

ERGODICITY OF ISOMETRIES MAPPING A SPHERE INTO ITSELF IN THE FIELD OF p -ADIC NUMBERS

SATTAROV ISKANDAR

V.I. ROMANOVSKIY INSTITUTE OF MATHEMATICS, TASHKENT, UZBEKISTAN
sattarovi-a@yandex.ru

ABSTRACT. In this paper we study measure-preserving dynamical systems generated by isometries acting on spheres in the field of p -adic numbers. We focus on the boundary case $\rho(r) = |f(x) - x|_p = r$, which is not covered by the proper invariant-ball argument. Anashin's finite-level criterion is formulated for 1-Lipschitz self-maps of $\mathbb{Z}_p$ and therefore applies, after affine normalization, to dynamical systems on invariant balls. For $p \geq 3$, an invariant sphere is not a ball: it is a disjoint union of $p - 1$ balls of the next smaller radius. Hence the criterion on $\mathbb{Z}_p$ is not directly applicable to a map preserving the whole sphere. We establish an intrinsic finite-factor criterion for isometries of arbitrary p -adic spheres: the system is ergodic if and only if the permutations induced on all canonical partitions of the sphere consist of single cycles. We further prove a metric cyclic-lifting criterion: once the first finite factor consists of a single cycle, cyclicity at every subsequent level is equivalent to the return equalities $|f^{jN_n}(x_*) - x_*|_p = r/p^n$, $j = 1, \dots, p - 1$, at one point and at every level, where $N_n = (p - 1)p^{n-1}$. We apply these criteria to the borderline invariant spheres of $(2, 2)$ -rational functions with two fixed points and of $(2, 1)$ -rational functions with a unique fixed point. In both families we give explicit ergodic examples over $\mathbb{Q}_3$. The resulting ergodic systems are also minimal and uniquely ergodic.

MSC (2020): 37A25; 37P05; 11S82; 28C10.

Key words: p -adic numbers, sphere, isometry, Haar measure, ergodicity, finite dynamical system.

Introduction

p -Adic ergodic theory is a part of non-Archimedean dynamics which studies measure-preserving transformations on ultrametric spaces. One of the important features of p -adic spaces is their hierarchical structure: every ball is a finite disjoint union of balls of smaller radius, and two balls either do not intersect or one of them contains the other. Therefore, the dynamics of a continuous or isometric map on a compact p -adic set can often be described by finite dynamical systems induced by partitions into balls.

For general background on ergodic transformations and non-Archimedean dynamical systems, see [1–4, 19]. Basic facts on p -adic analysis used in this paper can be found in [5, 6]. Rational p -adic dynamical systems and their ergodicity on invariant spheres were studied in [7–9, 13, 17]. Closely related p -adic mappings arising from statistical-mechanical models on Cayley trees, including Potts–Bethe and Ising–Potts type mappings, were investigated in [8, 10–12].

Most results on ergodicity of p -adic dynamical systems were first obtained for maps acting on the ring of p -adic integers $\mathbb{Z}_p$. This is natural, because $\mathbb{Z}_p$ is a compact topological group and carries a normalized Haar measure. At the same time, in rational p -adic dynamics the invariant sets of a map are often not equal to $\mathbb{Z}_p$; they can be balls and spheres with arbitrary centers and radii. For this reason, the study of Haar measure and ergodicity on arbitrary invariant spheres is a natural continuation of the classical problem on $\mathbb{Z}_p$.

In [18] the group structure of arbitrary balls and spheres was constructed, the Haar measure on such sets was described, and the dynamical system generated by an isometry on an invariant sphere was studied. In particular, if the quantity

$$\rho(r) = |f(x) - x|_p$$

does not depend on $x \in S_r(a)$ and satisfies $\rho(r) < r$, then the ergodicity problem was solved there. Namely, for $p \geq 3$ the system is not ergodic, and for $p = 2$ the system may be ergodic only in the case $r = 2\rho(r)$. Thus, the case $\rho(r) < r$ is not new; the novelty of the present work lies in the boundary case $\rho(r) = r$.

The aim of this paper is to study the remaining case

$$\rho(r) = r. \tag{1}$$

This case is more delicate, because the ball of radius $\rho(r)$ coincides with the whole sphere scale and does not automatically give a proper invariant ball of positive measure. To overcome this difficulty, we consider finite partitions of a sphere

into balls of smaller radii and reduce the ergodicity problem to the cyclicity of the induced permutations on these finite factors.

The distinction between dynamics on a ball and dynamics on a sphere is essential here. Anashin proved that a 1-Lipschitz self-map $F : \mathbb{Z}_p \rightarrow \mathbb{Z}_p$ is ergodic with respect to normalized Haar measure if and only if every reduction of F modulo p^n is a single-cycle permutation [1–3]. Since every p -adic ball is an affine copy of $\mathbb{Z}_p$, this criterion can be transferred, by normalization, to a map carrying a ball into itself. It also explains why the case $p = 2$ is exceptional: for every $k \in \mathbb{Z}$,

$$S_{2^{-k}}(a) = V_{2^{-(k+1)}}(a + 2^k),$$

so a 2-adic sphere is itself a ball; compare [4]. In contrast, for $p \geq 3$,

$$S_{p^{-k}}(a) = \bigsqcup_{i=1}^{p-1} V_{p^{-(k+1)}}(a + ip^k),$$

and the natural normalization $x = a + p^k u$ identifies the sphere with the unit group $\mathbb{Z}_p^*$ rather than with $\mathbb{Z}_p$. A map preserving the whole sphere may permute its $p - 1$ maximal ball components. Consequently, Anashin's criterion on $\mathbb{Z}_p$ is not directly applicable to the ergodicity problem for arbitrary sphere-preserving isometries when $p \geq 3$. Anashin also obtained ergodicity results for certain polynomial and locally analytic maps on spheres [1,2], but those results do not provide an intrinsic criterion for arbitrary isometries of a sphere.

The first main contribution of the present paper is therefore Theorem 5, an intrinsic finite-factor criterion formulated directly on $S_{p^{-k}}(a)$. Its canonical factor $X_{n,k}$ has

$$|X_{n,k}| = (p - 1)p^{n-1},$$

which reflects the decomposition of the sphere into the nonzero residue classes and their successive refinements. The second main contribution is the metric cyclic-lifting criterion of Theorem 6, which characterizes the passage from level n to level $n + 1$ by return-distance equalities at a single point. We then apply these criteria to the boundary case $\rho(r) = r$ for two classes of rational p -adic maps.

The paper is organized as follows. In Section 2 we recall basic notation. In Section 3 we formulate previous results on the group structure, Haar measure and the case $\rho(r) < r$. In Section 4 we consider finite factors of a sphere. In Section 5 we prove the ergodicity and cyclic-lifting criteria for the case $\rho(r) = r$. In Section 6 we give two model examples. In Section 7 we apply the criteria to two classes of rational functions.

Preliminaries

Let p be a prime number and let $\mathbb{Q}_p$ be the field of p -adic numbers with the p -adic absolute value $|\cdot|_p$. We write

$$\nu_p(x) = -\log_p |x|_p \quad (x \neq 0), \quad \nu_p(0) = +\infty,$$

and denote the group of p -adic units by

$$\mathbb{Z}_p^* = \{u \in \mathbb{Z}_p : |u|_p = 1\}.$$

For $a \in \mathbb{Q}_p$ and $r > 0$ we use the notation

$$U_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p < r\}, \quad V_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p \leq r\},$$

$$S_r(a) = \{x \in \mathbb{Q}_p : |x - a|_p = r\}.$$

In what follows, every radius is assumed to belong to the value group of $\mathbb{Q}_p$; equivalently, $r = p^m$ for some $m \in \mathbb{Z}$.

Let $(X, \mathcal{B}, \mu)$ be a probability space. A measurable map $T : X \rightarrow X$ is measure-preserving if

$$\mu(T^{-1}(E)) = \mu(E), \quad E \in \mathcal{B}.$$

It is called ergodic if every measurable set E satisfying

$$\mu(T^{-1}(E) \triangle E) = 0$$

has measure either 0 or 1. Thus invariant sets are understood modulo sets of measure zero.

In this paper, an isometry of $S_r(a)$ means a bijection $f : S_r(a) \rightarrow S_r(a)$ satisfying

$$|f(x) - f(y)|_p = |x - y|_p, \quad x, y \in S_r(a). \quad (2)$$

Previous results

We recall several results which were proved in [18]. They are used in the present paper without repeating their proofs.

Let $a \in \mathbb{Q}_p$ and $r \in \{p^m : m \in \mathbb{Z}\}$. On the ball $V_r(a)$ and the sphere $S_r(a)$ define the operations

$$\begin{aligned} x \oplus y &= x + y - a, & x, y \in V_r(a), \\ x \odot y &= r(x - a)(y - a) + a, & x, y \in S_r(a). \end{aligned} \quad (3)$$

Here the number r in (3) is considered as an element of $\mathbb{Q}_p$.

