \leq (1 - x_3)h_3 - (1 + b)x_2h_2 < 0$. Thus h_1, h_3, h_4 preserve their signs, while only h_2 may change sign. Consequently the first possible neighbouring region is F_2 .

Case F₂. Here $h_1 > 0$, $h_2 < 0$, $h_3 < 0$, $h_4 > 0$. First, $h_2(y) = h_2 - ax_1h_1 + bx_3h_3 - ex_4h_4 < 0$. Using (10) to eliminate x_2h_2 , we obtain

$$h_1(y) = h_1 + ax_2h_2 - cx_3h_3 = h_1(1 - ax_1) - (a + c)x_3h_3 - ax_4h_4 = h_1(1 - ax_1 - ex_4) - (a + c)x_3h_3,$$

where the last equality follows from $h_4 = (e/a)h_1$. Since $a, e \leq 1$,

$$1 - ax_1 - ex_4 \geq 1 - x_1 - x_4 = x_2 + x_3 > 0.$$

Thus $h_1(y) > 0$, and therefore $h_4(y) > 0$. Only h_3 may change sign, so the first possible neighbouring region is F_3 .

Case F₃. Here $h_1 > 0$, $h_2 < 0$, $h_3 > 0$, $h_4 > 0$. We immediately have $h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 > 0$. Using (10) to eliminate x_3h_3 , we get

$$h_2(y) = h_2 - ax_1h_1 + bx_3h_3 - ex_4h_4 = h_2(1 - bx_2) - (a + b)x_1h_1 - (b + e)x_4h_4.$$

Since $b \leq 1$, we have $1 - bx_2 \geq 1 - x_2 = x_1 + x_3 + x_4 > 0$, so every term in the last expression is negative. Hence $h_2(y) < 0$. Therefore h_2 and h_3 preserve their signs, whereas h_1 and h_4 , which always have the same sign, may change simultaneously. The first possible neighbouring region is F_4 .

Case F₄. Here $h_1 < 0$, $h_2 < 0$, $h_3 > 0$, $h_4 < 0$. Clearly, $h_1(y) = h_1 + ax_2h_2 - cx_3h_3 < 0$. Since $x_1h_1 < 0$ and $x_4h_4 < 0$, the inequalities $c, d \leq 1$ now give

$$h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 \geq h_3 + x_1h_1 - bx_2h_2 + x_4h_4.$$

Using (10) once more, $h_3(y) \geq (1 - x_3)h_3 - (1 + b)x_2h_2 > 0$. Hence h_1, h_3, h_4 preserve their signs, and only h_2 may change. The first possible neighbouring region is F_5 .

Case F₅. Here $h_1 < 0$, $h_2 > 0$, $h_3 > 0$, $h_4 < 0$. We have $h_2(y) = h_2 - ax_1h_1 + bx_3h_3 - ex_4h_4 > 0$. The same identity used in the case F_2 yields $h_1(y) = h_1(1 - ax_1 - ex_4) - (a + c)x_3h_3$. Since $1 - ax_1 - ex_4 > 0$, while $h_1 < 0$ and $h_3 > 0$, we obtain $h_1(y) < 0$, and hence $h_4(y) < 0$. Thus only h_3 may change sign, and the first possible neighbouring region is F_6 .

Case F₆. Here $h_1 < 0$, $h_2 > 0$, $h_3 < 0$, $h_4 < 0$. First, $h_3(y) = h_3 + cx_1h_1 - bx_2h_2 + dx_4h_4 < 0$. Using the identity obtained in the case F_3 , $h_2(y) = h_2(1 - bx_2) - (a + b)x_1h_1 - (b + e)x_4h_4$. Here $1 - bx_2 > 0$, $h_2 > 0$, and $h_1, h_4 < 0$,

so every term on the right-hand side is positive. Thus $h_2(y) > 0$. Hence h_2, h_3 preserve their signs, while h_1 and h_4 may change simultaneously. The first possible neighbouring region is F_1 .

The six cases prove that whenever a positive orbit leaves one strict signature region through a separating wall, the first neighbouring region it can enter is the next one in the cyclic order

$$F_1 \longrightarrow F_2 \longrightarrow F_3 \longrightarrow F_4 \longrightarrow F_5 \longrightarrow F_6 \longrightarrow F_1.$$

This proves the theorem.

Remark 1. A positive trajectory may remain in the same region for several iterates, hit a boundary, or converge toward an invariant subset without traversing the entire six-cycle. The theorem only restricts the first neighbouring region that can be entered when a crossing occurs.

