

SOME GENERALIZED METRIC PROPERTIES OF THE SPACE OF G -PERMUTATION DEGREE

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ABSTRACT. In this paper we investigate some generalized metric properties of the space of G -permutation degree $SP_G^n X$. We prove that an infinite T_1 -space X has a σ -locally finite network if and only if the space $SP_G^n X$ has a σ -locally finite network. We also establish that X has a σ -locally finite base if and only if $SP_G^n X$ has a σ -locally finite base. As an application, we obtain characterizations of metrizable and σ -spaces for the space $SP_G^n X$. Finally, we prove that an infinite regular space X is a $\mathfrak{P}_0$ -space if and only if $SP_G^n X$ is a $\mathfrak{P}_0$ -space.

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Key words: locally finite family, σ -locally finite network, σ -locally finite base, σ -space, $\mathfrak{P}_0$ -space.

Introduction

Generalized metric spaces form one of the central topics of modern general topology. Various classes of spaces defined by network properties, such as spaces with σ -locally finite networks, σ -locally finite bases, σ -spaces and $\mathfrak{P}_0$ -spaces, naturally generalize metrizable spaces and play an important role in the study of some topological structures. These classes provide useful tools for describing spaces that are not metrizable while still retaining many properties characteristic of metrizable spaces. In particular, the existence of a σ -locally finite base is closely related to metrizability through the classical Nagata- Smirnov Metrization Theorem.

Banach introduced $\mathfrak{P}_0$ -spaces as regular spaces possessing a countable Pytkeev network and established several fundamental properties of this class, including its relationships with $\aleph_0$ -spaces and other generalized metric spaces (see [6]). Gabrielyan and Kąkol continued this line of research by introducing $\mathfrak{P}$ -spaces and studying their connections with $\mathfrak{P}_0$ -spaces, $\aleph$ -spaces and $\aleph_0$ -spaces. They proved that every $\mathfrak{P}$ -space has the strong Pytkeev property and investigated numerous properties and characterizations of these spaces (see [5]).

An important problem in topology is to determine which topological properties are preserved under various topological constructions. One such construction is the space of G -permutation degree $SP_G^n X$. Many topological and cardinal properties of spaces of G -permutation degree have been investigated in several papers. In particular, separation axioms and some other topological properties of the space $SP_G^n X$ were studied by R.B. Beshimov and R.M. Juraev (see [4] and [7]). Furthermore, some generalized metric properties and network-type properties of the space $SP_G^n X$ were investigated in [8], [9] and [10].

In [10], several generalized metric properties of the space $SP_G^n X$ were investigated. In particular, it was proved that the functor SP_G^n preserves developable spaces, Moore spaces, M_1 -spaces, M_2 -spaces, stratifiable spaces, Lašnev spaces and Nagata spaces. These results show that the functor SP_G^n preserves a wide range of important generalized metric properties of topological spaces.

In addition, in [10] It was proved that the functor SP_G^n preserves the point- G_δ property, r -spaces, cosmic spaces, $C(\kappa)$ -cosmic spaces, $\aleph_0$ -spaces, α -spaces and strongly first countable spaces. Consequently, it was shown that the functor SP_G^n preserves a broad class of generalized metric properties of topological spaces.

The purpose of this paper is to study other generalized metric properties of the spaces $SP_G^n X$.

Preliminaries

Let G be a subgroup of the finite permutation group S_n and let X^n be the n -th power of a topological space X . Define a relation $\sim_G$ on X^n as follows:

For elements $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$ in X^n ,

$$\mathbf{x} \sim_G \mathbf{y} \iff \text{there exists } \sigma \in G \text{ such that } y_i = x_{\sigma(i)}, \quad 1 \leq i \leq n.$$

This relation is an equivalence relation on the set X^n and is called the G -symmetric equivalence relation, and the G -equivalence class (or orbit) of an element $\mathbf{x} = (x_1, \dots, x_n) \in X^n$ is denoted by $[\mathbf{x}]_G$ or $[(x_1, \dots, x_n)]_G$.