Theorem 1. (see [18]). The following statements hold:

- (i) $(V_r(a), \oplus)$ is a compact topological additive Abelian group;
- (ii) $(S_r(a), \odot)$ is a compact topological multiplicative Abelian group.

Theorem 2. (see [18]). Any two balls with positive radii are isomorphic as groups. Any two spheres with positive radii are also isomorphic as groups.

Let $\mathcal{B}$ be the Borel σ -algebra generated by clopen subsets of $V_r(a)$ or $S_r(a)$. The measure $\bar{\mu}$ defined by

$$\bar{\mu}(V_\rho(s)) = \rho$$

is the Haar measure on the corresponding compact group. On the sphere $S_r(a)$ we use the normalized Haar measure

$$\mu(V_\rho(s)) = \frac{\bar{\mu}(V_\rho(s))}{\bar{\mu}(S_r(a))} = \frac{p\rho}{(p-1)r}.$$

Theorem 3. (see [18]). Let $f : S_r(a) \rightarrow S_r(a)$ be an isometry and let μ be the normalized Haar measure on $S_r(a)$. Then f preserves μ , i.e.

$$\mu(f(V_\rho(s))) = \mu(V_\rho(s))$$

for every ball $V_\rho(s) \subset S_r(a)$. Moreover, the orbit of any non-fixed initial point is not convergent.

Assume that the value $|f(x) - x|_p$ does not depend on the choice of $x \in S_r(a)$. We denote

$$\rho(r) = |f(x) - x|_p, \quad x \in S_r(a). \quad (4)$$

Theorem 4. (see [18]). Let $f : S_r(a) \rightarrow S_r(a)$ be an isometry and let (4) hold. If $\rho(r) < r$, then

- (i) the dynamical system $(S_r(a), f, \mu)$ is not ergodic for all $p \geq 3$;
- (ii) for $p = 2$ the dynamical system $(S_r(a), f, \mu)$ may be ergodic only if $r = 2\rho(r)$.

The proof of this theorem is based on the fact that a ball of radius $\rho(r)$ is a minimal invariant ball. Therefore, if $\rho(r) < r$, a proper invariant ball appears automatically. This is the reason why in the present paper we consider only the case (1).

Remark 1. Some rational p -adic dynamical systems are isometries on their invariant spheres. For instance, such phenomena were studied for rational functions in [7–9,13,17]. In those works the ergodicity of concrete rational maps was analyzed on invariant spheres. In the present paper we formulate the result for arbitrary isometries satisfying (1).

Finite partitions of a sphere

Let $k \in \mathbb{Z}$ and consider the sphere $S_{p^{-k}}(a)$. For every $n \geq 1$ this sphere can be represented as a disjoint union of balls of radius $p^{-(k+n)}$ in the following form:

$$S_{p^{-k}}(a) = \bigcup_{i_1=1}^{p-1} \bigcup_{i_2=0}^{p-1} \cdots \bigcup_{i_n=0}^{p-1} V_{p^{-(k+n)}} \left(a + p^k \sum_{l=1}^n i_l p^{l-1} \right). \quad (5)$$

Denote by $X_{n,k}$ the set of all balls appearing in the decomposition (5). Then

$$|X_{n,k}| = p^{n-1}(p-1).$$

To see the decomposition explicitly, write every $x \in S_{p^{-k}}(a)$ in its unique form

$$x - a = p^k (i_1 + i_2 p + \cdots + i_n p^{n-1} + p^n z), \quad z \in \mathbb{Z}_p,$$

where $i_1 \in \{1, \dots, p-1\}$ and $i_2, \dots, i_n \in \{0, \dots, p-1\}$. Fixing the first n digits gives one ball of radius $p^{-(k+n)}$, while two different digit strings give disjoint balls. This also explains directly why the number of atoms is $(p-1)p^{n-1}$.

Let $\nu_{n,k}$ be the normalized counting measure on $X_{n,k}$, i.e.

$$\nu_{n,k}(A) = \frac{|A|}{|X_{n,k}|}, \quad A \subset X_{n,k}.$$

Remark 2. No positivity assumption on k is needed. Indeed, the affine isometry $u \mapsto a + p^k u$ maps $\mathbb{Z}_p^* = S_1(0)$ onto $S_{p^{-k}}(a)$, and the decomposition (5) is valid for every $k \in \mathbb{Z}$. Consequently, all results in this and the next section hold for arbitrary $k \in \mathbb{Z}$. In examples based on integral congruence expansions we may retain $k \geq 1$ for convenience.

Lemma 1. Let $k \in \mathbb{Z}$, $n \geq 1$, and let $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$ be an isometry. Then f induces a permutation on $X_{n,k}$.

Proof. Put

$$\varepsilon = p^{-(k+n)}$$

and let $B = V_\varepsilon(s)$ be an atom of $X_{n,k}$. If $x \in B$, then

$$|f(x) - f(s)|_p = |x - s|_p \leq \varepsilon.$$

Hence

$$f(B) \subset V_\varepsilon(f(s)). \quad (5a)$$

Because $\varepsilon < p^{-k} = |f(s) - a|_p$, the ultrametric inequality shows that every point of $V_\varepsilon(f(s))$ still belongs to $S_{p^{-k}}(a)$. Thus this ball is one of the atoms of $X_{n,k}$.

We now prove the reverse inclusion in (5a). Let $z \in V_\varepsilon(f(s))$. Since f is bijective, there is a point $x \in S_{p^{-k}}(a)$ such that $z = f(x)$. Using the isometry property once more, we get

$$|x - s|_p = |f(x) - f(s)|_p = |z - f(s)|_p \leq \varepsilon.$$

Therefore $x \in B$, and consequently $z = f(x) \in f(B)$. Hence

$$f(B) = V_\varepsilon(f(s)).$$

Thus f maps each atom of $X_{n,k}$ onto an atom of the same partition. If two atoms had the same image, their inverse images under the bijection f would coincide. Therefore the induced map on the finite set $X_{n,k}$ is injective, and hence it is a permutation.

We denote the induced permutation on $X_{n,k}$ by the same symbol f .

Remark 3. Let T be a permutation of a finite set X and let ν be the uniform measure. Every T -invariant subset of X is a union of cycles of T . Therefore (X, T, ν) is ergodic if and only if T has exactly one cycle. Applied to $(X_{n,k}, f, \nu_{n,k})$, this means that the induced permutation must act transitively on all balls of the partition (5).

Ergodicity in the case $\rho(r) = r$

In this section let $k \in \mathbb{Z}$ and let $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$ be an isometry. The finite-factor criteria below are valid without any displacement assumption. For applications to the boundary case we shall additionally impose

$$\rho(r) = |f(x) - x|_p = r, \quad r = p^{-k}, \quad x \in S_r(a). \quad (6)$$

The following criterion is formulated intrinsically on the sphere. Under the normalization

$$x = a + p^k u,$$

the sphere $S_{p^{-k}}(a)$ is identified with $\mathbb{Z}_p^*$, and the partition $X_{n,k}$ is identified with the unit residue classes $(\mathbb{Z}/p^n\mathbb{Z})^*$. Thus

$$|X_{n,k}| = |(\mathbb{Z}/p^n\mathbb{Z})^*| = (p-1)p^{n-1}.$$

For $p \geq 3$ these factors are not the residue rings $\mathbb{Z}/p^n\mathbb{Z}$ occurring in Anashin's criterion on $\mathbb{Z}_p$. Hence the next theorem is not obtained by reducing the system to a self-map of one invariant ball; it is proved directly from the canonical partitions of the entire sphere.

For later use, put

$$N_n = |X_{n,k}| = (p-1)p^{n-1}, \quad n \geq 1. \quad (7)$$

Theorem 5. Let $k \in \mathbb{Z}$, let $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$ be an isometry and let μ_k be the normalized Haar measure on $S_{p^{-k}}(a)$. Then the dynamical system

$$(S_{p^{-k}}(a), f, \mu_k)$$

is ergodic if and only if the finite dynamical system

$$(X_{n,k}, f, \nu_{n,k})$$

is ergodic for every $n \geq 1$.

Proof. We prove the two implications separately.

Assume first that $(S_{p-k}(a), f, \mu_k)$ is ergodic. Suppose, to the contrary, that for some $n \geq 1$ the induced permutation on $X_{n,k}$ has at least two cycles. Let Y be one of these cycles and define

$$A = \bigcup_{B \in Y} B.$$

The set A is clopen because it is a finite union of balls. It is also f -invariant, since the atoms in Y are permuted among themselves. Moreover, Y is nonempty and is not the whole of $X_{n,k}$, so

$$0 < \mu_k(A) < 1.$$

This contradicts ergodicity. Hence the induced permutation on every $X_{n,k}$ consists of a single cycle.

Conversely, assume that the induced permutation on $X_{n,k}$ consists of a single cycle for every $n \geq 1$. Let A be a measurable set invariant modulo measure zero, that is,

$$\mu_k(f^{-1}(A) \triangle A) = 0.$$

We shall prove that $\mu_k(A)$ is either 0 or 1.

