

WEIGHTED $\mathcal{P}_\alpha$ -CAPACITY

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ABSTRACT. In this paper, we study the weighted $\mathcal{P}_\alpha$ -capacity, which generalizes the classical $\mathcal{P}_\alpha$ -capacity. We establish its main properties, including continuity with respect to sets, convergence for compact sets, and subadditivity.

MSC (2020): 31B05; 31B10; 31B15.

Key words: α -subharmonic function, α -subharmonic measure, α -polar set, $\mathcal{P}_\alpha$ -capacity, weighted $\mathcal{P}_\alpha$ -capacity.

Introduction

Pluripotential theory constitutes an important branch of mathematics in which the study of classes of functions such as subharmonic (*sh*) and plurisubharmonic (*psh*) functions [12], as well as their generalizations, plays a fundamental role. These classes of functions are widely used in the investigation of differential forms, currents, capacity theory, and boundary value problems, and they are studied in close connection with one another.

An important generalization of the classical class of subharmonic functions is given by α -subharmonic functions (see [1]). Since subharmonic functions arise naturally in many branches of analysis and mathematical physics, their generalizations make it possible to extend classical methods and results to broader settings. For this reason, the study of α -subharmonic measures and the corresponding capacities is of particular interest in modern potential theory.

Recall that the class of α -subharmonic functions is defined by the positivity of the current $dd^c u \wedge \alpha$, where α is a closed, strictly positive differential form of degree $(n-1, n-1)$ in a bounded domain $D \subset \mathbb{C}^n$.

Definition 1. (see [1]) A function $u \in L^1_{loc}(D)$ is called α -subharmonic in D , if it is strongly upper semicontinuous in D , i.e., $u(z)$ is upper semicontinuous in D , in addition, for every $A \subset D$ which has full Lebesgue measure in a neighbourhood of $z^0 \in D$, the following relation

$$\limsup_{\substack{z \rightarrow z^0 \\ z \in A}} u(z) = u(z^0)$$

holds and the current $dd^c u \wedge \alpha$ is positive, i.e., for any positive test function ω in D , we have

$$\int u \alpha \wedge dd^c \omega \geq 0,$$

where $d = \partial + \bar{\partial}$, $d^c = \frac{\bar{\partial} - \partial}{4i}$.

The class of α -subharmonic functions in D is denoted by α -*sh*(D), and for convenience, the function $u(z) \equiv -\infty$ is also considered to be in this class.

Since the weighted (α, ψ) -subharmonic measure was introduced only recently [5], its associated capacity theory has not yet been developed. In this paper, we introduce the corresponding weighted $\mathcal{P}_\alpha$ -capacity and establish its fundamental properties, including continuity with respect to sets, convergence for compact sets, subadditivity, and its relation to α -polar sets. We now state the main results of the paper (see Section 2 and Section 3 for the definition of weighted $\mathcal{P}_\alpha$ -capacity and other relevant definitions).

Theorem 1. The following statements hold for weighted $\mathcal{P}_\alpha$ -capacity.

1. Assume that $\omega_\alpha = \omega_\alpha^*$ a.e on D . Let $\{K_j\}$ be a sequence of compact subsets of D such that $K_{j+1} \subset K_j$ and let ψ be bounded lower semicontinuous function in K_1 and $\sup_{K_1} \psi < 0$. Then, the relation

$$\mathcal{P}_\alpha(K, D, \psi) = \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D, \psi)$$

holds, where $K = \bigcap_{j=1}^{\infty} K_j$;

2. Let $U \subset D$ be an open set and let $U = \bigcup_{j=1}^{\infty} K_j$, where $K_j \subset K_{j+1}^{\circ}$ are compact sets. Assume that ψ is bounded upper semicontinuous function in U and $\sup_U \psi < 0$. Then

$$\lim_{j \rightarrow \infty} \mathcal{P}_{\alpha}(K_j, D, \psi) = \mathcal{P}_{\alpha}(U, D, \psi);$$