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Denote by $\text{SP}_G^n X$ the quotient set $X^n / \sim_G$ equipped with the quotient topology (of the product topology on X^n). The quotient mapping is denoted by $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ and defined as

$$\pi_{n,G}^s([\mathbf{x}]_G) = \pi_{n,G}^s((x_1, \dots, x_n)) = [(x_1, \dots, x_n)]_G, \quad \mathbf{x} = (x_1, \dots, x_n) \in X^n.$$

This space is called *the space of G -permutation degree of X* . Observe that the points of the space $\text{SP}_G^n X$ are finite subsets of the product X^n . If $G = S_n$, then one writes $\text{SP}^n X$ instead of $\text{SP}_G^n X$ and this space is simply called *the space of permutation degree of X* . When $G = \{e\}$, where e is an identity element of the permutation group S_n , then $\text{SP}_G^n X$ is the space X^n (see [1]).

Let X be a T_1 -space and n be a positive integer. Note that the quotient mapping $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ is an open, closed and surjective map (see [2]).

Main Results

Definition 1. [3] A family $\mathcal{P}$ of subsets of a topological space X is said to be *locally finite* if every point $x \in X$ has a neighborhood U such that a collection $\{P \in \mathcal{P} : P \cap U \neq \emptyset\}$ is finite.

Definition 2. [3] A family $\mathcal{P}$ of subsets of a topological space X is called *σ -locally finite*, if there exists a sequence $\{\mathcal{P}_m\}_{m \in \mathbb{N}}$ of locally finite families such that $\mathcal{P} = \bigcup_{m=1}^{\infty} \mathcal{P}_m$.

A base $\mathcal{B}$ of a topological space X is called a *σ -locally finite base* if there exists a sequence $\{\mathcal{B}_m\}_{m \in \mathbb{N}}$ of locally finite families of open subsets of X such that $\mathcal{B} = \bigcup_{m \in \mathbb{N}} \mathcal{B}_m$. In other words, $\mathcal{B}$ is the union of countably many locally finite families.

Theorem 1. Let X be a T_1 -space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has a σ -locally finite network if and only if the space $\text{SP}_G^n X$ has a σ -locally finite network.

Proof. ($\Rightarrow$) Suppose that topological space X has a σ -locally finite network $\mathcal{N} = \bigcup_{m=1}^{\infty} \mathcal{N}_m$, where each $\mathcal{N}_m$ is a locally finite family for each $m \in \mathbb{N}$. For every $m_1, \dots, m_n \in \mathbb{N}$ we can define a family $\mathcal{G}_{m_1, \dots, m_n} = \{[N_1 \times \dots \times N_n]_G : N_i \in \mathcal{N}_{m_i}, i = 1, \dots, n\}$. Put

$$\mathcal{G} = \bigcup_{m_1, \dots, m_n \in \mathbb{N}} \mathcal{G}_{m_1, \dots, m_n}.$$

We first prove that $\mathcal{G}$ is a network of $\text{SP}_G^n X$. Let $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G$ be an arbitrary point and let $\mathbf{U}$ be an arbitrary neighborhood of $[\mathbf{x}]_G$ in the space $\text{SP}_G^n X$.

Since $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ is continuous, $(\pi_{n,G}^s)^{-1}(\mathbf{U})$ is an open neighborhood of $(x_1, \dots, x_n)$ in X^n . Hence there exist open neighborhoods $V_1, \dots, V_n$ of $x_1, \dots, x_n$, respectively, such that $(x_1, \dots, x_n) \in V_1 \times \dots \times V_n \subset (\pi_{n,G}^s)^{-1}(\mathbf{U})$. Since $\mathcal{N}$ is a network of X , for every $i \in \{1, \dots, n\}$ there exists $N_i \in \mathcal{N}_{m_i}$ for some $m_i \in \mathbb{N}$ such that $x_i \in N_i \subset V_i$. Therefore, $[\mathbf{x}]_G \in [N_1 \times \dots \times N_n]_G \subset [V_1 \times \dots \times V_n]_G \subset \mathbf{U}$. Thus, $\mathcal{G}$ is a network of $\text{SP}_G^n X$.

It remains to prove that each $\mathcal{G}_{m_1, \dots, m_n}$ is locally finite. Fix $m_1, \dots, m_n \in \mathbb{N}$ and let $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G$ be an arbitrary point of $\text{SP}_G^n X$. For every $j, r \in \{1, \dots, n\}$, since $\mathcal{N}_{m_r}$ is locally finite, there exists an open neighborhood $W_{j,r}$ of x_j such that $\{N \in \mathcal{N}_{m_r} : N \cap W_{j,r} \neq \emptyset\}$ is finite. Put $W_j = \bigcap_{r=1}^n W_{j,r}$, $1 \leq j \leq n$. Then each W_j is a neighborhood of x_j , and for every $r \in \{1, \dots, n\}$ the family $\{N \in \mathcal{N}_{m_r} : N \cap W_j \neq \emptyset\}$ is finite.