First note that, for every integer $j \geq 1$,

$$\mu_k(f^{-j}(A) \triangle A) = 0. \quad (7a)$$

Indeed, the assertion is given for $j = 1$. If it holds for j , then

$$f^{-(j+1)}(A) \triangle A \subset f^{-1}(f^{-j}(A) \triangle A) \cup (f^{-1}(A) \triangle A).$$

Both sets on the right have measure zero because f preserves μ_k . Thus (7a) follows by induction.

Fix $n \geq 1$ and take any two atoms $B, C \in X_{n,k}$. Since the induced permutation is a single cycle, there exists an integer $j \geq 0$ such that $f^j(B) = C$. The map f is invertible and measure-preserving, and the invariance of A implies $f^{-j}(A) = A$ modulo a null set. Therefore

$$\begin{aligned} \mu_k(A \cap C) &= \mu_k(f^{-j}(A \cap C)) \\ &= \mu_k(f^{-j}(A) \cap B) \\ &= \mu_k(A \cap B). \end{aligned} \quad (7b)$$

Thus $\mu_k(A \cap B)$ has the same value for all atoms $B \in X_{n,k}$. There are $N_n = (p-1)p^{n-1}$ such atoms, each of measure $1/N_n$. Summing (7b) over all atoms gives

$$\mu_k(A \cap B) = \frac{\mu_k(A)}{N_n} \quad (B \in X_{n,k}),$$

and hence

$$\frac{\mu_k(A \cap B)}{\mu_k(B)} = \mu_k(A). \quad (7c)$$

Suppose, for contradiction, that

$$0 < a := \mu_k(A) < 1.$$

Finite unions of atoms of the partitions $X_{n,k}$ form an algebra that generates the Borel σ -algebra. Since Haar measure on the compact sphere is regular and the atoms form a clopen base, for every $\varepsilon > 0$ there exist a level n and a union U of atoms of $X_{n,k}$ such that

$$\mu_k(A \triangle U) < \varepsilon. \quad (7d)$$

Here are the details. By regularity, choose a compact set $K \subset A$ and an open set $O \supset A$ such that

$$\mu_k(O \setminus K) < \varepsilon.$$

For every $x \in K$, the partition balls form a neighbourhood base, so there is an atom $B_{n_x}(x)$ satisfying

$$x \in B_{n_x}(x) \subset O.$$

Compactness of K gives a finite subcover. Let n be at least as large as all levels occurring in this subcover and refine each selected ball into level- n atoms. If U is the union of the resulting atoms, then

$$K \subset U \subset O.$$

Consequently,

$$A \triangle U \subset O \setminus K,$$

which proves (7d).

Let $t = \mu_k(U)$. Since U is a union of level- n atoms, (7c) gives

$$\mu_k(A \cap U) = at.$$

Therefore

$$\begin{aligned} \mu_k(A \triangle U) &= \mu_k(A) + \mu_k(U) - 2\mu_k(A \cap U) \\ &= a + t - 2at. \end{aligned} \tag{7e}$$

If $0 < a \leq 1/2$, then $a + t - 2at \geq a$. If $1/2 \leq a < 1$, then $a + t - 2at \geq 1 - a$. Thus in all cases

$$\mu_k(A \triangle U) \geq \min\{a, 1 - a\} > 0,$$

which contradicts (7d) for $\varepsilon < \min\{a, 1 - a\}$. Hence $\mu_k(A)$ cannot lie strictly between 0 and 1. The system is therefore ergodic.

Corollary 1. For an isometry $f : S_{p^{-k}}(a) \rightarrow S_{p^{-k}}(a)$, the following statements are equivalent:

- (i) $(S_{p^{-k}}(a), f, \mu_k)$ is ergodic;
- (ii) the induced permutation on $X_{n,k}$ consists of a single cycle for every $n \geq 1$;
- (iii) every orbit is dense in $S_{p^{-k}}(a)$;
- (iv) at least one orbit is dense in $S_{p^{-k}}(a)$.

If these conditions hold, μ_k is the unique invariant Borel probability measure.

Proof. The equivalence (i) $\Leftrightarrow$ (ii) is precisely Theorem 5.

Assume (ii) and fix $x \in S_{p^{-k}}(a)$. Let B be any nonempty open subset of the sphere. Since the partition balls form a base of the p -adic topology, there exist $n \geq 1$ and an atom $C \in X_{n,k}$ such that $C \subset B$. The induced permutation on $X_{n,k}$ is a single cycle, so the orbit of the atom $B_n(x)$ eventually reaches C . Hence $f^j(x) \in C \subset B$ for some $j \geq 0$. Thus the orbit of every point is dense, and (ii) $\Rightarrow$ (iii).

The implication (iii) $\Rightarrow$ (iv) is immediate. Now assume (iv) and let the orbit of x_* be dense. For a fixed n , this orbit meets every atom of $X_{n,k}$. The sequence of atoms containing $f^j(x_*)$ is exactly the orbit of $B_n(x_*)$ under the induced finite permutation. Since it visits all atoms, that permutation consists of a single cycle. This holds for every n , so (iv) $\Rightarrow$ (ii).

It remains to prove unique ergodicity. Let ν be any invariant Borel probability measure on the sphere. Fix n . If $B, C \in X_{n,k}$, then $C = f^j(B)$ for some j . Since f is bijective, invariance under inverse images also gives invariance under forward images: for every Borel set E ,

$$\nu(f(E)) = \nu(f^{-1}(f(E))) = \nu(E).$$

Applying this identity repeatedly, we obtain

$$\nu(C) = \nu(f^j(B)) = \nu(B).$$

Thus all N_n atoms have the same ν -measure, necessarily $1/N_n$. The normalized Haar measure μ_k has the same value on every atom of every partition.

Let $\mathcal{P}$ consist of the empty set, the whole sphere, and all atoms of all partitions $X_{n,k}$. Any two nonempty members of $\mathcal{P}$ are either disjoint or one contains the other; hence $\mathcal{P}$ is a π -system. It is also a base of clopen sets and therefore generates the Borel σ -algebra. Since ν and μ_k agree on $\mathcal{P}$, the uniqueness theorem for probability measures on a generating π -system gives $\nu = \mu_k$.

We now give a metric form of the criterion. Recall the number N_n defined in (7). For $x \in S_{p^{-k}}(a)$, denote by $B_n(x)$ the unique element of $X_{n,k}$ containing x . If the induced permutation on $X_{n,k}$ consists of a single cycle, then after N_n iterates the ball $B_n(x)$ returns to itself. The return map therefore permutes the p balls of level $n + 1$ contained in $B_n(x)$.

Lemma 2. Assume that the induced permutation on $X_{n,k}$ consists of a single cycle. Then the following statements are equivalent:

- (i) the induced permutation on $X_{n+1,k}$ consists of a single cycle;

- (ii) the return map f^{N_n} is a p -cycle on the p balls of level $n+1$ contained in one element of $X_{n,k}$;
 (iii) for one $x \in S_{p-k}(a)$ one has

$$|f^{jN_n}(x) - x|_p = p^{-(k+n)}, \quad j = 1, \dots, p-1;$$

- (iv) the same equalities hold for every $x \in S_{p-k}(a)$.

Proof. Fix a parent ball $B \in X_{n,k}$ and denote by $\mathcal{C}(B)$ the set of its p children in $X_{n+1,k}$. Since the level- n permutation is a single cycle of length N_n , we have

$$f^{N_n}(B) = B.$$

Therefore f^{N_n} induces a permutation

$$R_B : \mathcal{C}(B) \longrightarrow \mathcal{C}(B), \quad R_B(C) = f^{N_n}(C),$$

which will be called the return permutation over B .

The return permutations over different parent balls have the same cycle structure. Indeed, if $B' = f^q(B)$, then the map

$$\Phi_q : \mathcal{C}(B) \rightarrow \mathcal{C}(B'), \quad \Phi_q(C) = f^q(C),$$

is a bijection, and the commutativity of the iterates of f gives

$$R_{B'} = \Phi_q \circ R_B \circ \Phi_q^{-1}. \quad (8a)$$

Thus the return permutations are conjugate.

We first prove (i) $\Leftrightarrow$ (ii). Let $C \in \mathcal{C}(B)$ and let ℓ be the length of the cycle of C under R_B . We claim that the cycle of C under the full level- $(n+1)$ permutation has length exactly ℓN_n . Indeed, if $f^q(C) = C$, then projection to the parent level gives $f^q(B) = B$. Since the parent-level cycle has length N_n , this forces $q = tN_n$ for some integer $t \geq 1$. The equality $f^{tN_n}(C) = C$ is exactly $R_B^t(C) = C$, whose least positive solution is $t = \ell$. Therefore the least return time of C is ℓN_n .

There are exactly pN_n atoms at level $n+1$. Hence the full level- $(n+1)$ permutation consists of a single cycle if and only if $\ell N_n = pN_n$, or equivalently $\ell = p$. The latter condition says that R_B is a p -cycle. By (8a), it is enough to check this over one parent ball. This proves (i) $\Leftrightarrow$ (ii).