3. Let $E \subset D$ and ψ be bounded function on E with $\sup_E \psi < 0$. Then, the weighted $\mathcal{P}_{\alpha}$ -capacity of E is zero if and only if E is α -polar in D , that is,

$$\mathcal{P}_{\alpha}(E, D, \psi) \equiv 0 \iff E - \alpha - \text{polar}.$$

The α -subharmonic measure, defined in the class of α -subharmonic functions, has been studied in [10], where a number of its properties have been established. The $\mathcal{P}_{\alpha}$ -capacity in the unweighted case has also been investigated in [10] and the properties of α -subharmonic measure has been used to establish the characteristics of the $\mathcal{P}_{\alpha}$ -capacity. In the weighted setting, the weighted $\mathcal{P}_{\alpha}$ -capacity exhibits properties analogous to those of the $\mathcal{P}_{\alpha}$ -capacity and the corresponding proofs rely on the properties of the weighted α -subharmonic measure established in [5].

This paper is organized as follows: in Section 2, we present the properties of the weighted α -subharmonic measure; in Section 3, we define the capacity in the weighted and unweighted settings and prove Theorem 1. Furthermore, we present the relationship between the weighted and unweighted $\mathcal{P}_{\alpha}$ -capacities at the end.

2. The weighted α - subharmonic measure

Let $D \subset \mathbb{C}^n$ be a bounded α -regular domain, let $E \subset D$ be a fixed subset, and let ψ be a bounded function on E such that

$$\sup_{z \in E} \psi(z) < 0.$$

We denote by $\mathcal{U}(E, D, \psi)$ the class of all functions $u \in \alpha - sh(D)$ satisfying

$$u|_E \leq \psi, \quad u < 0 \quad \text{on } D.$$

We define the extremal function

$$\omega_{\alpha}(z, E, D, \psi) = \sup\{u(z) : u \in \mathcal{U}(E, D, \psi)\}.$$

Definition 2. The upper regularization

$$\omega_{\alpha}^*(z, E, D, \psi) = \overline{\lim_{w \rightarrow z}} \omega_{\alpha}(w, E, D, \psi)$$

is called (α, ψ) -subharmonic measure of the set E with respect to D .

By definition and according to the property of α -subharmonic functions (see [1]), the function $\omega_{\alpha}^*(z, E, D, \psi)$ is α -subharmonic in D . Note that, when $\psi \equiv -1$, (α, ψ) - subharmonic measure coincides with α -subharmonic measure, that is,

$$\omega_{\alpha}^*(z, E, D, -1) \equiv \omega_{\alpha}^*(z, E, D),$$

where

$$\omega_{\alpha}^*(z, E, D) = (\sup\{u(z) : u \in \alpha - sh(D), u|_E \leq -1, u|_D < 0\})^*$$

The (α, ψ) - measure has the following properties.

Proposition 1. [5] The following monotonicity properties hold:

1. if $E_1 \subset E_2 \subset D_1 \subset D_2$ and ψ is defined on E_2 , with $\sup_{E_2} \psi < 0$, then

$$\omega_{\alpha}^*(z, E_2, D_2, \psi) \leq \omega_{\alpha}^*(z, E_1, D_2, \psi) \leq \omega_{\alpha}^*(z, E_1, D_1, \psi), \quad \forall z \in D_1;$$

2. if $\psi_1|_E \leq \psi_2|_E$, then $\omega_{\alpha}^*(z, E, D, \psi_1) \leq \omega_{\alpha}^*(z, E, D, \psi_2)$, $\forall z \in D$;

3. Let $A \subset D$ be an α -polar set. Then, for any set $E \subset D$,

$$\omega_{\alpha}^*(z, E, D, \psi) = \omega_{\alpha}^*(z, E \cup A, D, \psi)$$

for all $z \in D$, where ψ is defined on $E \cup A$.

4. If $E \subset\subset D$, then

$$\lim_{z \rightarrow \partial D, z \in D} \omega_{\alpha}^*(z, E, D, \psi) = 0.$$

Apart from these properties presented above, the weighted α -subharmonic measure also possesses the properties stated in the following theorem.

Theorem 2. [5] The following properties hold for the weighted α -subharmonic measure.

1. Let $\{K_j\}$ be a sequence of compact subsets of D such that $K_{j+1} \subset K_j$ and let ψ be lower semicontinuous in K_1 . Then

$$\lim_{j \rightarrow \infty} \omega_\alpha(z, K_j, D, \psi) = \omega_\alpha(z, \bigcap_{j=1}^{\infty} K_j, D, \psi)$$

for all $z \in D$.