Put $\mathbf{W} = [W_1 \times \dots \times W_n]_G$. Since $\pi_{n,G}^s$ is an open mapping, $\mathbf{W}$ is a neighborhood of $[\mathbf{x}]_G$ in $\text{SP}_G^n X$. Suppose that $[N_1 \times \dots \times N_n]_G \cap \mathbf{W} \neq \emptyset$ for some $N_i \in \mathcal{N}_{m_i}$. Then there exists a permutation $\tau \in G$ such that $N_{\tau(i)} \cap W_i \neq \emptyset$, $i = 1, \dots, n$. For each i , the set of members of $\mathcal{N}_{m_{\tau(i)}}$ which intersect W_i is finite. Since $i = 1, \dots, n$ ranges over only finitely many indices, there are only finitely many possible choices for $(N_1, \dots, N_n)$. Hence only finitely many members of $\mathcal{G}_{m_1, \dots, m_n}$ intersect $\mathbf{W}$.

Thus, $\mathcal{G}_{m_1, \dots, m_n}$ is locally finite. Since $\mathbb{N}^n$ is countable, $\mathcal{G} = \bigcup_{(m_1, \dots, m_n) \in \mathbb{N}^n} \mathcal{G}_{m_1, \dots, m_n}$ is a σ -locally finite network of $\text{SP}_G^n X$.

($\Leftarrow$) Conversely, suppose that $\mathcal{U} = \bigcup_{m=1}^{\infty} \mathcal{U}_m$ is a σ -locally finite network of $\text{SP}_G^n X$, where every $\mathcal{U}_m$ is locally finite. Consider a subset $\Delta = \{[(x, \dots, x)]_G : x \in X\}$ of $\text{SP}_G^n X$. Clearly, $(\pi_{n,G}^s)^{-1}([(x, \dots, x)]_G)$ consists of exactly one point. Hence the mapping $i : X \rightarrow \Delta$, $i(x) = [(x, \dots, x)]_G$ is a homeomorphism.

For every $m \in \mathbb{N}$, define $\mathcal{V}_m = \{U \cap \Delta : U \in \mathcal{U}_m\}$. Since the restriction of a locally finite family to a subspace is locally finite, each $\mathcal{V}_m$ is locally finite in Δ . Moreover, $\mathcal{V} = \bigcup_{m=1}^{\infty} \mathcal{V}_m$ is a network of the subspace Δ .

Now put $\mathcal{N}_m = \{i^{-1}(V) : V \in \mathcal{V}_m\}$. Since homeomorphisms preserve local finiteness, each $\mathcal{N}_m$ is locally finite in X . Let $\mathcal{N} = \bigcup_{m=1}^{\infty} \mathcal{N}_m$. We prove that $\mathcal{N}$ is a network of X . Let $x \in X$ and let O be an arbitrary neighborhood of x . Then $i(O)$ is open in Δ and contains $i(x)$. Since $\mathcal{V}$ is a network of Δ , there exists $V \in \mathcal{V}$ such that $i(x) \in V \subset i(O)$. Applying i^{-1} , we obtain $x \in i^{-1}(V) \subset O$. Hence $\mathcal{N}$ is a network of X . Therefore $\mathcal{N} = \bigcup_{m=1}^{\infty} \mathcal{N}_m$ is a σ -locally finite network of X . This completes the proof of Theorem 1.

Proposition 1. Let X be a T_1 -space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has a σ -locally finite base if and only if the space $\text{SP}_G^n X$ has a σ -locally finite base.

Proof. ($\Rightarrow$) Suppose X has a σ -locally finite base. Let $\mathcal{B} = \bigcup_{m \in \mathbb{N}} \mathcal{B}_m$ be a σ -locally finite base of X . For fixed indices $m_1, \dots, m_n \in \mathbb{N}$, define the family:

$$\mathcal{C}_{m_1, \dots, m_n} = \{[B_1 \times \dots \times B_n]_G : B_i \in \mathcal{B}_{m_i}, i = 1, \dots, n\}.$$

Then, define the countable union $\mathcal{C} = \bigcup_{m_1, \dots, m_n \in \mathbb{N}} \mathcal{C}_{m_1, \dots, m_n}$.

