

# THE DERIVATION ALGEBRA OF THE EXTENDED CONFORMAL GALILEI ALGEBRA

NURATDINOV KAZBEK\*

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
kazbeknur11@gmail.com

NO'MONXONOVA SOJIDAXON

NATIONAL UNIVERSITY OF UZBEKISTAN NAMED AFTER M.ULUGBEK, TASHKENT, UZBEKISTAN  
nomonxonovasojida@gmail.com

**ABSTRACT.** We investigate the derivation algebra of the mass-extended conformal Galilei algebra over the field of complex numbers. The algebra is realized as a one-dimensional central extension of the conformal Galilei algebra. We determine the derivations and identify the distinguished outer derivation which acts by one half on the translation module and by the identity on the central element. We then establish completeness of the derivation algebra and describe the associated derivation tower.

**MSC (2020):** 17B30; 17B40; 17B56.

**Key words:** Lie algebra, derivation, inner derivation.

## Introduction

Conformal Galilei algebras arise as nonrelativistic extensions of the Galilei algebra incorporating an  $\mathfrak{sl}_2$ -symmetry in the time variable. They provide a family of finite-dimensional Lie algebras describing nonrelativistic systems with conformal-type symmetries and have been studied extensively in mathematical physics, representation theory, and the theory of invariant differential equations. For a fixed spatial dimension  $d$ , these algebras form a family parametrized by a non-negative half-integer  $\ell$ , which may be interpreted as the spin of the corresponding  $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})$ -module. The case  $\ell = \frac{1}{2}$  is particularly important: it gives the Schrödinger algebra, while higher values of  $\ell$  lead to higher-order conformal extensions of the Galilei algebra. Finite-dimensional nonrelativistic conformal Lie algebras can also be understood in terms of polynomial vector fields and are related to nonrelativistic conformal transformations and Newton–Hooke geometries.

Derivations of a Lie algebra encode infinitesimal symmetries of its algebraic structure. For a Lie algebra  $\mathfrak{g}$ , we denote by  $\text{Der}(\mathfrak{g})$  the Lie algebra of all derivations of  $\mathfrak{g}$ , while  $\text{ad}(\mathfrak{g})$  denotes the ideal of inner derivations. The quotient

$$\text{Out}(\mathfrak{g}) = \text{Der}(\mathfrak{g}) / \text{ad}(\mathfrak{g})$$

is the outer derivation algebra. Determining these spaces for non-semisimple Lie algebras is generally more subtle than in the semisimple case, where all derivations are inner by the classical Whitehead theorem.

In the present paper, we study the derivation algebra of the mass-extended conformal Galilei algebra  $\widehat{\text{cga}}_\ell(d, \mathbb{C})$ . The presence of the central element  $M$  introduces additional compatibility conditions for a derivation, since the bracket  $[P_{n,i}, P_{m,j}]$  is no longer zero but is proportional to  $M$ . Consequently, a derivation must simultaneously preserve the action of the Levi factor, the Heisenberg-type structure of the radical, and the central extension. Our aim is to determine  $\text{Der}(\widehat{\text{cga}}_\ell(d, \mathbb{C}))$  and to identify explicitly its inner and outer parts. The resulting description also clarifies how the mass central extension affects the derivation structure compared with the unextended conformal Galilei algebra.

## Preliminaries

Let  $\text{cga}_\ell(d, \mathbb{C})$  denote the conformal Galilei algebra in spatial dimension  $d$ . It has a Levi subalgebra

$$\mathfrak{s} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})$$

and an abelian ideal

$$V = \text{span}\{P_{n,i} \mid 0 \leq n \leq 2\ell, 1 \leq i \leq d\}.$$

Thus, over a field of characteristic zero, one may write

$$\widehat{\text{cga}}_\ell(d, \mathbb{C}) = (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})) \ltimes V.$$