2. Let $E \subset \subset D_1$ and let $\{D_j\}$ be an increasing sequence of domains such that $D_j \subset D_{j+1}$, $D = \bigcup_{j=1}^{\infty} D_j$. Then

$$\lim_{j \rightarrow \infty} \omega_\alpha^*(z, E, D_j, \psi) = \omega_\alpha^*(z, E, D, \psi).$$

3. Let $E \subset D$ be an arbitrary set and let ψ be lower semicontinuous on E . Then, there exists a decreasing sequence of open sets $U_j \supset E$ such that

$$\left(\lim_{j \rightarrow \infty} \omega_\alpha^*(z, U_j, D, \tilde{\psi}) \right)^* = \omega_\alpha^*(z, E, D, \psi),$$

where $\tilde{\psi}$ is lower semicontinuous in some open neighborhood V of E and satisfies

$$\tilde{\psi}|_E = \psi|_E, \quad \sup_{z \in V} \tilde{\psi}(z) < 0.$$

4. Let $U \subset D$ be an open set and let $U = \bigcup_{j=1}^{\infty} K_j$, where $K_j \subset K_{j+1}^\circ$ are compact sets. Assume that ψ is upper semicontinuous in U . Then

$$\omega_\alpha^*(z, K_j, D, \psi) \downarrow \omega_\alpha^*(z, U, D, \psi).$$

The following lemma, which establishes the connection between weighted and unweighted α -subharmonic measures, is crucial for the study of $\mathcal{P}_\alpha$ -capacities.

Lemma 1. [5] For any set $E \subset D$, the following inequalities hold for all $z \in D$

$$-\inf_{w \in E} \psi(w) \omega_\alpha^*(z, E, D) \leq \omega_\alpha^*(z, E, D, \psi) \leq -\sup_{w \in E} \psi(w) \omega_\alpha^*(z, E, D).$$

3. The weighted $\mathcal{P}_\alpha$ -capacity and its properties

We first recall the definition of the $\mathcal{P}_\alpha$ -capacity. Let us be given any subset $E \subset D$ and its α -subharmonic measure $\omega_\alpha^*(z, E, D)$.

Definition 3. (see [10]). The quality

$$\mathcal{P}_\alpha(E, D) = - \int_D \omega_\alpha^*(z, E, D) dV$$

is called $\mathcal{P}_\alpha$ -capacity of E with respect to D , where dV is volume form in $\mathbb{C}^n$.

By the definition, the following estimate holds for the $\mathcal{P}_\alpha$ -capacity

$$0 \leq \mathcal{P}_\alpha(E, D) \leq m(D),$$

where m is Lebesgue measure in $\mathbb{C}^n$.

The $\mathcal{P}_\alpha$ -capacity was studied in [10] and the following properties were established.

1. the $\mathcal{P}_\alpha$ -capacity is monotone increasing and subadditive with respect to sets, i.e.,

$$E_1 \subset E_2 \implies \mathcal{P}_\alpha(E_1, D) \leq \mathcal{P}_\alpha(E_2, D)$$

and

$$\mathcal{P}_\alpha \left(\bigcup_{j=1}^{\infty} E_j, D \right) \leq \sum_{j=1}^{\infty} \mathcal{P}_\alpha(E_j, D);$$

2. for any subset $E \subset D$ and for any $\varepsilon > 0$, there exists an open set $U \supset E$ such that

$$\mathcal{P}_\alpha(U, D) - \mathcal{P}_\alpha(E, D) < \varepsilon;$$

3. for any decreasing sequence of compact sets $K_1 \supset K_2 \supset \dots$, the following relation holds:

$$\mathcal{P}_\alpha\left(\bigcap_{j=1}^{\infty} K_j, D\right) = \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D);$$

4. for any increasing sequence of open sets $G_1 \subset G_2 \subset \dots$, the following relation holds:

$$\mathcal{P}_\alpha\left(\bigcup_{j=1}^{\infty} G_j, D\right) = \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(G_j, D);$$

Nevertheless, there remain numerous properties of the $\mathcal{P}_\alpha$ -capacity in the weighted setting that have not yet been fully explored, and the study of these properties continues to be of considerable interest in modern analysis. In the present work, we aim to contribute to this line of research by establishing and discussing several of its properties in the weighted setting.

