

INTEGRATION OF THE EXTENDED FIFTH-ORDER KORTEWEG-DE VRIES EQUATION WITH
TIME-DEPENDENT COEFFICIENTS AND A SELF-CONSISTENT SOURCE IN THE CLASS OF RAPIDLY
DECREASING FUNCTIONS

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ABSTRACT. In this article, the problem of integrating the extended Korteweg-de Vries equation with time-dependent coefficients and a self-consistent source in the class of rapidly decreasing functions using the inverse scattering method for the self-adjoint Sturm-Liouville operator is studied. Furthermore, an integration algorithm based on this method is developed, and an illustrative example is solved using the proposed algorithm.

MSC (2020): 34L25, 35P25, 47A40, 37K15.

Key words: Korteweg-de Vries equation, Jost solutions, scattering data, Gelfand-Levitan-Marchenko integral equation.

Introduction

The inverse scattering method was first studied in the work of Gardner, Green, Kruskal, and Miura[1] in 1967. They succeeded in finding a global solution of the Cauchy problem for the following Korteweg-de Vries(KdV) equation

$$u_t - 6uu_x + u_{xxx} = 0$$

using the method of inverse problems of scattering theory for the self-adjoint Sturm-Liouville operator on the class of rapidly decreasing functions. In doing so, they used the method of solving the proper and inverse problem of scattering theory, studied by L.D. Faddeev [2], for a given Sturm-Liouville operator on the entire axis. After that, P. Lax [3] showed that the method used by these scientists was universal and found a higher-order analogue of the nonlinear Korteweg-de Vries equation. In the work of V.K. Melnikov[4], integration of the Korteweg-de Vries equation with a self-consistent source in the class of rapidly decreasing functions is achieved. Apart from the Korteweg-de Vries equation, using the method of inverse problems of scattering theory, equations such as Hirota [5] equation, Modified Korteweg-de Vries equation [6, 7], Sine-Gordon [8], and mKdF-Sine-Gordon equation have been integrated [9].

The classical third-order KdV model has shortcomings in representing several important properties of nonlinear waves observed in laboratory experiments, such as the non-monotonic relationship between the velocity and amplitude of a solitary wave or the table-top shape of large-amplitude solitary waves [10]. Therefore, there is a need to study the fifth-order Korteweg-de Vries equation and its extended form.

In [11], the dependence of the wave amplitude, width, and velocity of the solution to the fifth-order Korteweg-de Vries equation on the equation parameters is studied. Some forms of the fifth-order KdV equation, with some coefficients set to zero, have also been used to study plasma waves [12] electromagnetic waves in discrete transmission lines [13], gravity-capillary water waves [14,15] and waves in floating ice sheets.

Several methods are used to integrate the Korteweg-de Vries equation and its higher-order analogues, in addition to the inverse scattering transform method. Examples include the Hirota method, the tanh-coth method.

Received: 14.08.2026, **Accepted:** 22.09.2026

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The application of Hirota's method to KdV and higher-order KdV equations was studied in [16]. In [17], N-soliton solutions for the fifth-order KdV equation were studied under the Bargmann constraint via the Hirota method and the Wronsky technique, respectively. In [18], the solution of the fifth-order Korteweg-de Vries (Lax, Sawada-Kotera, Kaup-Kuperschmidt) equation in several forms was found using the tanh method.