**Received:** 18.09.2026, **Accepted:** 20.09.2026

\*Corresponding author

As a module for  $\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})$ , the ideal  $V$  is naturally identified with

$$V \cong L(2\ell) \otimes \mathbb{C}^d,$$

where  $L(2\ell)$  is the irreducible  $\mathfrak{sl}_2(\mathbb{C})$ -module of highest weight  $2\ell$  and  $\mathbb{C}^d$  is the natural  $\mathfrak{so}(d, \mathbb{C})$ -module. In particular, the structure of  $\mathfrak{cga}_\ell(d, \mathbb{C})$  combines a semisimple Levi factor with an abelian module, making these algebras non-semisimple and providing a natural setting for questions concerning their cohomology, deformations, central extensions, and derivations. This semidirect-product description is also useful from the representation-theoretic point of view.

Now, we presents the nontrivial brackets of  $\mathfrak{cga}_\ell(d, \mathbb{C})$ . Fix a basis  $\{H, D, C, M_{ij}, P_{n,i}\}$  of  $\mathfrak{cga}_\ell(d, \mathbb{C})$  where  $1 \leq i, j \leq d$ ,  $0 \leq n \leq 2\ell$ . The generators  $H, D, C$  span a copy of  $\mathfrak{sl}_2(\mathbb{C})$ , while the elements  $M_{ij}$  span  $\mathfrak{so}(d, \mathbb{C})$ . The elements  $P_{n,i}$  span the abelian ideal  $V$ . The we get the following nonzero brackets

$$\begin{aligned} [M_{ij}, M_{kr}] &= -\delta_{ik}M_{jr} - \delta_{jr}M_{ik} + \delta_{ir}M_{jk} + \delta_{jk}M_{ir}, \\ [D, H] &= 2H, \quad [C, H] = D, \quad [D, C] = -2C, \\ [M_{ij}, P_{n,k}] &= -\delta_{ik}P_{n,j} + \delta_{jk}P_{n,i}, \\ [H, P_{n,i}] &= -nP_{n-1,i}, \\ [D, P_{n,i}] &= 2(\ell - n)P_{n,i}, \\ [C, P_{n,i}] &= (2\ell - n)P_{n+1,i}. \end{aligned}$$

Generators outside the permitted range of the index  $n$  are understood to be zero.

In addition to their intrinsic Lie algebra structure, conformal Galilei algebras admit nontrivial one-dimensional central extensions for particular values of the parameters. Of particular importance is the so-called *mass central extension*. For every spatial dimension  $d$  and every half-integer  $\ell$ , the abelian ideal  $V$  admits an invariant alternating bilinear form. Consequently, one may introduce a central element  $M$  and replace the trivial bracket on  $V$  by

$$[P_{n,i}, P_{m,j}] = \delta_{ij} \delta_{n+m, 2\ell} (-1)^{n+\ell+\frac{1}{2}} n!(2\ell - n)! M.$$

All other brackets remain unchanged, and  $M$  is central  $[M, \widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})] = 0$ . The resulting Lie algebra will be denoted by  $\widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})$ .

From the algebraic viewpoint, passing from  $\mathfrak{cga}_\ell(d, \mathbb{C})$  to its mass extension changes an important structural feature: the ideal generated by the  $P_{n,i}$  is no longer abelian. Instead, it becomes a Heisenberg-type algebra with one-dimensional center spanned by  $M$ . In particular, one can view the mass-extended algebra as

$$\widehat{\mathfrak{cga}}_\ell(d, \mathbb{C}) \cong (\mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C})) \ltimes \mathfrak{h},$$

where  $\mathfrak{h} = V \oplus \mathbb{C}M$  and the bracket on  $V$  is determined by the invariant symplectic form defining the central extension. This description is especially useful when studying automorphisms and derivations of the extended algebra.