For a negative trajectory, if $V(y) = x$, then

$$y_j - x_j = -y_j h_j(y). \quad (10)$$

Thus every backward coordinate displacement has the opposite sign from the corresponding forward displacement from y to x .

Proposition 3. If a predecessor y of x lies in one of the six relative interiors, the admissible backward crossing order is $F_1 \leftarrow F_2 \leftarrow F_3 \leftarrow F_4 \leftarrow F_5 \leftarrow F_6 \leftarrow F_1$.

Proof. Equation (10) reverses each forward coordinate direction, and the reverse crossing order follows from Theorem 3.

Theorem 4. Let $x^{(0)} \in \text{relint } S^3$. If $x^{(0)} \notin [C_1, C_2]$, then the positive trajectory $\{V^n(x^{(0)})\}_{n \geq 0}$ does not converge in S^3 . Equivalently, a convergent positive trajectory starting in $\text{relint } S^3$ is necessarily stationary, and its initial point belongs to $\text{relint}[C_1, C_2]$.

Proof. First observe that $\text{relint } S^3$ is forward invariant. Indeed, for $x \in \text{relint } S^3$ and $0 < a, b, c, d, e \leq 1$,

$$\begin{aligned} 1 + h_1(x) &= 1 + ax_2 - cx_3 \geq 1 - cx_3 > 0, \\ 1 + h_2(x) &= 1 - ax_1 + bx_3 - ex_4 \geq 1 - (x_1 + x_4) = x_2 + x_3 > 0, \\ 1 + h_3(x) &= 1 + cx_1 - bx_2 + dx_4 \geq 1 - bx_2 > 0, \\ 1 + h_4(x) &= 1 + ex_2 - dx_3 \geq 1 - dx_3 > 0. \end{aligned}$$

Hence all coordinates of $V(x)$ remain positive. Assume that $x^{(n)} = V^n(x^{(0)})$ converges to a point $p \in S^3$. Since V is continuous, $p = \lim_{n \rightarrow \infty} x^{(n+1)} = \lim_{n \rightarrow \infty} V(x^{(n)}) = V(p)$, so $p \in \text{Fix}(V)$. By Theorem 2, $p \in [e_1, e_4] \cup [C_1, C_2] \cup \{e_2, e_3\}$. We exclude these possibilities one by one. Suppose first that $p \in [C_1, C_2]$. Put $I(p) = \{i : p_i > 0\}$, $\Phi_p(x) = \prod_{i \in I(p)} \left(\frac{x_i}{p_i}\right)^{p_i}$.

Since $p \in \ker A \cap S^3$ and A is skew-symmetric, $p^T A = 0$. Therefore, for every $x \in \text{relint } S^3$, the weighted arithmetic-geometric mean inequality yields

$$\frac{\Phi_p(V(x))}{\Phi_p(x)} = \prod_{i \in I(p)} (1 + h_i(x))^{p_i} \leq \sum_{i \in I(p)} p_i (1 + h_i(x)) = 1 + p^T A x = 1.$$

Thus Φ_p is non-increasing along every positive trajectory in the relative interior. On the other hand,

$$\Phi_p(x) \leq \sum_{i \in I(p)} p_i \frac{x_i}{p_i} = \sum_{i \in I(p)} x_i \leq 1.$$

If $p \in \text{relint}[C_1, C_2]$, equality in AM-GM holds only for $x = p$; if $p = C_1$ or $p = C_2$, then one coordinate of p is zero while all coordinates of x are positive, so the last inequality is strict. Thus $x \neq p$ implies $\Phi_p(x) < 1$. Consequently, if $x^{(0)} \neq p$, then $\Phi_p(x^{(n)}) \leq \Phi_p(x^{(0)}) < 1$ ($n \geq 0$), whereas $x^{(n)} \rightarrow p$ would imply $\Phi_p(x^{(n)}) \rightarrow \Phi_p(p) = 1$, a contradiction. Hence a non-stationary interior trajectory cannot converge to any point of $[C_1, C_2]$.