We next relate the return permutation to the metric equalities. Fix $x \in S_{p-k}(a)$, put $B = B_n(x)$, and let $C = B_{n+1}(x) \in \mathcal{C}(B)$. For every $j \geq 1$, both x and $f^{jN_n}(x)$ belong to the parent ball B , while $f^{jN_n}(x)$ belongs to the child $R_B^j(C)$. Two points in the same level- n ball belong to the same level- $(n+1)$ child exactly when their distance is at most $p^{-(k+n+1)}$. If they belong to different children, their distance is larger than $p^{-(k+n+1)}$ but no larger than $p^{-(k+n)}$. Since the nonzero values of the p -adic norm are integral powers of p , the only possibility is that their distance equals $p^{-(k+n)}$. Hence

$$|f^{jN_n}(x) - x|_p = p^{-(k+n)} \iff R_B^j(C) \neq C. \quad (8b)$$

The equalities in (iii) hold for $j = 1, \dots, p-1$ if and only if the child C does not return to itself during the first $p-1$ iterates of R_B . Since R_B acts on a set of p elements, this is equivalent to saying that the orbit of C has length p . In that case R_B is a p -cycle. Thus (iii) $\Leftrightarrow$ (ii).

Finally, if (iii) holds for one point, then R_B is a p -cycle. By (8a), every return permutation is a p -cycle, and (8b) then holds for every child and every point. Hence (iii) $\Rightarrow$ (iv). The reverse implication is immediate. All four statements are therefore equivalent.

Theorem 6. Fix $x_* \in S_{p-k}(a)$. All finite dynamical systems $(X_{n,k}, f, \nu_{n,k})$ are ergodic if and only if the following conditions hold:

- (i) $(X_{1,k}, f, \nu_{1,k})$ is ergodic;
 (ii) for every $n \geq 1$ and every $j = 1, \dots, p-1$,

$$|f^{jN_n}(x_*) - x_*|_p = p^{-(k+n)}. \quad (8)$$

Proof. Assume first that every finite factor is ergodic. Then, by Remark 3, the induced permutations on both $X_{n,k}$ and $X_{n+1,k}$ are single cycles for every $n \geq 1$. Condition (i) is immediate. Applying Lemma 2 to the fixed point x_* gives

$$|f^{jN_n}(x_*) - x_*|_p = p^{-(k+n)}, \quad j = 1, \dots, p-1,$$

for every n , so condition (ii) also holds.

Conversely, assume (i) and (ii). By (i), the induced permutation on $X_{1,k}$ is a single cycle. Suppose inductively that the permutation on $X_{n,k}$ is a single cycle. Condition (ii) supplies exactly the metric equalities in part (iii) of

Lemma 2 at the point x_* . Lemma 2 therefore implies that the permutation on $X_{n+1,k}$ is also a single cycle. Starting from $n = 1$ and repeating this argument proves by induction that every finite factor is ergodic.

Remark 4. For $p \geq 5$, checking only the equality in (8) corresponding to $j = 1$ at a single point is not sufficient: it merely shows that the child containing that point is not fixed. Even if the $j = 1$ equality is verified at every point, it only excludes fixed children, whereas a fixed-point-free permutation of p elements may decompose into several cycles. The full family $j = 1, \dots, p-1$ in (8) is therefore a simple uniform criterion, although it is not claimed to be a minimal set of tests. For $p = 2$, the single value $j = 1$ is sufficient. For $p = 3$, the equality corresponding to $j = 2$ at one point already rules out return cycles of lengths 1 and 2 and is therefore sufficient; alternatively, one may check the $j = 1$ equality at every point.

Combining Theorems 5 and 6, we obtain the following criterion for the case $\rho(r) = r$.

Corollary 2. Let $f : S_{p-k}(a) \rightarrow S_{p-k}(a)$ be an isometry and assume that

$$|f(x) - x|_p = p^{-k}, \quad x \in S_{p-k}(a).$$

Then the system $(S_{p-k}(a), f, \mu_k)$ is ergodic if and only if conditions (i) and (ii) of Theorem 6 hold.

Proof. By Theorem 5, ergodicity of the system on the sphere is equivalent to ergodicity of all finite factors. By Theorem 6, the latter is equivalent to conditions (i) and (ii) stated there. The displacement assumption identifies this result with the boundary case $\rho(r) = r$.

For $p = 3$ the criterion has the following convenient form.

Corollary 3. Let $p = 3$ and assume that $|f(x) - x|_3 = 3^{-k}$ on $S_{3-k}(a)$. Then the first finite factor is the transposition of its two balls. The system is ergodic if and only if, for one point $x_* \in S_{3-k}(a)$ and every $n \geq 1$,

$$|f^{2 \cdot 3^{n-1}}(x_*) - x_*|_3 = |f^{4 \cdot 3^{n-1}}(x_*) - x_*|_3 = 3^{-(k+n)}. \quad (9)$$

Proof. The first partition of a 3-adic sphere has $p-1 = 2$ atoms, each of radius $3^{-(k+1)}$. If x and $f(x)$ belonged to the same atom, then

$$|f(x) - x|_3 \leq 3^{-(k+1)},$$

contrary to the assumption $|f(x) - x|_3 = 3^{-k}$. Thus every point is sent to the other atom, and the induced permutation on $X_{1,k}$ is the transposition. In particular, condition (i) of Theorem 6 is automatic. For $p = 3$ we have

$$N_n = (p-1)p^{n-1} = 2 \cdot 3^{n-1}$$

and $j = 1, 2$. Substituting these values into condition (ii) of Theorem 6 gives exactly the two equalities in (9). The assertion now follows from Corollary 2.

The following proposition will be used in the 3-adic examples.

Proposition 1. Let $m \geq 2$ and suppose that T is analytic on a ball containing $V_{3-m}(x_*)$. Assume that on this ball T has a Taylor expansion

$$T(x_* + h) = T(x_*) + Ah + \sum_{q=2}^{\infty} c_q h^q, \quad A, c_q \in \mathbb{Z}_3,$$

and that

$$\nu_3(T(x_*) - x_*) = m, \quad A = T'(x_*) \equiv 1 \pmod{3}. \quad (10)$$

Then

$$\nu_3(T^j(x_*) - x_*) = m, \quad j = 1, 2, \quad (11)$$

and

$$\nu_3(T^3(x_*) - x_*) = m + 1, \quad (T^3)'(x_*) \equiv 1 \pmod{3}. \quad (12)$$

Proof. Put

$$d = T(x_*) - x_*, \quad A = T'(x_*).$$

By assumption, $\nu_3(d) = m$. Since all higher Taylor coefficients are 3-adic integers, the expansion may be written in the form

$$T(x_* + h) = T(x_*) + Ah + h^2 E(h), \quad E(h) \in \mathbb{Z}_3 \quad (13)$$

whenever $\nu_3(h) \geq m$.

Taking $h = d$ in (13), we obtain

$$\begin{aligned} T^2(x_*) - x_* &= d + Ad + d^2 E(d) \\ &= d(1 + A + dE(d)). \end{aligned}$$

Now $A \equiv 1 \pmod{3}$ and $d \equiv 0 \pmod{3}$, so

$$1 + A + dE(d) \equiv 2 \pmod{3}.$$

The factor in parentheses is therefore a 3-adic unit. Hence

$$\nu_3(T^2(x_*) - x_*) = \nu_3(d) = m.$$

Together with the hypothesis for $j = 1$, this proves (11).

Let $h_2 = T^2(x_*) - x_*$. The preceding calculation gives

$$h_2 = d(1 + A) + O(d^2),$$

where $O(d^2)$ denotes a term of valuation at least $2m$. Applying (13) to $h = h_2$ yields

$$\begin{aligned} T^3(x_*) - x_* &= d + Ah_2 + O(h_2^2) \\ &= d(1 + A + A^2) + O(d^2). \end{aligned}$$

Write $A = 1 + 3c$ with $c \in \mathbb{Z}_3$. Then

$$1 + A + A^2 = 3(1 + 3c + 3c^2),$$

so

$$\nu_3(1 + A + A^2) = 1.$$

Consequently, the main term $d(1 + A + A^2)$ has valuation $m + 1$. The remainder has valuation at least $2m$, and

$$2m \geq m + 2 > m + 1$$

because $m \geq 2$. The remainder therefore cannot cancel the main term, and

$$\nu_3(T^3(x_*) - x_*) = m + 1.$$

It remains to calculate the derivative. Differentiating the Taylor series shows that, for $\nu_3(h) \geq 1$,

$$T'(x_* + h) = A + hE_1(h), \quad E_1(h) \in \mathbb{Z}_3.$$

The points $T(x_*)$ and $T^2(x_*)$ differ from x_* by elements of valuation $m \geq 2$. Hence

$$T'(T(x_*)) \equiv T'(T^2(x_*)) \equiv A \equiv 1 \pmod{3}.$$

By the chain rule,

$$(T^3)'(x_*) = T'(T^2(x_*)) T'(T(x_*)) T'(x_*) \equiv 1 \pmod{3}.$$

This proves (12).

Example 1.