Every member of $\mathcal{C}$ is open in $\text{SP}_G^n X$ because $B_1 \times \dots \times B_n$ is open in X^n and the quotient map $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ is open. To show the base property, let $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G$ be an arbitrary point in $\text{SP}_G^n X$ and let U be a neighborhood of $[\mathbf{x}]_G$. Choose a neighborhood $V_1 \times \dots \times V_n$ of the representative $\mathbf{x} = (x_1, \dots, x_n)$ contained in $(\pi_{n,G}^s)^{\leftarrow}(U)$. Since $\mathcal{B}$ is a base for X , for each $i \in \{1, \dots, n\}$, we can choose $B_i \in \mathcal{B}$ such that $x_i \in B_i \subset V_i$. It follows that $[\mathbf{x}]_G \in [B_1 \times \dots \times B_n]_G \subset U$. Thus, $\mathcal{C}$ is a base for $\text{SP}_G^n X$.

To show that each $\mathcal{C}_{m_1, \dots, m_n}$ is locally finite, let $[(x_1, \dots, x_n)]_G$ be an arbitrary point of $\text{SP}_G^n X$. Its inverse image under the mapping $\pi_{n,G}^s$ consists of finitely many points, $(\pi_{n,G}^s)^{\leftarrow}([(x_1, \dots, x_n)]_G) = \{\mathbf{y}^1, \mathbf{y}^2, \dots, \mathbf{y}^k\}$, where each $\mathbf{y}^j = (y_1^j, \dots, y_n^j)$. By using the corrected indexing argument for local finiteness, for each coordinate $j \in \{1, \dots, n\}$ and each $r \in \{1, \dots, n\}$, choose a neighborhood $W_{j,r}$ of x_j meeting only finitely many members of $\mathcal{B}_{m_r}$. Put $W_j = \bigcap_{r=1}^n W_{j,r}$. Let $\mathbf{W} = [W_1 \times \dots \times W_n]_G$, which is a neighborhood of $[(x_1, \dots, x_n)]_G$. By construction, every W_j meets only finitely many members of each family $\mathcal{B}_{m_1}, \dots, \mathcal{B}_{m_n}$. Consequently, by the permutation argument, only finitely many members of $\mathcal{C}_{m_1, \dots, m_n}$ intersect $\mathbf{W}$. Therefore, each $\mathcal{C}_{m_1, \dots, m_n}$ is locally finite, making $\mathcal{C}$ a σ -locally finite base.

($\Leftarrow$) Suppose $\mathcal{U} = \bigcup_{m=1}^{\infty} \mathcal{U}_m$ is a σ -locally finite base of $\text{SP}_G^n X$, where each $\mathcal{U}_m$ is locally finite. Consider the diagonal subset $\Delta = \{[(x, \dots, x)]_G : x \in X\} \subset \text{SP}_G^n X$. Since X is a T_1 -space, the mapping $i : X \rightarrow \Delta$ defined by $i(x) = [(x, \dots, x)]_G$ is a homeomorphism.

For every $m \in \mathbb{N}$, define $\mathcal{V}_m = \{U \cap \Delta : U \in \mathcal{U}_m\}$. Since the restriction of a locally finite family to a subspace remains locally finite, each $\mathcal{V}_m$ is locally finite in Δ , and $\mathcal{V} = \bigcup_{m=1}^{\infty} \mathcal{V}_m$ forms a base for the subspace Δ . Putting $\mathcal{B}_m = \{i^{\leftarrow}(V) : V \in \mathcal{V}_m\}$, and since homeomorphisms preserve local finiteness, each $\mathcal{B}_m$ is locally finite in X . Thus, $\mathcal{B} = \bigcup_{m=1}^{\infty} \mathcal{B}_m$ forms a σ -locally finite base for X . This completes the proof of Proposition 1.

Theorem 2. [4] Let X be a topological space, n be a natural number, and G be an arbitrary subgroup of the permutation group S_n . The topological space X is a T_i -space if and only if the space $\text{SP}_G^n X$ is a T_i -space, where $i = 1, 2, 3, 3\frac{1}{2}$.

Theorem 2 implies that a topological space X is regular if and only if the space $\text{SP}_G^n X$ is regular.

Theorem 3(Nagata-Smirnov Metrization Theorem). [3] A topological space X is metrizable if and only if X is regular and has a σ -locally finite base.

By applying Theorem 1, Theorem 2 and Theorem 3 we obtain following result.