### The space of derivations of $\widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})$

From now on, we use the following notations:

$$\mathfrak{g} = \mathfrak{cga}_\ell(d, \mathbb{C}), \quad \widehat{\mathfrak{g}} = \widehat{\mathfrak{cga}}_\ell(d, \mathbb{C}), \quad \mathfrak{s} = \mathfrak{sl}_2(\mathbb{C}) \oplus \mathfrak{so}(d, \mathbb{C}), \quad V = L(2\ell) \otimes \mathbb{C}^d, \quad W = V \oplus \mathbb{C}M.$$

Recall that a linear map  $D : \mathfrak{a} \rightarrow \mathfrak{a}$  is a derivation if

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all  $x, y \in \mathfrak{a}$ . The set of all derivations of  $\mathfrak{a}$  forms a vector space and is denoted by  $\text{Der}(\mathfrak{a})$ . Moreover,  $\text{Der}(\mathfrak{a})$  is a Lie algebra with respect to the commutator of linear operators. The operators  $\text{ad}_x$  defined as  $\text{ad}_x(y) = [x, y]$  are derivations. Such derivations are called inner derivations. The set of inner derivations is an ideal of  $\text{Der}(\mathfrak{a})$  and is denoted by  $\text{ad}(\mathfrak{a})$ . We now determine the derivations of  $\widehat{\mathfrak{g}}$ .

**Lemma 2.1.** Let  $D \in \text{Der}(\widehat{\mathfrak{g}})$ . Then  $D(M) \in \mathbb{C}M$ .

*Proof.* Let  $D$  be a derivation. Since  $M$  is central, we have

$$0 = D([M, x]) = [D(M), x]$$

for all  $x \in \widehat{\mathfrak{cga}}_\ell(d, \mathbb{C})$ . Therefore we obtain  $D(M) \in Z(\widehat{\mathfrak{g}}) = \mathbb{C}M$ . □

The next step is to remove the inner component of the restriction of  $D$  to  $\mathfrak{s}$ . Since  $\mathfrak{s}$  is semisimple, the first Whitehead lemma gives

$$H^1(\mathfrak{s}, \mathfrak{s}) = 0, \quad H^1(\mathfrak{s}, V) = 0, \quad H^1(\mathfrak{s}, \mathbb{C}) = 0.$$

Consequently, after subtracting a suitable inner derivation, a derivation can be assumed to vanish on  $\mathfrak{s}$ .

**Lemma 2.2.** For every  $D \in \text{Der}(\widehat{\mathfrak{g}})$  there exists  $x \in \widehat{\mathfrak{g}}$  such that  $D_0 = D - \text{ad}_x$  satisfies  $D_0|_{\mathfrak{s}} = 0$ .

*Proof.* Let us restrict  $D$  to  $\mathfrak{s}$ . Relative to  $\widehat{\mathfrak{g}} = \mathfrak{s} \oplus V \oplus \mathbb{C}M$ , the restriction has three components. The derivation identity for pairs of elements in  $\mathfrak{s}$  shows that these components define 1-cocycles with coefficients in  $\mathfrak{s}$ ,  $V$ , and the trivial module  $\mathbb{C}$ , respectively. By the Whitehead lemma all such 1-cocycles are coboundaries. Therefore the restriction  $D|_{\mathfrak{s}}$  differs from an inner derivation by zero. Subtracting the corresponding inner derivation gives a derivation  $D_0$  satisfying  $D_0|_{\mathfrak{s}} = 0$ .  $\square$

Note that the radical of a Lie algebra is a characteristic ideal, i.e. any derivation maps the radical into itself (see [5]). Therefore any derivation  $D$  of  $\widehat{\mathfrak{g}}$  induces a map  $\widetilde{D} : W \rightarrow W$ , where  $W = V \oplus \mathbb{C}$ .

**Proposition 2.3.** Suppose  $D \in \text{Der}(\widehat{\mathfrak{g}})$  satisfies  $D|_{\mathfrak{s}} = 0$ . Then the induced map  $\widetilde{D} : W \rightarrow W$  commutes with the  $\mathfrak{s}$ -action.