Let us consider the following extended fifth-order Korteweg-de Vries equation with time-dependent coefficients and a self-consistent source:

$$\begin{cases} u_t + \alpha(t)(u_{xxxxx} - 20u_x u_{xxx} - 10u u_{xxx} + 30u^2 u_x) + \beta(t)(u_{xxx} - 6u u_x) + \\ + \gamma(t)u_x = 4 \sum_{m=1}^N \omega_m(t) \frac{\partial}{\partial x} |\varphi_m|^2, \\ -\varphi_m'' + u(x, t)\varphi_m = k_m^2 \varphi_m. \end{cases} \quad (1)$$

Where $\alpha(t), \beta(t), \gamma(t)$ and $\omega_m(t)$, $m = \overline{1, N}$ are the previously given continuously differentiable functions. In the case under consideration, the functions $\varphi_m = \varphi_m(x, t)$, $m = \overline{1, N}$ are the eigenfunctions corresponding to the eigenvalues of the self-adjoint Sturm-Liouville operator

$$L(t) := -\frac{d^2}{dx^2} + u(x, t), \quad t > 0$$

and satisfy the following conditions

$$\int_{-\infty}^{+\infty} |\varphi_m(x, t)|^2 dx = A_m(t), \quad m = \overline{1, N}. \quad (2)$$

Where $A_m(t)$, $m = \overline{1, N}$ are positive functions given. We study the given equation (1) with the following initial condition

$$u(x, 0) = u_0(x), \quad x \in \mathbb{R}. \quad (3)$$

Where $u_0(x)$ is a real function with the following properties:

1)

$$\int_{-\infty}^{\infty} (1 + |x|) |u_0(x)| dx < \infty \quad (4)$$

2) The operator $L(0) = -\frac{d^2}{dx^2} + u_0(x)$, $x \in \mathbb{R}$ has N eigenvalues $\lambda_1(0), \lambda_2(0), \dots, \lambda_N(0)$ and does not possess spectral singularities.

Let us assume that the function $u(x, t)$ has the required smoothness and rather quickly tends to its limits at $x \rightarrow \pm\infty$, i.e

$$\int_{-\infty}^{\infty} \left[(1 + |x|) |u(x, t)| + \sum_{j=1}^5 \left| \frac{\partial^j u(x, t)}{\partial x^j} \right| \right] dx < \infty, \quad \forall t \geq 0. \quad (5)$$

The main goal of the work is to obtain representations for the solution problem $u(x, t)$, $\varphi_m(x, t)$, $m = \overline{1, N}$ (1)-(5) in the framework of the method of the inverse scattering problem for the operator $L(t)$.

Preliminaries

Consider the Sturm-Liouville operator

$$L(0)y := -y'' + u_0(x)y = k^2 y, \quad x \in \mathbb{R} \quad (6)$$

where the potential $u_0(x)$ satisfies condition (3). In this section, information necessary for further presentation concerning the direct and inverse scattering problems for equation (1). Denote by $f(x, k)$ and $g(x, k)$ the Jost solutions of equation (1) with asymptotics

$$\begin{aligned} f(x, k) &= e^{ikx} + \bar{o}(1), \quad x \rightarrow +\infty, \quad f'(x, k) = ike^{ikx} + \bar{o}(1), \quad x \rightarrow +\infty (\operatorname{Im} k = 0); \\ g(x, k) &= e^{-ikx} + \bar{o}(1), \quad x \rightarrow -\infty, \quad g'(x, k) = -ike^{-ikx} + \bar{o}(1), \quad x \rightarrow -\infty (\operatorname{Im} k = 0) \end{aligned} \quad (7)$$

Under the condition (3), such solutions exist and are uniquely determined by the asymptotics (7). Solutions $f(x, k), g(x, k)$ have the representations

$$f(x, k) = e^{ikx} + \int_x^\infty K^+(x, t) e^{ikt} dt, \quad g(x, k) = e^{-ikx} + \int_{-\infty}^x K^-(x, t) e^{-ikt} dt \quad (8)$$

where kernels $K^+(x, t), K^-(x, t)$ are real functions related to the potential $u_0(x)$ as follows:

$$u_0(x) = \mp 2 \frac{dK^\pm(x, x)}{dx}$$

For real k , pairs of functions $\{f(x, k), f(x, -k)\}$ and $\{g(x, k), g(x, -k)\}$ are pairs of linearly independent solutions to the equation (6), therefore

$$f(x, k) = a(k)g(x, -k) + b(k)g(x, k),$$

$$g(x, k) = -b(-k)f(x, k) + a(k)f(x, -k). \quad (9)$$

The functions $r^+(k) = -\frac{b(-k)}{a(k)}$ and $r^-(k) = \frac{b(k)}{a(k)}$ are called reflection coefficients (right and left, respectively). The coefficients $a(k), b(k)$ and $r^+(k)$ have the following properties (see [19], p.121):