Let

$$p = 3, \quad k \in \mathbb{Z},$$

and consider the sphere $S_{3^{-k}}(a)$, where $a \in \mathbb{Q}_3$. Define

$$f(x) = a + 2(x - a). \tag{14}$$

For $x \in S_{3^{-k}}(a)$,

$$|f(x) - a|_3 = |2|_3 |x - a|_3 = 3^{-k},$$

so the sphere is invariant. For any x, y on the sphere,

$$|f(x) - f(y)|_3 = |2(x - y)|_3 = |x - y|_3,$$

and hence f is an isometry. Moreover,

$$|f(x) - x|_3 = |x - a|_3 = 3^{-k},$$

so this is exactly the case $\rho(r) = r$.

Under the coordinate change $x = a + 3^k u$, the sphere is identified with $\mathbb{Z}_3^*$ and the map becomes $u \mapsto 2u$. On the first finite factor the two possible nonzero residues are 1 and 2, and

$$1 \mapsto 2, \quad 2 \mapsto 1.$$

Thus the first finite factor consists of a single cycle.

For $N_n = 2 \cdot 3^{n-1}$, direct iteration of (14) gives

$$f^{jN_n}(x) - x = (2^{jN_n} - 1)(x - a), \quad j = 1, 2. \quad (15)$$

Since

$$|x - a|_3 = 3^{-k}, \quad (16)$$

it remains to calculate the valuation of $2^{jN_n} - 1$. We have

$$2^{jN_n} - 1 = 4^{j3^{n-1}} - 1.$$

The lifting-the-exponent formula for the odd prime 3 gives

$$\begin{aligned} \nu_3(4^{j3^{n-1}} - 1) &= \nu_3(4^j - 1) + \nu_3(3^{n-1}) \\ &= 1 + (n - 1) = n, \end{aligned}$$

for $j = 1, 2$, because $4 - 1 = 3$ and $4^2 - 1 = 15$ both have 3-adic valuation one. Therefore

$$\nu_3(2^{jN_n} - 1) = n, \quad j = 1, 2. \quad (17)$$

Combining (15)–(17), we obtain

$$|f^{jN_n}(x) - x|_3 = |2^{jN_n} - 1|_3 |x - a|_3 = 3^{-n} 3^{-k} = 3^{-(k+n)}, \quad j = 1, 2.$$

All conditions of Corollary 3 are satisfied. Hence

$$(S_{3-k}(a), f, \mu_k)$$

is ergodic. By Corollary 1, it is also minimal and uniquely ergodic.

Example 2.

Let

$$p = 3, \quad k \geq 1, \quad \theta \in \mathbb{Q}_3, \quad |\theta|_3 > 1,$$

and consider

$$f(x) = \frac{\theta^2 x^2 + 2x + 1}{x^2 + 2x + \theta^2}, \quad x \in S_{3-k}(1). \quad (18)$$

Put

$$u = x - 1, \quad |u|_3 = 3^{-k}, \quad \lambda = \theta^{-2}.$$

Since $|\theta|_3 > 1$, we have $\nu_3(\lambda) \geq 2$, and hence $\lambda \in 9\mathbb{Z}_3$. A direct substitution gives

$$f(1 + u) - 1 = u \frac{(1 - \lambda)(2 + u)}{1 + \lambda(3 + 4u + u^2)}. \quad (19)$$

We first verify invariance. Since $u \in 3\mathbb{Z}_3$, the numbers $1 - \lambda$ and $2 + u$ are 3-adic units. Also,

$$|\lambda(3 + 4u + u^2)|_3 < 1,$$

so the denominator in (19) is a unit. Therefore

$$|f(x) - 1|_3 = |u|_3 = 3^{-k}.$$

Thus $S_{3-k}(1)$ is invariant.

We next prove that f is an isometry on this sphere. For $x, y \in S_{3-k}(1)$, a direct calculation gives

$$f(x) - f(y) = \frac{(\theta - 1)(\theta + 1)(x - y)(\theta^2 x + \theta^2 y + 2xy + x + y + 2)}{(\theta^2 + x^2 + 2x)(\theta^2 + y^2 + 2y)}. \quad (20)$$

Because $x \equiv y \equiv 1 \pmod{3}$, we have

$$|x|_3 = |y|_3 = |x + y|_3 = 1.$$

Moreover, $|\theta \pm 1|_3 = |\theta|_3$ and

$$|\theta^2 + x^2 + 2x|_3 = |\theta^2 + y^2 + 2y|_3 = |\theta|_3^2.$$

In the remaining factor of the numerator, the term $\theta^2(x+y)$ has norm $|\theta|_3^2$, whereas $2xy + x + y + 2$ has norm at most 1. Hence

$$|\theta^2x + \theta^2y + 2xy + x + y + 2|_3 = |\theta|_3^2.$$

Taking norms in (20), all powers of $|\theta|_3$ cancel and we obtain

$$|f(x) - f(y)|_3 = |x - y|_3.$$

Thus f is an isometry.

Formula (19) also gives the displacement. The quotient multiplying u is congruent to 2 modulo 3, so that quotient minus 1 is a unit. Hence

$$|f(x) - x|_3 = |f(1+u) - 1 - u|_3 = |u|_3 = 3^{-k}.$$

Reducing (19) modulo 3^{k+1} gives

$$f(1+u) - 1 \equiv 2u \pmod{3^{k+1}}. \quad (21)$$

Indeed, $u^2 \in 3^{2k}\mathbb{Z}_3 \subset 3^{k+1}\mathbb{Z}_3$, while every term containing λu belongs to $3^{k+2}\mathbb{Z}_3$. After writing $u = 3^k v$ with $v \in \mathbb{Z}_3^*$, congruence (21) becomes $v \mapsto 2v$ modulo 3. Therefore the first finite factor is the transposition of its two atoms.

For the higher-level lifting, set

$$G(u) = f(1+u) - 1 = u \frac{(1-\lambda)(2+u)}{1+\lambda(3+4u+u^2)}. \quad (22)$$

Since the denominator is a unit and $\lambda \in 9\mathbb{Z}_3$, there exists an integral analytic function $E(u)$ such that

$$\frac{1-\lambda}{1+\lambda(3+4u+u^2)} = 1 + \lambda E(u).$$

Consequently,

$$G(u) \equiv 2u + u^2 \pmod{3^{k+2}}. \quad (23)$$

Put $P(u) = 2u + u^2 = u(2+u)$. Since $2+u$ is a unit, $P(u)$ belongs to the same sphere $|u|_3 = 3^{-k}$. By (23), the points $G(u)$ and $P(u)$ are congruent modulo 3^{k+2} . Since G is an isometry on this sphere, $G(G(u))$ and $G(P(u))$ are congruent modulo the same power. Applying (23) once more, now at the point $P(u)$, gives

$$G(P(u)) \equiv P(P(u)) \pmod{3^{k+2}}.$$

Combining the last two congruences, we obtain

$$G^2(u) \equiv P(P(u)) \pmod{3^{k+2}}.$$

A direct calculation gives

$$\begin{aligned} P(P(u)) - u &= 3u + 6u^2 + 4u^3 + u^4 \\ &= 3u(1+2u) + u^3(4+u). \end{aligned}$$

Since $k \geq 1$, we have $\nu_3(u^3) \geq 3k \geq k+2$. Thus, for $H = G^2$,

$$H(u) - u \equiv 3u(1+2u) \pmod{3^{k+2}}. \quad (24)$$

The factor $1+2u$ is a unit, so

$$\nu_3(H(u) - u) = k+1.$$

A direct differentiation of (22), followed by reduction modulo 3, gives

$$G'(u) \equiv 2 \pmod{3}.$$

Since $G(u)$ also has valuation k , the chain rule yields

$$H'(u) = G'(G(u))G'(u) \equiv 2^2 \equiv 1 \pmod{3}. \quad (25)$$

The denominator in (22) is a unit on $3\mathbb{Z}_3$. To justify the integrality assertion used below, fix $u_0 \in 3\mathbb{Z}_3$ and write the denominator at $u_0 + h$ as $D(u_0) + hD_1(h)$, where $D(u_0) \in \mathbb{Z}_3^*$ and $D_1(h) \in \mathbb{Z}_3[[h]]$. Then

$$\frac{1}{D(u_0 + h)} = \frac{1}{D(u_0)} \sum_{q=0}^{\infty} \left(-\frac{hD_1(h)}{D(u_0)} \right)^q,$$

so the local Taylor coefficients of G belong to $\mathbb{Z}_3$. The same property is preserved under composition; hence it holds for every iterate of G . Define

$$R_1 = H = G^2, \quad R_{n+1} = R_n^3 = G^{2 \cdot 3^n}.$$

For R_1 , formulae (24)–(25) give

$$\nu_3(R_1(u) - u) = k + 1, \quad R_1'(u) \equiv 1 \pmod{3}.$$

Applying Proposition 1 to R_1 gives the same valuation for $R_1^2(u) - u$ and increases the valuation by one for $R_2(u) - u = R_1^3(u) - u$; it also gives $R_2'(u) \equiv 1 \pmod{3}$. Repeating the same argument inductively yields

$$R_n = G^{2 \cdot 3^{n-1}}$$

and, for every $n \geq 1$ and $j = 1, 2$,

$$|R_n^j(u) - u|_3 = 3^{-(k+n)}. \quad (26)$$

Corollary 3 now implies that

$$(S_{3^{-k}}(1), f, \mu_k)$$

is ergodic. By Corollary 1, it is also minimal and uniquely ergodic.