*Proof.* For  $x \in \mathfrak{s}$  and  $v \in V$ , we have

$$D([x, v]) = [D(x), v] + [x, D(v)] = [x, D(v)].$$

Since  $[x, v] \in V$ , this proves that  $D(v)$  must transform under  $\mathfrak{s}$  in the same way as  $v$ . The central component is allowed because  $M$  is an  $\mathfrak{s}$ -trivial module. The noncentral component therefore satisfies  $\widetilde{D}([x, v]) = [x, \widetilde{D}(v)] = x \cdot \widetilde{D}(v)$  for all  $x \in \mathfrak{s}$  and  $v \in V$ .  $\square$

**Proposition 2.4.** Let  $D \in \text{Der}(\widehat{\mathfrak{g}})$ , such that  $D|_{\mathfrak{s}} = 0$ . Then there exists  $\lambda \in \mathbb{C}$  such that  $D(v) = \lambda v$ , ( $v \in V$ ) and  $D(M) = 2\lambda M$ .

*Proof.* By the preceding proposition and Schur's lemma, we have

$$D(v) = av + \phi(v)M,$$

where  $\phi : V \rightarrow \mathbb{C}$  is an  $\mathfrak{s}$ -module homomorphism. Since  $V$  is a nontrivial irreducible  $\mathfrak{s}$ -module, we get  $\text{Hom}_{\mathfrak{s}}(V, \mathbb{C}) = 0$ . Hence, we obtain  $\phi = 0$ . Therefore  $D(v) = \lambda v$  for all  $v \in V$ . Now choose  $v, w \in V$  for which  $[v, w] \neq 0$ . Since  $[v, w] = \alpha M$ ,  $\alpha \in \mathbb{C}$ , the derivation identity gives

$$\begin{aligned} D([v, w]) &= [D(v), w] + [v, D(w)] \\ &= a[v, w] + a[v, w] = 2a[v, w]. \end{aligned}$$

On the other hand,  $D([v, w]) = \alpha D(M)$ . Because  $\alpha \neq 0$ , we obtain  $D(M) = 2\lambda M$ .  $\square$

Now, we classify the derivations of  $\widehat{\mathfrak{g}}$ . Define a linear map  $\tau : \widehat{\mathfrak{g}} \rightarrow \widehat{\mathfrak{g}}$  by

$$\tau|_{\mathfrak{s}} = 0, \quad \tau|_V = \frac{1}{2} \text{Id}_V, \quad \tau(M) = M. \quad (33)$$

In terms of the standard generators,  $\tau$  acts on basis elements as follows

$$\tau(H) = 0, \quad \tau(D) = 0, \quad \tau(C) = 0, \quad \tau(M_{ij}) = 0, \quad \tau(P_{n,i}) = \frac{1}{2} P_{n,i}, \quad \tau(M) = M.$$

**Proposition 2.5.** The map  $\tau$  is an outer derivation of  $\widehat{\mathfrak{g}}$ .

*Proof.* The derivation identity is immediate on  $\mathfrak{s}$ , since  $\tau$  vanishes there. For  $x \in \mathfrak{s}$ ,  $v \in V$ , we have

$$\tau([x, v]) = \frac{1}{2} [x, v]$$

while

$$[\tau(x), v] + [x, \tau(v)] = 0 + \frac{1}{2} [x, v] = \tau([x, v]).$$

Finally, for  $v, w \in V$ , we have

$$\tau([v, w]) = \tau(\beta M) = \beta M.$$

On the other hand,

$$[\tau(v), w] + [v, \tau(w)] = \frac{1}{2}[v, w] + \frac{1}{2}[v, w] = [v, w].$$

Thus the derivation identity holds.