Corollary 1. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X is metrizable if and only if $\text{SP}_G^n X$ is metrizable.

Definition 3. [5] A regular topological space X is called σ -space, if X has a σ -locally finite network.

By Definition 3 and Theorem 1 we obtain the following result.

Proposition 2. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X is σ -space if and only if $\text{SP}_G^n X$ is σ -space.

A family $\mathcal{F}$ of subsets of a topological space X is said to be *closure preserving* if for every subfamily $\mathcal{G} \subseteq \mathcal{F}$, $\bigcup \mathcal{G} = \bigcup \{\overline{G} : G \in \mathcal{G}\}$. A family $\mathcal{F}$ of subsets of a topological space X is called σ -closure preserving if $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, where each family $\mathcal{F}_n$ is closure preserving.

A family $\mathcal{F}$ of subsets of a topological space X is said to be *discrete* if every point $x \in X$ has a neighborhood U that intersects at most one member of $\mathcal{F}$. A family $\mathcal{F}$ of subsets of a topological space X is called σ -discrete if $\mathcal{F} = \bigcup_{n \in \mathbb{N}} \mathcal{F}_n$, where each family $\mathcal{F}_n$ is discrete.

Theorem 4. [11] For a regular space X the following are equivalent:

- i) X has a σ -closure preserving network;
- ii) X is a σ -space;
- iii) X has a σ -discrete network.

Proposition 2 and Theorem 4 yield the following corollaries.

Corollary 2. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has σ -closure preserving network if and only if $\text{SP}_G^n X$ has σ -closure preserving network.

Corollary 3. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X has σ -discrete network if and only if $\text{SP}_G^n X$ has σ -discrete network.

We say that a subset A of a topological space X *accumulates at a point* $x \in X$ if each neighborhood $O_x \subset X$ of x contains infinitely many points of the set A . Each accumulation point belongs to the closure $\overline{A}$ of the set A in X . In a T_1 -space X a point $x \in X$ is an accumulation point of a set $A \subset X$ if and only if $x \in \overline{A} \setminus \{x\}$.

Definition 4. [6] Let X be a topological space and $x \in X$ be a point. A family $\mathcal{N}$ of subsets of X is called a *Pytkeev network at x* if $\mathcal{N}$ is a network at x and for every neighborhood $O_x \subset X$ of x and every subset $A \subset X$ accumulating at x there is a set $N \in \mathcal{N}$ such that $N \subset O_x$, $N \cap A$ is an infinite set. A family $\mathcal{N}$ is a Pytkeev network of X if it is a Pytkeev network at each point of X .

A regular topological space X is called $\mathfrak{P}_0$ -space [6] if X has a countable Pytkeev network.

Theorem 5. Let X be a regular space, G be a subgroup of permutation group S_n and $n \in \mathbb{N}$. The topological space X is a $\mathfrak{P}_0$ -space if and only if $\text{SP}_G^n X$ is a $\mathfrak{P}_0$ -space.

Proof. ($\Rightarrow$) Let X be a regular space with a countable Pytkeev network $\mathcal{P}$. By Theorem 3.5 of [6] we have that finite product of $\mathfrak{P}_0$ -spaces is also $\mathfrak{P}_0$ -space and hence a family $\mathcal{P}^n = \{P_1 \times \dots \times P_n : P_i \in \mathcal{P}, 1 \leq i \leq n\}$ is a countable Pytkeev network of X^n .

Now we prove that a family $\text{SP}_G^n \mathcal{P} = \{[P_1 \times \dots \times P_n]_G : P_i \in \mathcal{P}, 1 \leq i \leq n\}$ is a countable Pytkeev network of the space $\text{SP}_G^n X$, where $[P_1 \times \dots \times P_n]_G = \pi_{n,G}^s(P_1 \times \dots \times P_n)$. Let $\mathbf{A} \subset \text{SP}_G^n X$ be any set accumulating at a point $[\mathbf{x}]_G = [(x_1, \dots, x_n)]_G \in \text{SP}_G^n X$. Take any neighborhood $\mathbf{U}$ of $[\mathbf{x}]_G$.