Defenition. The following set $\{r^+(k, 0), \chi_1(0), \chi_2(0), \dots, \chi_N(0), B_1(0), B_2(0), \dots, B_N(0)\}$ is called the data of the scattering theory of the operator $L(0)$.

(8) The kernel $K^+(x, y)$ in the Levin representation consists of the unique global solution of this Gelfand-Levitan-Marchenko integral equation

$$K^+(x, y) + F^+(x + y) + \int_x^\infty K^+(x, z) F^+(z + y) dz = 0, \quad (y > x). \quad (10)$$

Where

$$F^+(x) = \sum_{j=1}^N \alpha_j^+(0) e^{-\chi_j(0)x} + \frac{1}{2\pi} \int_{-\infty}^\infty r^+(k, 0) e^{ikx} dk, \quad \alpha_j^+(0) = \frac{iB_j(0)}{i \frac{da(z)}{dz} \Big|_{z=i\chi_j}}.$$

A. For real $k \neq 0$

$$\overline{a(k)} = a(-k), \quad \overline{b(k)} = b(-k), \quad |a(k)|^2 = 1 + |b(k)|^2, \quad a(k) = -\frac{1}{2ik} W\{f(x, k), g(x, k)\},$$

$$b(k) = \frac{1}{2ik} W\{f(x, k), g(x, -k)\}; \quad a(k) = 1 + O\left(\frac{1}{k}\right), \quad b(k) = O\left(\frac{1}{k}\right), \quad k \rightarrow \pm\infty;$$

where $W\{f(x, k), g(x, k)\} \equiv f'(x, k)g(x, k) - f(x, k)g'(x, k)$;

B. The function $a(k)$ extends analytically to the half-plane $\operatorname{Im} k > 0$ and there it has a finite number of zeros $k_m = i\chi_m$, ($\chi_n > 0$), $m = 1, 2, \dots, N$, these zeros are simple, and $\lambda_m = -\chi_m^2$, $m = 1, 2, \dots, N$ are the eigenvalues of the operator $L(0)$. In addition, the following relation holds

$$g(x, i\chi_j) = B_j f(x, i\chi_j), \quad j = 1, 2, \dots, N.$$

C. For real $k \neq 0$, the function $r^+(k)$ is continuous,

$$\overline{r^+(k)} = r^+(-k), \quad |r^+(k)| < 1, \quad r^+(k) = o(k^{-1}), \quad |k| \rightarrow \infty, \quad k^2 \left[1 - |r^+(k)|^2\right]^{-1} = O(1), \quad |k| \rightarrow 0;$$

D. This function $k(a(k) - 1)$ is continuously bounded in $\text{Im } k > 0$, and the following relations hold:

$$a^{-1}(k) = O(1), \quad |k| \rightarrow 0, \quad \text{Im } k \geq 0, \quad \lim_{k \rightarrow 0} ka(k) (r^+(k) + 1) = 0, \quad \text{Im } k = 0,$$

where the function $a(k)$ is defined by the following formula

$$a(k) = \prod_{j=1}^N \frac{k - i\chi_j}{k + i\chi_j} \exp \left\{ -\frac{1}{2i\pi} \int_{-\infty}^{\infty} \frac{\ln(1 - |r^+(\xi)|^2)}{\xi - k} d\xi \right\}, \quad \text{Im } k > 0.$$