The same rational function has another invariant sphere on which its behaviour is different. Let $|\theta|_3 = 3^m$, $m \geq 1$, and write $\theta = 3^{-m}\tau$ with $\tau \in \mathbb{Z}_3^*$. Every point of $S_{|\theta|_3^2}(1)$ has the form

$$x = 1 + 3^{-2m}y, \quad y \in \mathbb{Z}_3^*.$$

A direct calculation gives

$$3^{2m}(f(x) - 1) = \frac{y(y + 2 \cdot 3^{2m})(\tau^2 - 3^{2m})}{y^2 + 3^{2m}(\tau^2 + 4y) + 3^{4m+1}}. \quad (27)$$

The denominator is congruent to y^2 modulo 3, while the numerator is congruent to $y^2\tau^2$. Since every unit square is congruent to 1 modulo 3, we obtain

$$3^{2m}(f(x) - 1) \equiv 1 \pmod{3}. \quad (28)$$

In the normalized coordinate y , the first partition has two atoms, corresponding to $y \equiv 1$ and $y \equiv 2 \pmod{3}$. Formula (28) shows that the image of the entire sphere is contained in the first of these atoms. That atom has normalized Haar measure $1/2$. If it is denoted by B , then $f^{-1}(B)$ is the whole sphere, and hence

$$\mu(f^{-1}(B)) = 1 \neq \frac{1}{2} = \mu(B).$$

Thus f is not measure-preserving on $S_{|\theta|_3^2}(1)$ and therefore cannot be ergodic there.

Remark 5. For any odd prime p , the same reduction of (19) gives multiplication by 2 on the first factor $\mathbb{F}_p^*$. Therefore a necessary condition for ergodicity is that the residue class of 2 generate the cyclic group $\mathbb{F}_p^*$.

Remark 6. Consider the affine isometry

$$f(x) = a + g(x - a), \quad g \in \mathbb{Z}_p^*.$$

After the normalization $x = a + p^k u$, the map becomes $u \mapsto gu$ on $\mathbb{Z}_p^*$. The atoms of the level- n partition correspond exactly to the unit residue classes modulo p^n , and the induced permutation is multiplication by g on $(\mathbb{Z}/p^n\mathbb{Z})^*$. It is a single cycle precisely when the class of g has order equal to the number of units, namely

$$\text{ord}_{p^n}(g) = (p - 1)p^{n-1}.$$

Consequently, the affine system is ergodic if and only if this equality holds for every $n \geq 1$.

Applications to rational functions

We now apply Theorem 6 to two classes of rational functions. In both classes the displacement on the considered invariant sphere is equal to the radius.

First consider the nondegenerate $(2, 2)$ -rational function

$$f(x) = \frac{ax^2 + bx}{x^2 + dx + b}, \quad ab \neq 0, \quad a \neq d, \quad a^2 - ad + b \neq 0. \quad (29)$$

Assume that the denominator splits over $\mathbb{Q}_p$, and let $\xi_1, \xi_2 \in \mathbb{Q}_p$ be its poles. Put

$$\alpha = \min\{|\xi_1|_p, |\xi_2|_p\}, \quad \beta = \max\{|\xi_1|_p, |\xi_2|_p\}, \quad \delta = |a|_p, \quad s = |a - d|_p.$$

The fixed points are 0 and $a - d$. If the denominator is irreducible over $\mathbb{Q}_p$, then its two roots are conjugate over $\mathbb{Q}_p$. The p -adic absolute value has a unique extension to every finite algebraic extension, so conjugation preserves that absolute value. The two roots therefore have the same norm. Hence the strict inequality $\alpha < \beta$ considered below can occur only in the split case.

Theorem 7. Let $p \geq 3$, let $\xi_1, \xi_2 \in \mathbb{Q}_p$, and assume that

$$\delta = \beta, \quad \alpha < r < \beta, \quad s = \beta. \quad (30)$$

Choose $\lambda \in \mathbb{Q}_p^*$ with $|\lambda|_p = r$ and define

$$G_\lambda(u) = \lambda^{-1} f(\lambda u) = \frac{a\lambda u^2 + bu}{\lambda^2 u^2 + d\lambda u + b}. \quad (31)$$

Fix $u_* \in \mathbb{Z}_p^*$ and put $N_n = (p-1)p^{n-1}$. Then $(S_r(0), f, \mu_{0,r})$ is ergodic if and only if

- (i) the residue class $\overline{a/d}$ is a generator of $\mathbb{F}_p^*$;
- (ii) for every $n \geq 1$ and every $j = 1, \dots, p-1$,

$$|G_\lambda^{jN_n}(u_*) - u_*|_p = p^{-n}. \quad (32)$$

If these conditions hold, the system is minimal and uniquely ergodic.

Proof. The invariant spheres of this family were studied in [17]. For completeness, we verify directly all properties needed here.

Relabel the poles, if necessary, so that

$$|\xi_1|_p = \alpha, \quad |\xi_2|_p = \beta.$$

Since

$$x^2 + dx + b = (x - \xi_1)(x - \xi_2),$$

Vieta's formulae and the strict inequality $\alpha < \beta$ give

$$|d|_p = |\xi_1 + \xi_2|_p = \beta, \quad |b|_p = |\xi_1 \xi_2|_p = \alpha\beta. \quad (32a)$$

Let $x \in S_r(0)$. From $\alpha < r < \beta$ we obtain

$$|x - \xi_1|_p = r, \quad |x - \xi_2|_p = \beta. \quad (32b)$$

Also,

$$|ax|_p = \beta r > |b|_p = \alpha\beta,$$

so $|ax + b|_p = \beta r$. Consequently,

$$|f(x)|_p = \frac{|x|_p |ax + b|_p}{|x - \xi_1|_p |x - \xi_2|_p} = \frac{r \cdot \beta r}{r\beta} = r.$$

Thus $S_r(0)$ is invariant.

We now prove the isometry property. A direct subtraction gives

$$f(x) - f(y) = \frac{(x - y)Q(x, y)}{(x^2 + dx + b)(y^2 + dy + b)}, \quad (32c)$$

where

$$Q(x, y) = ab(x + y) + adxy + b^2 - bxy.$$

For $x, y \in S_r(0)$, the term $adxy$ has norm

$$|adxy|_p = \beta^2 r^2.$$

All other terms have strictly smaller norm:

$$\begin{aligned} |ab(x + y)|_p &\leq \alpha\beta^2 r < \beta^2 r^2, \\ |b^2|_p &= \alpha^2 \beta^2 < \beta^2 r^2, \\ |bxy|_p &= \alpha\beta r^2 < \beta^2 r^2. \end{aligned}$$

Therefore $|Q(x, y)|_p = \beta^2 r^2$. By (32b), each denominator in (32c) has norm $r\beta$, and hence

$$|f(x) - f(y)|_p = |x - y|_p.$$

Thus the restriction of f to $S_r(0)$ is an isometry.

The displacement is obtained from the identity

$$f(x) - x = \frac{x^2(a - d - x)}{(x - \xi_1)(x - \xi_2)}.$$

Since $|a - d|_p = \beta > r$, we have $|a - d - x|_p = \beta$. Together with (32b), this gives

$$|f(x) - x|_p = \frac{r^2\beta}{r\beta} = r. \quad (32d)$$

Now consider the affine isometry

$$h_\lambda : \mathbb{Z}_p^* \rightarrow S_r(0), \quad h_\lambda(u) = \lambda u.$$

It maps the canonical partitions of $\mathbb{Z}_p^*$ onto the partitions of $S_r(0)$ and transports normalized Haar measure to $\mu_{0,r}$. The conjugate map is

$$h_\lambda^{-1} \circ f \circ h_\lambda = G_\lambda.$$

Using (32a), we rewrite (31) as

$$G_\lambda(u) = \frac{a}{d}u \frac{1 + b/(a\lambda u)}{1 + \lambda u/d + b/(d\lambda u)}.$$

For $u \in \mathbb{Z}_p^*$, the correction terms satisfy

$$\left| \frac{b}{a\lambda u} \right|_p = \left| \frac{b}{d\lambda u} \right|_p = \frac{\alpha}{r} < 1, \quad \left| \frac{\lambda u}{d} \right|_p = \frac{r}{\beta} < 1.$$

Thus both correction factors are congruent to 1 modulo p , and

$$G_\lambda(u) \equiv \frac{a}{d}u \pmod{p}. \quad (32e)$$

Since $|a/d|_p = 1$, multiplication by $\overline{a/d}$ permutes $\mathbb{F}_p^*$. This permutation is a single cycle if and only if $\overline{a/d}$ generates the cyclic group $\mathbb{F}_p^*$. Hence condition (i) is exactly the first-level condition of Theorem 6.