Since X is regular, it is a T_1 -space. By Theorem 2, $\text{SP}_G^n X$ is also regular and hence T_1 . Choose, for every $\mathbf{y} \in \mathbf{A} \setminus \{[\mathbf{x}]_G\}$, a point $z_{\mathbf{y}} \in (\pi_{n,G}^s)^{\leftarrow}(\mathbf{y})$, and put $B = \{z_{\mathbf{y}} : \mathbf{y} \in \mathbf{A} \setminus \{[\mathbf{x}]_G\}\}$. Then $\pi_{n,G}^s(B) = \mathbf{A} \setminus \{[\mathbf{x}]_G\}$. Since $\mathbf{A}$ accumulates at $[\mathbf{x}]_G$ and $\text{SP}_G^n X$ is T_1 , we have $[\mathbf{x}]_G \in \overline{\mathbf{A} \setminus \{[\mathbf{x}]_G\}}$. Therefore, $[\mathbf{x}]_G \in \overline{\pi_{n,G}^s(B)}$. The mapping $\pi_{n,G}^s : X^n \rightarrow \text{SP}_G^n X$ is closed. Hence $\pi_{n,G}^s(\overline{B})$ is a closed subset of $\text{SP}_G^n X$ containing $\pi_{n,G}^s(B)$. Consequently, $[\mathbf{x}]_G \in \pi_{n,G}^s(\overline{B})$. Thus, there exists $z \in (\pi_{n,G}^s)^{\leftarrow}([\mathbf{x}]_G) \cap \overline{B}$. On the other hand, $B \subset (\pi_{n,G}^s)^{\leftarrow}(\mathbf{A} \setminus \{[\mathbf{x}]_G\})$, and therefore $B \cap (\pi_{n,G}^s)^{\leftarrow}([\mathbf{x}]_G) = \emptyset$. Hence $z \notin B$. Thus, $z \in \overline{B} \setminus B$.

Since X is T_1 , the product X^n is also a T_1 -space. If some neighborhood of z contained only finitely many points of B , then, using the T_1 property, we could remove these finitely many points and obtain a neighborhood of z disjoint from B , contradicting $z \in \overline{B}$. Hence every neighborhood of z contains infinitely many points of B . Thus, z is an accumulation point of B .

Since $\pi_{n,G}^s$ is continuous, $(\pi_{n,G}^s)^{\leftarrow}(\mathbf{U})$ is a neighborhood of z in X^n . Since $\mathcal{Q}$ is a Pytkeev network for X^n , there exists $Q \in \mathcal{Q}$ such that $z \in Q \subset (\pi_{n,G}^s)^{\leftarrow}(\mathbf{U})$ and $|Q \cap B|$ is infinite. It follows immediately that $\pi_{n,G}^s(Q) \subset \mathbf{U}$. Moreover, the mapping $\pi_{n,G}^s$ is finite-to-one, since every fiber has cardinality at most $|G|$. Therefore, the image of an infinite subset of X^n under $\pi_{n,G}^s$ is an infinite. Hence $|\pi_{n,G}^s(Q \cap B)|$ is an infinite. Since $\pi_{n,G}^s(Q \cap B) \subset \pi_{n,G}^s(Q) \cap \mathbf{A}$, we obtain $|\pi_{n,G}^s(Q) \cap \mathbf{A}|$ is an infinite. Therefore, for every neighborhood $\mathbf{U}$ of $[\mathbf{x}]_G$ and every subset $\mathbf{A}$ accumulating at $[\mathbf{x}]_G$, there exists $\pi_{n,G}^s(Q) \in \text{SP}_G^n \mathcal{Q}$ such that $[\mathbf{x}]_G \in \pi_{n,G}^s(Q) \subset \mathbf{U}$ and $|\pi_{n,G}^s(Q) \cap \mathbf{A}|$ is an infinite. Thus, $\text{SP}_G^n \mathcal{Q}$ is a countable Pytkeev network of $\text{SP}_G^n X$. Hence $\text{SP}_G^n X$ is a $\mathfrak{P}_0$ -space.

($\Leftarrow$) Let $\mathcal{N}$ be a countable Pytkeev network for the space $\text{SP}_G^n X$. For T_1 -space X , X is homeomorphic to the closed subspace $\Delta = \{[(x, \dots, x)]_G : x \in X\}$ of $\text{SP}_G^n X$. Therefore, it is enough to show that the subspace Δ has a countable Pytkeev network. By Proposition 3.1 of [6] Pytkeev network is preserved under every subspace. Hence, Δ has a countable Pytkeev network. It completes the proof of Theorem 5.

The relationships among the above classes of spaces can be summarized as follows.

Separable metrizable spaces $\longrightarrow \mathfrak{P}_0$ -spaces $\longrightarrow \sigma$ -spaces.

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