E. These

$$R^{\pm}(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r^{\pm}(k) e^{\pm ikx} dk$$

functions satisfy the following conditions for each $a > -\infty$:

$$(1 + |x|) |R^{\pm}(\pm x)| \in L^1(a, \infty)$$

Lemma 1. Let the functions $y(x, \lambda)$ and $z(x, \mu)$ be solutions of the following equations $Ly(x, \lambda) = \lambda y(x, \lambda)$, $Lz(x, \mu) = \mu z(x, \mu)$ respectively. In this case, the following equalities are true

$$\frac{d}{dx} W \{y(x, \lambda), z(x, \mu)\} = (\lambda - \mu) y(x, \lambda) z(x, \mu).$$

Theorem 1 [(see [19, p. 121])]. The potential of the Sturm-Liouville operator $u_0(x)$ is defined as a single value by the data of its scattering theory.

Evolution of scattering data

Let us consider the following equation

$$u_t + \alpha(t) (u_{xxxxx} - 20u_x u_{xx} - 10u u_{xxx} + 30u^2 u_x) + \beta(t) (u_{xxx} - 6u u_x) = G(x, t). \quad (11)$$

Where we introduce the following notation $G(x, t) = -\gamma(t)u_x + 4 \sum_{m=1}^N \omega_m(t) \frac{\partial}{\partial x} |\varphi_m|^2$.

For equation (11), the Lax pair is in the following form:

$$\begin{aligned} -\phi''_{xx}(x, k, t) + (u(x, t) - k^2) \phi(x, k, t) &= 0, \\ \frac{\partial \phi(x, k, t)}{\partial t} &= 8\alpha(t) \left(2\phi_{xxxxx} - 5u\phi_{xxx} + \frac{15}{4}uu_x\phi - \frac{25}{4}u_{xx}\phi_x - \frac{15}{8}\phi u_{xxx} + \frac{15}{4}u^2\phi_x \right) + \\ &+ 60k^2\alpha(t)u_x\phi + 16ik^5\alpha(t)\phi + \beta(t) (-u_x + 4ik^3) \phi + \beta(t) (2u + 4k^2) \phi_x + F(x, t). \end{aligned} \quad (12)$$

Now, based on relation $\phi_{txx} = \phi_{xxt}$ and using equalities (11)-(12), we get the following equation

$$-F_{xx} + (u - k^2) F = -G(x, t)\phi. \quad (13)$$

Where $\phi(x, k, t) = g(x, k, t)$. To solve this equation, we use the method of variation of constants. That is, if we put expression

$$F(x, t) = B(x, t)g(x, k, t) + C(x, t)g(x, -k, t) \quad (14)$$

into equation (13), we get this system of equations

$$\begin{cases} B'_x(x, t)g(x, k, t) + C'_x(x, t)g(x, -k, t) = 0 \\ B'_x(x, t)g'_x(x, k, t) + C'_x(x, t)g'_x(x, -k, t) = G(x, t)g(x, k, t). \end{cases} \quad (15)$$

If we solve the last system of equations with respect to the unknowns $B(x, t)$ and $C(x, t)$, then we get the following equalities

$$B(x, t) = -\frac{1}{2ik} \int_{-\infty}^x G(s, t)g(s, k, t)g(s, -k, t)ds, C(x, t) = \frac{1}{2ik} \int_{-\infty}^x G(s, t)g^2(s, k, t)ds.$$

Then, by substituting the obtained $B(x, t)$ and $C(x, t)$ into equality (14) and using relation (12), we obtain the following expression:

$$\begin{aligned} \frac{\partial g(x, k, t)}{\partial t} = & 8\alpha(t) \left(2g_{xxxxx} - 5ug_{xxx} + \frac{15}{4}uu_xg - \frac{25}{4}u_{xx}g_x - \frac{15}{8}gu_{xxx} + \frac{15}{4}u^2g_x \right) + 60k^2\alpha(t)u_xg + \\ & + 16ik^5\alpha(t)g + \beta(t)(-u_x + 4ik^3)g + \beta(t)(2u + 4k^2)g_x - \\ & - \frac{g(x, k, t)}{2ik} \int_{-\infty}^x G(s, t)g(s, k, t)g(s, -k, t)ds + \frac{g(x, -k, t)}{2ik} \int_{-\infty}^x G(s, t)g^2(s, k, t)ds \end{aligned} \quad (16)$$