Under the conjugacy h_λ , the metric equality

$$|G_\lambda^{jN_n}(u_*) - u_*|_p = p^{-n}$$

is equivalent to

$$|f^{jN_n}(h_\lambda(u_*)) - h_\lambda(u_*)|_p = rp^{-n}.$$

Therefore condition (ii) is precisely the cyclic-lifting condition of Theorem 6 for the original sphere. Theorem 5 and Theorem 6 now prove the equivalence stated in the theorem. Finally, Corollary 1 gives minimality and unique ergodicity.

Example 3.

Consider

$$f(x) = \frac{2x^2 + 9x}{x^2 + x + 9} \quad \text{on } S_{1/3}(0) \subset \mathbb{Q}_3. \quad (33)$$

We verify the conditions of Theorem 7 with the fixed normalized point $u_* = 1$.

The discriminant of the denominator is

$$\Delta = 1 - 36 = -35.$$

Since $-35 \equiv 1 \pmod{3}$ and the derivative of $z^2 + 35$ at $z = 1$ is $2 \not\equiv 0 \pmod{3}$, Hensel's lemma gives a square root of -35 in $\mathbb{Q}_3$. Hence the denominator splits over $\mathbb{Q}_3$. Its two roots have sum -1 and product 9 . Suppose that their norms were equal to some number t . From the product relation we would have $t^2 = |9|_3 = 1/9$, hence $t = 1/3$. The ultrametric inequality would then give

$$|\xi_1 + \xi_2|_3 \leq \frac{1}{3},$$

contrary to $|\xi_1 + \xi_2|_3 = |-1|_3 = 1$. Thus the two norms are unequal. For two summands of unequal norm, the norm of their sum equals the larger norm; consequently the larger root norm is 1. The product relation then gives the smaller root norm $1/9$. Therefore

$$\alpha = \frac{1}{9}, \quad \beta = 1.$$

Furthermore,

$$\delta = |2|_3 = 1, \quad s = |2 - 1|_3 = 1.$$

Thus (30) holds with $r = 1/3$.

Putting $x = 3u$ gives the normalized map

$$G(u) = \frac{u(2u + 3)}{3u^2 + u + 3}. \quad (34)$$

For $u \in \mathbb{Z}_3^*$, the denominator is congruent to u modulo 3 and is therefore a unit. Reducing (34) modulo 3 yields

$$G(u) \equiv 2u \pmod{3}.$$

Thus the first factor is the transposition of 1 and 2.

At the next modulus, direct substitution gives the single cycle

$$1 \mapsto 2 \mapsto 4 \mapsto 8 \mapsto 7 \mapsto 5 \mapsto 1 \pmod{9}. \quad (35)$$

In particular,

$$G^2(1) \equiv 4 \pmod{9}, \quad G^4(1) \equiv 7 \pmod{9},$$

so

$$\nu_3(G^2(1) - 1) = \nu_3(G^4(1) - 1) = 1. \quad (36)$$

These are exactly the two return equalities required at level $n = 1$.

A direct calculation modulo 27 gives

$$G^6(1) \equiv 10 \pmod{27}.$$

Hence

$$\nu_3(G^6(1) - 1) = \nu_3(9) = 2. \quad (36a)$$

Moreover,

$$G'(u) = -\frac{7u^2 - 12u - 9}{(3u^2 + u + 3)^2} \equiv 2 \pmod{3}. \quad (37)$$

Every iterate of a unit remains a unit because $G(u) \equiv 2u \pmod{3}$. Therefore, by the chain rule,

$$(G^6)'(1) = \prod_{i=0}^5 G'(G^i(1)) \equiv 2^6 \equiv 1 \pmod{3}. \quad (37a)$$

The denominator of G is a unit on $\mathbb{Z}_3^*$. Expanding its reciprocal by the same geometric-series argument used in Example 2 shows that the Taylor coefficients of G on every unit residue ball belong to $\mathbb{Z}_3$. This property is preserved under composition, so it also holds for all iterates of G .

Set

$$R_2 = G^6.$$

Equations (36a) and (37a) allow us to apply Proposition 1 to R_2 at the point 1 with $m = 2$. We obtain

$$\nu_3(R_2^j(1) - 1) = 2, \quad j = 1, 2,$$

and $R_3 = R_2^3 = G^{18}$ satisfies

$$\nu_3(R_3(1) - 1) = 3, \quad R_3'(1) \equiv 1 \pmod{3}.$$

Repeating this argument inductively with

$$R_n = G^{2 \cdot 3^{n-1}}$$

gives

$$\nu_3(R_n^j(1) - 1) = n, \quad n \geq 2, \quad j = 1, 2. \quad (38)$$

Together with (36), these equalities verify condition (ii) of Theorem 7 for every $n \geq 1$. Consequently,

$$(S_{1/3}(0), f, \mu_{0,1/3})$$

is ergodic, minimal and uniquely ergodic.

Now consider the $(2, 1)$ -rational function

$$f(x) = \frac{x^2 + ax + b}{x + c}, \quad a \neq c, \quad c^2 - ac + b \neq 0. \quad (39)$$

Its unique fixed point is

$$x_0 = \frac{b}{c-a}.$$

Put

$$\eta = x_0 + a, \quad \zeta = x_0 + c, \quad \alpha = |\eta|_p, \quad \beta = |\zeta|_p.$$

Theorem 8. Let $p \geq 3$ and assume that

$$\alpha = \beta, \quad |a - c|_p = \beta, \quad 0 < r < \beta. \quad (40)$$

Choose $\lambda \in \mathbb{Q}_p^*$ with $|\lambda|_p = r$ and define

$$G_\lambda(t) = \lambda^{-1}(f(x_0 + \lambda t) - x_0) = t \frac{\lambda t + \eta}{\lambda t + \zeta}. \quad (41)$$

Put $q = \eta/\zeta$, fix $t_* \in \mathbb{Z}_p^*$, and let $N_n = (p-1)p^{n-1}$. Then $(S_r(x_0), f, \mu_{x_0, r})$ is ergodic if and only if

- (i) the residue class $\bar{q}$ is a generator of $\mathbb{F}_p^*$;
- (ii) for every $n \geq 1$ and every $j = 1, \dots, p-1$,

$$|G_\lambda^{jN_n}(t_*) - t_*|_p = p^{-n}. \quad (42)$$

If these conditions hold, the system is minimal and uniquely ergodic.

Proof. The invariant spheres of this family were studied in [14]. We give a direct verification of all properties used in the ergodicity criterion.

Write

$$u = x - x_0.$$

A direct substitution of $x = x_0 + u$ into (39), using $x_0 = b/(c-a)$, gives the centred identity

$$f(x) - x_0 = u \frac{u + \eta}{u + \zeta}. \quad (42a)$$

Let $x \in S_r(x_0)$, so $|u|_p = r$. Since $r < |\eta|_p = |\zeta|_p = \beta$, we have

$$|u + \eta|_p = |u + \zeta|_p = \beta.$$

Taking norms in (42a) gives

$$|f(x) - x_0|_p = r,$$

so the sphere is invariant.

We next prove that the restriction is an isometry. For $u = x - x_0$ and $v = y - x_0$, subtraction of the two expressions in (42a) gives

$$f(x) - f(y) = (u - v) \frac{\eta\zeta + uv + \zeta(u + v)}{(u + \zeta)(v + \zeta)}. \quad (42b)$$

The term $\eta\zeta$ has norm β^2 , while

$$|uv|_p = r^2 < \beta^2, \quad |\zeta(u + v)|_p \leq \beta r < \beta^2.$$

Thus the numerator factor in (42b) has norm β^2 , and the denominator also has norm β^2 . Therefore

$$|f(x) - f(y)|_p = |u - v|_p = |x - y|_p.$$

To compute the displacement from x , subtract $u = x - x_0$ from (42a):

$$\begin{aligned} f(x) - x &= u \left(\frac{u + \eta}{u + \zeta} - 1 \right) \\ &= u \frac{\eta - \zeta}{u + \zeta} \\ &= u \frac{a - c}{u + \zeta}. \end{aligned} \quad (42c)$$

Since $|a - c|_p = \beta$ and $|u + \zeta|_p = \beta$, it follows that

$$|f(x) - x|_p = r. \quad (42d)$$

Now define

$$h_\lambda : \mathbb{Z}_p^* \rightarrow S_r(x_0), \quad h_\lambda(t) = x_0 + \lambda t.$$

This is an isometry carrying the canonical partitions and normalized Haar measure of $\mathbb{Z}_p^*$ to those of the sphere. Its conjugate map is $G_\lambda = h_\lambda^{-1} \circ f \circ h_\lambda$, as in (41). Since $|\lambda|_p = r < \beta$, we can write

$$G_\lambda(t) = qt \frac{1 + \lambda t/\eta}{1 + \lambda t/\zeta}, \quad q = \frac{\eta}{\zeta},$$

and both correction terms have norm less than 1. Hence

$$G_\lambda(t) \equiv qt \pmod{p}. \quad (42e)$$

The element q is a unit because $|\eta|_p = |\zeta|_p$. Multiplication by $\bar{q}$ is a single-cycle permutation of $\mathbb{F}_p^*$ exactly when $\bar{q}$ is a generator. Thus condition (i) is the first-level condition of Theorem 6.