Every inner derivation annihilates the center, i.e.  $\text{ad}_x(M) = [x, M] = 0$  for all  $x \in \mathfrak{g}$ . However, we have  $\tau(M) = M \neq 0$ . Therefore  $\tau \notin \text{ad}(\widehat{\mathfrak{g}})$ . Hence  $\tau$  is an outer derivation.  $\square$

We can now state one of the main results of the present paper.

**Theorem 2.6.** Let  $\ell \in \frac{1}{2} + \mathbb{N}_0$ . Then for the mass-extended conformal Galilei algebra  $\widehat{\mathfrak{g}}$ , one has

$$\text{Der}(\widehat{\mathfrak{g}}) = \text{ad}(\widehat{\mathfrak{g}}) \oplus \mathbb{C}\tau. \quad (34)$$

Consequently,  $\text{Out}(\widehat{\mathfrak{g}}) \cong \mathbb{C}$ .

*Proof.* Let  $D \in \text{Der}(\widehat{\mathfrak{g}})$ . By the Lemma 2.2, there exists  $x \in \widehat{\mathfrak{g}}$  such that  $D_0 = D - \text{ad}_x$  vanishes on  $\mathfrak{s}$ . By the proposition above, we get  $D_0(v) = \lambda v$  ( $v \in V$ ) and  $D_0(M) = 2\lambda M$ . Thus we obtain  $D_0 = 2\lambda\tau$ . Hence  $D = \text{ad}_x + 2\lambda\tau$ . This proves  $\text{Der}(\widehat{\mathfrak{g}}) = \text{ad}(\widehat{\mathfrak{g}}) + \mathbb{C}\tau$ .

We claim that the sum is direct. Indeed, if  $\text{ad}_x = c\tau$ , then evaluating at  $M$  gives

$$0 = \text{ad}_x(M) = c\tau(M) = cM,$$

which implies  $c = 0$ . Since the center is  $\mathbb{C}M$ , from  $\text{ad}_x = 0$  we obtain  $x \in \mathbb{C}M$ . Thus the intersection is zero.  $\square$

Let  $\mathcal{D} = \text{Der}(\widehat{\mathfrak{g}})$ . be the Lie algebra of derivations of  $\widehat{\mathfrak{g}}$ . By the main theorem, we have  $\mathcal{D} = \text{ad}(\widehat{\mathfrak{g}}) \oplus \mathbb{C}\tau$ . Since  $\ker(\text{ad}) = Z(\widehat{\mathfrak{g}}) = \mathbb{C}M$ , we obtain  $\text{ad}(\widehat{\mathfrak{g}}) \cong \widehat{\mathfrak{g}}/\mathbb{C}M$ . Hence  $\mathcal{D} \cong (\widehat{\mathfrak{g}}/\mathbb{C}M) \rtimes \mathbb{C}\tau$ . On the other hand, we have  $\mathfrak{g}_0 = \widehat{\mathfrak{g}}/\mathbb{C}M \cong \mathfrak{s} \ltimes V$ , and the action of  $\tau$  on  $\mathfrak{g}_0$  is  $[\tau, x] = 0$  ( $x \in \mathfrak{s}$ ) and  $[\tau, v] = \frac{1}{2}v$ ,  $v \in V$ . Thus

$$\mathcal{D} \cong (\mathfrak{s} \ltimes V) \rtimes \mathbb{C}\tau. \quad (35)$$

A Lie algebra  $\mathfrak{a}$  is called *complete* if  $Z(\mathfrak{a}) = 0$  and  $\text{Der}(\mathfrak{a}) = \text{ad}(\mathfrak{a})$ . A theorem of Su and Zhu states that the derivation algebra of a centerless perfect Lie algebra is complete.

**Theorem 2.7.**[1] Let  $\mathfrak{a}$  be a centerless perfect Lie algebra over a field of arbitrary characteristic. Then  $\text{Der}(\mathfrak{a})$  is complete.