We now begin the study of capacity in the weighted case.

Definition 4. The quality

$$\mathcal{P}_\alpha(E, D, \psi) = - \int_D \omega_\alpha^*(z, E, D, \psi) dV$$

is called the weighted $\mathcal{P}_\alpha$ -capacity of E with respect to D .

Note that, when $\psi \equiv -1$, the weighted and unweighted capacity coincide, that is,

$$\mathcal{P}_\alpha(E, D, -1) = \mathcal{P}_\alpha(E, D).$$

Since the weighted α -subharmonic measure $\omega_\alpha^*(z, E, D, \psi)$ satisfies the relations

$$\inf_{z \in E} \psi(z) \leq \omega_\alpha^*(z, E, D, \psi) \leq 0,$$

by definition of the weighted $\mathcal{P}_\alpha$ -capacity, we have

$$0 \leq \mathcal{P}_\alpha(E, D, \psi) \leq - \inf_{z \in E} \psi(z) \cdot m(D).$$

In addition, the following proposition characterizing the monotonicity properties of $\mathcal{P}_{(\alpha, \psi)}$ -capacity also holds.

Proposition 2. The following statements are true:

1. if $E_1 \subset E_2 \subset D$ and ψ is defined on E_2 , with $\sup_{E_2} \psi < 0$, then $\mathcal{P}_\alpha(E_1, D, \psi) \leq \mathcal{P}_\alpha(E_2, D, \psi)$;
2. if $\psi_1 \leq \psi_2$ and $\sup_E \psi_2 < 0$, then $\mathcal{P}_\alpha(E, D, \psi_2) \leq \mathcal{P}_\alpha(E, D, \psi_1)$;
3. if $E \subset D_1 \subset D_2$, then $\mathcal{P}_\alpha(E, D_1, \psi) \leq \mathcal{P}_\alpha(E, D_2, \psi)$ holds.

One can easily get the proof of the Proposition 2 by the definition and using the properties of the weighted (α, ψ) -subharmonic measure. This proposition shows the monotonicity properties of the weighted $\mathcal{P}_\alpha$ -capacity with respect to sets, domains and the weight functions, playing an important role in studying convergence properties.

Next, we give the proof of the Theorem 1.

Proof of Theorem 1. Now, we show the proof of the first assertion. By the theorem's assumption, Lebesgue measure of the set $\{\omega_\alpha < \omega_\alpha^*\}$ is 0, thus we find

$$\mathcal{P}_\alpha(K_j, D, \psi) = - \int_D \omega_\alpha^*(z, K_j, D, \psi) dV = - \int_D \omega_\alpha(z, K_j, D, \psi) dV$$

and

$$\mathcal{P}_\alpha(K, D, \psi) = - \int_D \omega_\alpha^*(z, K, D, \psi) dV = - \int_D \omega_\alpha(z, K, D, \psi) dV.$$

By the property 1 in Theorem 2,

$$\omega_\alpha(z, K_j, D, \psi) \uparrow \omega_\alpha(z, K, D, \psi).$$

Setting

$$f_j = \omega_\alpha(z, K_j, D, \psi) - \omega_\alpha(z, K_1, D, \psi) \geq 0,$$

and applying Levi's Theorem, we get

$$\lim_{j \rightarrow \infty} \int_D \omega_\alpha(z, K_j, D, \psi) dV = \int_D \omega_\alpha(z, K, D, \psi) dV.$$

Hence,

$$\lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D, \psi) = \mathcal{P}_\alpha(K, D, \psi).$$

Next, we prove the second assertion. According to the property 4 in Theorem 2, we get the following relation

$$\omega_\alpha^*(z, K_j, D, \psi) \downarrow \omega_\alpha^*(z, U, D, \psi)$$

for all $z \in D$. Therefore, the nonnegative sequence $\{-\omega_\alpha^*(z, K_j, D, \psi)\}_{j=1}^\infty$ is increasing with respect to j . Thus, Levi's Theorem (see [11]) gives

$$\begin{aligned} \lim_{j \rightarrow \infty} \mathcal{P}_\alpha(K_j, D, \psi) &= - \lim_{j \rightarrow \infty} \int_D \omega_\alpha^*(z, K_j, D, \psi) dV \\ &= - \int_D \omega_\alpha^*(z, U, D, \psi) dV = \mathcal{P}_\alpha(U, D, \psi). \end{aligned}$$