If we use (5),(7),(9) and the last equality, then pass to the limit $x \rightarrow +\infty$, we obtain the following equalities:

$$\begin{aligned} \frac{db(-k, t)}{dt} = & \left(32ik^5\alpha(t) + 8ik^3\beta(t) - \frac{1}{2ik} \int_{-\infty}^{+\infty} G(s, t)g(s, k, t)g(s, -k, t)ds \right) b(-k, t) - \\ & - \frac{a(-k, t)}{2ik} \int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds, \\ \frac{da(k, t)}{dt} = & -\frac{a(k, t)}{2ik} \int_{-\infty}^{+\infty} G(s, t)g(s, k, t)g(s, -k, t)ds - \frac{b(k, t)}{2ik} \int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds \end{aligned} \quad (17)$$

Using the last equations, we find the time evolution of the scattering function $r^+(k, t)$. That is, as follows:

$$\frac{dr^+(k, t)}{dt} = (32ik^5\alpha(t) + 8ik^3\beta(t))r^+(k, t) + \frac{1}{2ika^2(k, t)} \int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds. \quad (18)$$

Lemma 2. If functions $g(x, k, t)$ and $g(x, -k, t)$ are solutions of equation

$$-\phi''_{xx}(x, k, t) + (u(x, t) - k^2)\phi(x, k, t) = 0,$$

then the following properties

$$\int_{-\infty}^{+\infty} G(s, t)g^2(s, k, t)ds = -4k^2a(k, t)b(-k, t), \quad (19)$$

$$\int_{-\infty}^{+\infty} G(s, t)g(s, k, t)g(s, -k, t)ds = 4k^2b(k, t)b(-k, t) \quad (20)$$

hold.

Proof. To prove this lemma, we first calculate this integral using the relations (5), (7) and (9):

$$\int_{-\infty}^{+\infty} g^2(s, k, t)u'_s(s, t)ds = g^2(s, k, t)u(s, t) \Big|_{-\infty}^{+\infty} - \int_{-\infty}^{+\infty} 2g(s, k, t)g'_s(s, k, t)u(s, t)ds =$$

$$\begin{aligned} &= -2 \int_{-\infty}^{+\infty} (g''_{ss}(s, k, t) + k^2 g(s, k, t)) g'_s(s, k, t) ds = \int_{-\infty}^{+\infty} \left(-k^2 (g^2(s, k, t))'_s - (g'^2_s(s, k, t))'_s \right) ds = \\ &= \left(-k^2 g^2(s, k, t) - (g'_s(s, k, t))^2 \right) \Big|_{-\infty}^{+\infty} = 4k^2 a(k, t) b(-k, t). \end{aligned}$$

Therefore, we have obtained the following equality

$$\int_{-\infty}^{+\infty} g^2(s, k, t) u'_s(s, t) ds = 4k^2 a(k, t) b(-k, t). \quad (21)$$

Now we calculate the following integral using lemma 1.