Next, if $p = e_2$, then $h_1(e_2) = a > 0$. By continuity, there exist $\eta > 0$ and N such that $h_1(x^{(n)}) \geq \eta$ for all $n \geq N$. Hence $x_1^{(n+1)} \geq (1 + \eta)x_1^{(n)}$, $n \geq N$, which is impossible because $x_1^{(n)} > 0$ and convergence to e_2 requires $x_1^{(n)} \rightarrow 0$. Similarly, if $p = e_3$, then $h_2(e_3) = b > 0$, so the same argument applied to the second coordinate gives a contradiction. Finally, let $p \in [e_1, e_4]$. Then $p = (s, 0, 0, 1 - s)$ for some $0 \leq s \leq 1$, and $h_3(p) = cs + d(1 - s) > 0$. Hence $h_3(x^{(n)}) \geq \eta > 0$ for all sufficiently large n , so $x_3^{(n)}$ must eventually increase at least geometrically. This contradicts $x_3^{(n)} \rightarrow 0$. Therefore no fixed point can be the limit of a non-stationary positive trajectory starting in $\text{relint } S^3$. The theorem follows.

Theorem 5. Let $O^-(x^{(0)}) = \{x^{(-n)} : n \geq 0\}$ be a global negative trajectory contained in $\text{relint } S^3$. Then there exists a unique point $q = q(x^{(0)}) \in \text{relint}[C_1, C_2]$ such that $x^{(-n)} \rightarrow q$ as $n \rightarrow \infty$. Consequently, $\alpha(x^{(0)}) = \{q\}$, $\text{dist}(x^{(-n)}, [C_1, C_2]) \rightarrow 0$.

Proof. Put $z_n = x^{(-n)}$, so that $V(z_{n+1}) = z_n$. Fix $p \in \text{relint}[C_1, C_2]$ and define

$\Phi_p(x) = \prod_{i=1}^4 \left(\frac{x_i}{p_i}\right)^{p_i}$, with the convention $0^{p_i} = 0$. By Theorem 1, $Ap = 0$, and since $A^T = -A$, we also have $p^T A = 0$. Hence, by the weighted AM–GM inequality,

$$\frac{\Phi_p(V(x))}{\Phi_p(x)} = \prod_{i=1}^4 (1 + h_i(x))^{p_i} \leq \sum_{i=1}^4 p_i (1 + h_i(x)) = 1 + p^T A x = 1.$$

Therefore $\Phi_p(z_n) \leq \Phi_p(z_{n+1}) \leq 1$, so $\Phi_p(z_n) \rightarrow L_p$ for some $L_p > 0$.

Set $R_p(x) = \prod_{i=1}^4 (1 + h_i(x))^{p_i}$. Since $\Phi_p(z_n) = \Phi_p(z_{n+1})R_p(z_{n+1})$, we have $R_p(z_{n+1}) \rightarrow 1$. By compactness of S^3 , let $z_{n_k} \rightarrow q$ for some accumulation point q . Then $R_p(q) = 1$. Equality in the weighted AM–GM inequality implies

$$1 + h_1(q) = 1 + h_2(q) = 1 + h_3(q) = 1 + h_4(q) = 1,$$

and hence $Aq = 0$. By Theorem 1, $q \in [C_1, C_2]$. Moreover, $\Phi_p(q) = L_p > 0$, whereas $\Phi_p(C_1) = \Phi_p(C_2) = 0$. Thus $q \in \text{relint}[C_1, C_2]$. It remains to prove uniqueness. Define $\Phi_q(x) = \prod_{i=1}^4 \left(\frac{x_i}{q_i}\right)^{q_i}$. As above, $\Phi_q(z_n)$ is non-decreasing and bounded above by 1. Since $z_{n_k} \rightarrow q$,

$$\Phi_q(z_{n_k}) \rightarrow \Phi_q(q) = 1,$$

and therefore $\Phi_q(z_n) \rightarrow 1$. On the other hand, weighted AM–GM gives $\Phi_q(x) \leq 1$, with equality if and only if $x = q$. Hence every accumulation point of $\{z_n\}$ must be q . By compactness, this implies $z_n \rightarrow q$. Thus $x^{(-n)} \rightarrow q$, and consequently $\alpha(x^{(0)}) = \{q\}$. Finally, since $q \in [C_1, C_2]$,

$$\text{dist}(x^{(-n)}, [C_1, C_2]) \leq d(x^{(-n)}, q) \rightarrow 0.$$

The proof is complete.

Remark 2. Theorems 4 and 5 show a clear asymmetry between the two time directions in the relative interior of the simplex: a non-stationary positive trajectory cannot converge, whereas every global negative trajectory contained in $\text{relint } S^3$ converges to a unique point of $\text{relint}[C_1, C_2]$. This is a negative-time statement and does not mean that $[C_1, C_2]$ is attracting for the forward dynamics in the sense of Definition 7.