Now, we turn to prove the property 3. First, let's assume that $\mathcal{P}_\alpha(E, D, \psi) = 0$. Since

$$\mathcal{P}_\alpha(E, D, \psi) = 0$$

and

$$\mathcal{P}_\alpha(E, D, \psi) = - \int_D \omega_\alpha^*(z, E, D, \psi) dV,$$

while

$$\omega_\alpha^*(z, E, D, \psi) \leq 0 \quad \text{in } D,$$

we obtain

$$\omega_\alpha^*(z, E, D, \psi) = 0 \quad \text{for almost every } z \in D.$$

We claim that

$$\omega_\alpha^*(z, E, D, \psi) \equiv 0 \quad \text{in } D.$$

Indeed, suppose, to the contrary, that there exists $z_0 \in D$ such that

$$\omega_\alpha^*(z_0, E, D, \psi) < 0.$$

Since $\omega_\alpha^*(z, E, D, \psi)$ is upper semicontinuous in D , there exist $\varepsilon > 0$ and a neighborhood $B(z_0, r) \Subset D$ such that

$$\omega_\alpha^*(z, E, D, \psi) < -\varepsilon \quad \text{for all } z \in B(z_0, r).$$

Consequently,

$$\int_D \omega_\alpha^*(z, E, D, \psi) dV \leq -\varepsilon m(B(z_0, r)) < 0,$$

which contradicts

$$\int_D \omega_\alpha^*(z, E, D, \psi) dV = 0.$$

Hence,

$$\omega_\alpha^*(z, E, D, \psi) \equiv 0 \quad \text{in } D.$$

By Lemma 1, we find that $\omega_\alpha^*(z, E, D) \equiv 0$ which implies that E is α -polar in D .

Now, we prove the converse. Suppose that E is α -polar in D . Then, by the property of the unweighted α -subharmonic measure, $\omega_\alpha^*(z, E, D) \equiv 0$ on D [10]. Again, using Lemma 1, we find that

$$\omega_\alpha^*(z, E, D, \psi) \equiv 0 \quad \text{on } D.$$

Therefore, by the definition of the weighted $\mathcal{P}_\alpha$ -capacity, we have

$$\mathcal{P}_\alpha(E, D, \psi) = - \int_D \omega_\alpha^*(z, E, D, \psi) dV = 0.$$

The proven property indicates that sets of zero $\mathcal{P}_\alpha$ -capacity are necessarily very small in a measure-theoretic sense; in particular, such sets have Lebesgue measure zero.

The following lemma on the countable subadditivity of weighted capacity and the relation between weighted and unweighted $\mathcal{P}_\alpha$ -capacities is crucial for establishing their properties and relationships.

Lemma 2. The following statements hold for weighted $\mathcal{P}_\alpha$ -capacity.

1. Let $E_j \subset D$ for $j = 1, 2, \dots$, $E = \bigcup_{j=1}^{\infty} E_j$ and ψ is defined on E , with $\sup_E \psi < 0$. Assume that the sets

$$\{z \in D : \omega_\alpha(z, E_j, D, \psi) < \omega_\alpha^*(z, E_j, D, \psi)\}$$

and

$$\{z \in D : \omega_\alpha(z, E, D, \psi) < \omega_\alpha^*(z, E, D, \psi)\}$$

have Lebesgue measure zero. Then

$$\mathcal{P}_\alpha\left(\bigcup_{j=1}^{\infty} E_j, D, \psi\right) \leq \sum_{j=1}^{\infty} \mathcal{P}_\alpha(E_j, D, \psi);$$

2. Let E be any subset of D . Then, the following relations hold

$$-\sup_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D) \leq \mathcal{P}_\alpha(E, D, \psi) \leq -\inf_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D).$$