We prove that Lie algebra of derivations of is complete. This result shows that Theorem 2.7 is true for perfect Lie algebras with trivial centers.

**Theorem 2.8.** The derivation algebra  $\mathcal{D} = \text{Der}(\widehat{\mathfrak{g}})$  is complete.

*Proof.* The quotient  $\mathfrak{g}_0 = \widehat{\mathfrak{g}}/Z(\widehat{\mathfrak{g}})$  is centerless and perfect. Moreover, the classification of derivations gives  $\text{Der}(\widehat{\mathfrak{g}}) \cong \text{Der}(\mathfrak{g}_0)$ . Under this identification,  $\mathcal{D} \cong \text{Der}(\mathfrak{g}_0)$ . By Theorem 2.7,  $\text{Der}(\mathfrak{g}_0)$  is complete. Hence  $\mathcal{D}$  is complete.  $\square$

This theorem shows that the completeness of the Lie algebra of derivations holds for also perfect Lie algebras with nontrivial center.

**Remark 2.9.** The identification  $\text{Der}(\widehat{\mathfrak{g}}) \cong \text{Der}(\mathfrak{g}_0)$  should not be interpreted as a formal consequence of quotienting by the center alone. The point is that, in the present situation, the derivation classification explicitly determines the lifting and shows that the derivation algebra is precisely the derivation algebra of the centerless quotient. This is the structural step needed before applying the Su–Zhu theorem.

## Conclusion

We have determined the derivation algebra of the finite-dimensional mass-extended conformal Galilei algebra in the nonplanar cases. We proved that every derivation is the sum of an inner derivation and a scalar multiple of the distinguished outer derivation

$$\tau|_{\mathfrak{s}} = 0, \quad \tau|_V = \frac{1}{2}I_V, \quad \tau(M) = M.$$

Consequently, we obtained that  $\text{Der}(\widehat{\mathfrak{g}}) = \text{ad}(\widehat{\mathfrak{g}}) \oplus \mathbb{C}\tau$ , and  $\text{Out}(\widehat{\mathfrak{g}}) \cong \mathbb{C}$ . After factoring out the center, one obtains the centerless perfect Lie algebra  $\mathfrak{g}_0 = \widehat{\mathfrak{g}}/Z(\widehat{\mathfrak{g}}) \cong \mathfrak{s} \ltimes V$ . The completeness theorem of Su and Zhu then implies completeness of the derivation algebra. As a consequence, the derivation tower stabilizes at the first derivation algebra:

$$\widehat{\mathfrak{g}} \longrightarrow \text{Der}(\widehat{\mathfrak{g}}) \cong \text{Der}(\text{Der}(\widehat{\mathfrak{g}})) \cong \dots$$

The planar case  $d = 2$  is structurally exceptional because  $\mathfrak{so}_2(\mathbb{C})$  is abelian and the relevant module decomposes over  $\mathbb{C}$ . Therefore its derivation and derivation-tower structure requires a separate calculation rather than a direct application of the generic perfectness argument.

# REFERENCES

1. Y. Su and L. Zhu, *Derivation algebras of centerless perfect Lie algebras are complete*, J. Algebra **290** (2005), 484–494.
2. S. Alauadinov and Sh. Yusupov, *Local derivations of conformal Galilei algebra*, Commun. Algebra **52** (2024), 2489–2508.
3. Y. Gao, Y. Liu and Y. Pei, *Structure of the planar Galilean conformal algebra*, Rep. Math. Phys. **78** (2016), 107–122.
4. D. B. Fuks, *Cohomology of Infinite-Dimensional Lie Algebras*, Springer, New York, 1986.
5. N. Jacobson, *Lie Algebras*, Dover Publications, New York, 1979.

[Cite this article](#)

Nuratdinov K., No'monxonova S. *The Algebra of Derivations of the Extended Conformal Galilei Algebra*, Acta NUUZ, 2026, No 2/3, pp. 164-168.