Finally,

$$|G_\lambda^{jN_n}(t_*) - t_*|_p = p^{-n}$$

is equivalent, after multiplying by $|\lambda|_p = r$, to the return equality at the point $h_\lambda(t_*)$ on $S_r(x_0)$. Hence condition (ii) is precisely the cyclic-lifting condition of Theorem 6. Theorems 5 and 6 prove the stated ergodicity criterion, and Corollary 1 gives minimality and unique ergodicity.

Example 4.

Consider

$$f(x) = \frac{x^2 + x}{x + 2} \quad \text{on } S_{1/3}(0) \subset \mathbb{Q}_3. \quad (43)$$

Here

$$x_0 = 0, \quad \eta = 1, \quad \zeta = 2,$$

and therefore

$$\alpha = |\eta|_3 = 1, \quad \beta = |\zeta|_3 = 1, \quad |a - c|_3 = |1 - 2|_3 = 1.$$

Thus (40) holds for $r = 1/3$. Taking $\lambda = 3$, we obtain

$$G(t) = \frac{t(3t + 1)}{3t + 2}. \quad (44)$$

The denominator is a unit for every $t \in \mathbb{Z}_3^*$. Modulo 3,

$$G(t) \equiv \frac{t}{2} \equiv 2t \pmod{3},$$

so the first finite factor is the transposition of the two unit residues.

Direct calculations give

$$G^2(1) \equiv 7 \pmod{9}, \quad G^4(1) \equiv 4 \pmod{9}, \quad G^6(1) \equiv 19 \pmod{27}. \quad (45)$$

Therefore

$$\nu_3(G^2(1) - 1) = \nu_3(6) = 1, \quad \nu_3(G^4(1) - 1) = \nu_3(3) = 1,$$

and

$$\nu_3(G^6(1) - 1) = \nu_3(18) = 2. \quad (45a)$$

Moreover,

$$G'(t) = \frac{9t^2 + 12t + 2}{(3t + 2)^2} \equiv 2 \pmod{3}. \quad (46)$$

Since all iterates of 1 are units, the chain rule gives

$$(G^6)'(1) = \prod_{i=0}^5 G'(G^i(1)) \equiv 2^6 \equiv 1 \pmod{3}. \quad (46a)$$

The denominator of G is a unit on $\mathbb{Z}_3^*$. A local geometric-series expansion of its reciprocal therefore gives integral Taylor coefficients on every unit residue ball, and composition preserves this property for all iterates of G .

Set $R_2 = G^6$. By (45a)–(46a), Proposition 1 applies to R_2 at 1 with $m = 2$. It gives

$$\nu_3(R_2^j(1) - 1) = 2, \quad j = 1, 2,$$

and shows that $R_3 = R_2^3$ has displacement valuation 3 and derivative congruent to 1 modulo 3. Repeating the argument for

$$R_n = G^{2 \cdot 3^{n-1}}$$

yields

$$\nu_3(R_n^j(1) - 1) = n, \quad n \geq 2, \quad j = 1, 2.$$

Together with the two level-1 valuations in (45), this verifies condition (42) for every $n \geq 1$. Therefore

$$(S_{1/3}(0), f, \mu_{0,1/3})$$

is ergodic, minimal and uniquely ergodic.

Remark 7. For the function $ax/(x^2 + a)$ and for the $(3, 1)$ -rational family with a unique fixed point, the displacement on every invariant sphere is strictly smaller than its radius; see [15,16]. Hence these families do not give additional cases with $\rho(r) = r$.

Conclusion

In this paper we studied isometric dynamical systems on p -adic spheres in the case $\rho(r) = r$. We established an intrinsic finite-factor criterion for isometries preserving an arbitrary p -adic sphere. This criterion is essential for odd primes: when $p \geq 3$, a sphere is a disjoint union of $p - 1$ maximal balls and its natural normalized model is $\mathbb{Z}_p^*$, not $\mathbb{Z}_p$. Therefore the classical criterion for self-maps of $\mathbb{Z}_p$, or equivalently of a single invariant ball, cannot be applied directly to the whole sphere. We also proved the metric cyclic-lifting criterion: after the first finite factor consists of a single cycle, the return map must act as a p -cycle on the children of every partition ball. A convenient one-point criterion valid uniformly for all primes is obtained by checking the return equalities for $j = 1, \dots, p - 1$.

We applied the criterion to affine isometries and to rational functions. In particular, we described the borderline invariant spheres of $(2, 2)$ -rational functions with two fixed points and the equal-radius boundary spheres of $(2, 1)$ -rational functions with a unique fixed point. The two explicit 3-adic rational examples show that ergodicity in the case $\rho(r) = r$ actually occurs. In all resulting ergodic cases the system is also minimal and uniquely ergodic.

REFERENCES

1. Anashin, V.S. Ergodic transformations in the space of p -adic integers. *p-Adic Mathematical Physics*, AIP Conf. Proc. **826**, Amer. Inst. Phys., Melville, NY, 2006, 3–24.
2. Anashin, V.S.; Khrennikov, A.Yu. *Applied Algebraic Dynamics*. de Gruyter Expositions in Mathematics, Vol. 49, Walter de Gruyter, Berlin–New York, 2009.
3. Anashin, V.S. Non-Archimedean ergodic theory and pseudorandom generators. *The Computer Journal* **53**(4) (2010), 370–392.
4. Anashin, V.S.; Khrennikov, A.Yu.; Yurova, E.I. Ergodicity criteria for non-expanding transformations of 2-adic spheres. *Discrete and Continuous Dynamical Systems* **34**(2) (2014), 367–377.
5. Koblitz, N. *p-adic Numbers, p-adic Analysis and Zeta-Functions*. Springer, Berlin, 1977.
6. Katok, S. *p-Adic Analysis Compared with Real*. AMS, 2007.
7. Khamraev, M.; Mukhamedov, F.M. On a class of rational p -adic dynamical systems. *Journal of Mathematical Analysis and Applications* **315** (2006), 76–89.
8. Khakimov, O.N.; Mukhamedov, F.M. Chaotic behavior of the p -adic Potts–Bethe mapping II. *Ergodic Theory and Dynamical Systems* **42**(11) (2022), 3433–3457.
9. Mukhamedov, F.M.; Rozikov, U.A. On rational p -adic dynamical systems. *Methods of Functional Analysis and Topology* **10**(2) (2004), 21–31.
10. Mukhamedov, F.M.; Khakimov, O.N. Phase transition and chaos: p -adic Potts model on a Cayley tree. *Chaos, Solitons and Fractals* **87** (2016), 190–196.
11. Mukhamedov, F.M.; Khakimov, O.N. On Julia set and chaos in p -adic Ising model on the Cayley tree. *Mathematical Physics, Analysis and Geometry* **20** (2017), Article 23.
12. Mukhamedov, F.M.; Khakimov, O.N. Chaotic behavior of the p -adic Potts–Bethe mapping. *Discrete and Continuous Dynamical Systems* **38**(1) (2018), 231–245.
13. Rozikov, U.A.; Sattarov, I.A. p -adic dynamical systems of $(2, 2)$ -rational functions with unique fixed point. *Chaos, Solitons and Fractals* **105** (2017), 260–270.

14. Sattarov, I.A. Ergodicity properties of p -adic $(2, 1)$ -rational dynamical systems with unique fixed point. In: *Algebra, Complex Analysis, and Pluripotential Theory*, Springer Proceedings in Mathematics and Statistics, Vol. 264, 2018, 217–226.
15. Rozikov, U.A.; Sattarov, I.A.; Yam, S. p -adic dynamical systems of the function $\frac{ax}{x^2+a}$. *p-Adic Numbers, Ultrametric Analysis and Applications* **11**(1) (2019), 77–87.
16. Luna, A.; Rozikov, U.A.; Sattarov, I.A. p -adic dynamical systems of $(3, 1)$ -rational functions with unique fixed point. *p-Adic Numbers, Ultrametric Analysis and Applications* **12**(3) (2020), 210–230.
17. Rozikov, U.A.; Sattarov, I.A. Dynamical systems of the p -adic $(2, 2)$ -rational functions with two fixed points. *Results in Mathematics* **75**(3) (2020), Paper No. 100, 37 pp.
18. Sattarov, I.A. Group structure of the p -adic ball and dynamical system of isometry on a sphere. *p-Adic Numbers, Ultrametric Analysis and Applications* **16**(2) (2024), 128–135.
19. Walters, P. *An Introduction to Ergodic Theory*. Springer, Berlin–Heidelberg–New York, 1982.

[Cite this article](#)

Sattarov I. *Ergodicity of isometries mapping a sphere into itself in the field of p -adic numbers*, **Acta NUUZ**, 2026, 2/3, pp. 202-220.