Proof. To prove the first assertion, we take an arbitrary function $u_j(z)$ from each class $\mathcal{U}(E_j, D, \psi)$ and consider the function $u(z) = \sum_{j=1}^{\infty} u_j(z)$. Since for every $j \in \mathbb{N}$, $u_j|_D < 0$ is valid, so the sequence $S_k(z) = \sum_{j=1}^k u_j(z)$ is decreasing with respect to k and $S_k \in \alpha - sh(D)$. Therefore, as the limit function of decreasing sequence of α -subharmonic functions, $u(z)$ is α -subharmonic in D [1]. If we take a point $z \in E$, there exists a number $j_0 \in \mathbb{N}$ such that $z \in E_{j_0}$. Since $u_j|_D < 0$ for all j , the relation

$$u(z) = \sum_{j=1}^{\infty} u_j(z) \leq u_{j_0}(z) \leq \psi(z)$$

holds, which implies that

$$u \in \mathcal{U}\left(\bigcup_{j=1}^{\infty} E_j, D, \psi\right).$$

Therefore, for all $z \in D$, we have the following inequality

$$u(z) = \sum_{j=1}^{\infty} u_j(z) \leq \omega_\alpha\left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi\right). \quad (1)$$

Fix $\varepsilon > 0$ and $z \in D$. By the definition of ω_α , for each $j \in \mathbb{N}$, we may choose u_j such that

$$u_j(z) > \omega_\alpha(z, E_j, D, \psi) - \frac{\varepsilon}{2^j}.$$

It follows that

$$\sum_{j=1}^{\infty} u_j(z) > \sum_{j=1}^{\infty} \omega_\alpha(z, E_j, D, \psi) - \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary and the inequality (1) holds, we find that

$$\omega_\alpha\left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi\right) \geq \sum_{j=1}^{\infty} \omega_\alpha(z, E_j, D, \psi). \quad (2)$$

By the theorem's condition, the sets

$$\{z \in D : \omega_\alpha(z, E_j, D, \psi) < \omega_\alpha^*(z, E_j, D, \psi)\}$$

and

$$\{z \in D : \omega_\alpha(z, \bigcup_{j=1}^{\infty} E_j, D, \psi) < \omega_\alpha^*(z, \bigcup_{j=1}^{\infty} E_j, D, \psi)\}$$

have Lebesgue measure zero. Then by the inequality (2) and according to the Levi's Theorem (see, for example [11]), we get

$$\begin{aligned} \mathcal{P}_\alpha \left(\bigcup_{j=1}^{\infty} E_j, D, \psi \right) &= - \int_D \omega_\alpha^* \left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi \right) dV = - \int_D \omega_\alpha \left(z, \bigcup_{j=1}^{\infty} E_j, D, \psi \right) dV \\ &\leq - \int_D \left(\sum_{j=1}^{\infty} \omega_\alpha(z, E_j, D, \psi) \right) dV = - \sum_{j=1}^{\infty} \int_D \omega_\alpha(z, E_j, D, \psi) dV \\ &= - \sum_{j=1}^{\infty} \int_D \omega_\alpha^*(z, E_j, D, \psi) dV = \sum_{j=1}^{\infty} \mathcal{P}_\alpha(E_j, D, \psi). \end{aligned}$$

which proves the countably subadditiveness of the weighted $\mathcal{P}_\alpha$ -capacity.

Now, we prove the second assertion. According to Lemma 1, we have the following relation between the weighted and unweighted α -subharmonic measures

$$- \inf_{w \in E} \psi(w) \omega_\alpha^*(z, E, D) \leq \omega_\alpha^*(z, E, D, \psi) \leq - \sup_{w \in E} \psi(w) \omega_\alpha^*(z, E, D).$$

Multiplying the last inequality to (-1) and integrating over D , we get that

$$\sup_{w \in E} \psi(w) \int_D \omega_\alpha^*(z, E, D) dV \leq - \int_D \omega_\alpha^*(z, E, D, \psi) dV \leq \inf_{w \in E} \psi(w) \int_D \omega_\alpha^*(z, E, D) dV,$$

which is equivalent to

$$- \sup_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D) \leq \mathcal{P}_\alpha(E, D, \psi) \leq - \inf_{w \in E} \psi(w) \mathcal{P}_\alpha(E, D).$$

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Salaeva D. *Weighted $\mathcal{P}_\alpha$ -capacity*, *Acta NUUZ*, 2026, 2/3, pp. 195-201.