For $V_i(x) = x_i(1 + h_i(x))$, the Jacobian $J(x) = DV(x)$ has entries $J_{ij}(x) = \delta_{ij}(1 + h_i(x)) + x_i a_{ij}$. Therefore,

$$J(x) = \begin{pmatrix} 1 + ax_2 - cx_3 & ax_1 & -cx_1 & 0 \\ -ax_2 & 1 - ax_1 + bx_3 - ex_4 & bx_2 & -ex_2 \\ cx_3 & -bx_3 & 1 + cx_1 - bx_2 + dx_4 & dx_3 \\ 0 & ex_4 & -dx_4 & 1 + ex_2 - dx_3 \end{pmatrix}.$$

On $[C_1, C_2]$, $Ax = 0$, so $J(x) = I + \text{diag}(x)A$. Using (5),

$$J(X(t)) = \begin{pmatrix} 1 & aX_1(t) & -cX_1(t) & 0 \\ -aX_2(t) & 1 & bX_2(t) & -eX_2(t) \\ cX_3(t) & -bX_3(t) & 1 & dX_3(t) \\ 0 & eX_4(t) & -dX_4(t) & 1 \end{pmatrix}.$$

Theorem 6. For $x \in [C_1, C_2]$, the eigenvalues of $J(x)$ are 1, 1, $1 \pm i\omega_*(x)$, where

$$\omega_*(x)^2 = a^2 x_1 x_2 + c^2 x_1 x_3 + b^2 x_2 x_3 + e^2 x_2 x_4 + d^2 x_3 x_4. \quad (11)$$

Here $\omega_*(x)$ denotes a spectral frequency and should not be confused with the ω -limit set notation of Definition 3.

Proof. Put $D = \text{diag}(x)$ and $B = DA$, so that $J(x) = I + B$. Let $P = D^{1/2}$ and $Q = D^{1/2}A$. Then

$$PQ = DA = B, \quad QP = D^{1/2}AD^{1/2} = K.$$

The matrices PQ and QP have the same characteristic polynomial; hence B and K have the same eigenvalues, including at the endpoints where some coordinates vanish. The matrix K is real skew-symmetric. Moreover, $\text{rank } K \leq \text{rank } A = 2$, because $ce = ad$, while $K \neq 0$ on the whole fixed segment. Therefore the eigenvalues of K are $0, 0, \pm i\omega_*(x)$ for some $\omega_*(x) > 0$.

For a real skew-symmetric matrix, $\text{tr}(K^2) = -2 \sum_{i < j} K_{ij}^2$. On the other hand, the above spectrum gives $\text{tr}(K^2) = -2\omega_*(x)^2$. Comparing these two identities yields exactly formula (11). Finally, adding the identity shifts the spectrum of B by 1, which yields the asserted eigenvalues of $J(x)$.

The eigenvalues of $J(C_1)$ are $1, 1, 1 \pm i\sqrt{\frac{abc}{a+b+c}}$. For the eigenvalue 1 , two independent eigenvectors are C_1 and C_2 , viewed as vectors in $\mathbb{R}^4$. If $\mu_{\pm} = \pm i\sqrt{\frac{abc}{S}}$, $\lambda_{\pm} = 1 + \mu_{\pm}$, then corresponding eigenvectors are $w_{\pm}^{(1)} = (S\mu_{\pm} - ab, -ac - S\mu_{\pm}, a(b+c), 0)^T$.

The eigenvalues of $J(C_2)$ are $1, 1, 1 \pm i\sqrt{\frac{bde}{b+d+e}}$. For the eigenvalue 1 , two independent eigenvectors are again C_1 and C_2 . If $\nu_{\pm} = \pm i\sqrt{\frac{bde}{T}}$, $\lambda_{\pm} = 1 + \nu_{\pm}$, then corresponding eigenvectors are

$$w_{\pm}^{(2)} = (0, T\nu_{\pm} - bd, -be - T\nu_{\pm}, b(d+e))^T.$$

Proposition 4. No point of the fixed segment $[C_1, C_2]$ is an attracting fixed point in the sense of Definition 7.

Proof. Let $p \in [C_1, C_2]$. Every neighbourhood of p in the relative topology of S^3 contains another point $q \in [C_1, C_2]$, $q \neq p$. By Theorem 1, q is fixed, so $V^n(q) = q$ for every $n \geq 0$. Hence

$$d(V^n(q), p) = d(q, p) > 0,$$

and therefore $V^n(q)$ does not converge to p . Thus p is not attracting. Since p was arbitrary, no point of $[C_1, C_2]$ is attracting.

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