$$\begin{aligned} &4 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) \frac{\partial}{\partial s} \left(|\varphi_m(s, t)|^2 \right) g^2(s, k, t) ds = 2 \sum_{m=1}^N \omega_m(t) |\varphi_m(s, t)|^2 g^2(s, k, t) \Big|_{-\infty}^{+\infty} - \\ &-4 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) g(s, k, t) g'_s(s, k, t) |\varphi_m(s, t)|^2 ds + 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) g^2(s, k, t) \frac{\partial \left(|\varphi_m(s, t)|^2 \right)}{\partial s} ds = \\ &= 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \omega_m(t) [g \bar{\varphi}_m W \{g, \varphi_m\} + g \varphi_m W \{g, \bar{\varphi}_m\}] ds = \\ &= 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \frac{\omega_m(t)}{\lambda - \lambda_m} \left[\frac{dW \{g, \bar{\varphi}_m\}}{ds} W \{g, \varphi_m\} + W \{g, \bar{\varphi}_m\} \frac{dW \{g, \varphi_m\}}{ds} \right] ds = \\ &= 2 \int_{-\infty}^{+\infty} \sum_{m=1}^N \frac{\omega_m(t)}{\lambda - \lambda_m} \frac{d}{ds} \{W \{g, \bar{\varphi}_m\} W \{g, \varphi_m\}\} ds = \sum_{m=1}^N \frac{2\omega_m(t)}{\lambda - \lambda_m} W \{g, \bar{\varphi}_m\} W \{g, \varphi_m\} \Big|_{-\infty}^{+\infty} = 0. \end{aligned}$$

Thus, based on the last equality and equality (21), we were able to derive the following relationship:

$$\int_{-\infty}^{+\infty} G(s, t) g^2(s, k, t) ds = -4k^2 a(k, t) b(-k, t) \gamma(t).$$

Now, we derive the equality (20) in Lemma 2. To do this, we evaluate the following integral using the relations (5), (7) and (9):

$$\begin{aligned} &\int_{-\infty}^{+\infty} G(s, t) g(s, k, t) g(s, -k, t) ds = -\gamma(t) \int_{-\infty}^{+\infty} u'_s g(s, k, t) g(s, -k, t) ds + \\ &+ 4 \int_{-\infty}^{+\infty} \sum_{m=1}^N \alpha_m(t) \frac{\partial |\varphi_m(s, t)|^2}{\partial s} g(s, k, t) g(s, -k, t) ds = \gamma(t) I_1 + \sum_{m=1}^N \alpha_m(t) I_m. \quad (22) \\ &I_1 = - \int_{-\infty}^{+\infty} g(s, k, t) g(s, -k, t) u'_s(s, t) ds = \int_{-\infty}^{+\infty} g'_s(s, k, t) g(s, -k, t) u(s, t) ds + \\ &+ \int_{-\infty}^{+\infty} g(s, k, t) g'_s(s, -k, t) u(s, t) ds = \int_{-\infty}^{+\infty} (g''_{ss}(s, -k, t) + k^2 g(s, -k, t)) g'_s(s, k, t) ds + \end{aligned}$$

$$+ \int_{-\infty}^{+\infty} (g''_{ss}(s, k, t) + k^2 g(s, k, t)) g'_s(s, -k, t) ds = \frac{\partial g(s, k, t)}{\partial s} \cdot \frac{\partial g(s, -k, t)}{\partial s} \Big|_{-\infty}^{+\infty} +$$

$$+ k^2 g(s, k, t) g(s, -k, t) \Big|_{-\infty}^{+\infty} = 4k^2 b(k, t) b(-k, t).$$

We can also derive this expression using lemma 1.

$$I_m = 4 \int_{-\infty}^{+\infty} \frac{\partial}{\partial s} (|\varphi_m(s, t)|^2) g(s, k, t) g(s, -k, t) ds = 0.$$

If we substitute the found I_1 and I_m into expression (22), we obtain equality (20). **The lemma is proved.**

According to equality (18) and lemma 2, we form the following equality:

$$\frac{dr^+(k, t)}{dt} = (32ik^5 \alpha(t) + 8ik^3 \beta(t) - 2ik\gamma(t)) r^+(k, t). \quad (23)$$

Furthmore, based on equality (17) and lemma 2, the following relation $a_t(k, t) = 0$ hold. According to last equality, $a(k, t) = a(k, 0)$, $k_m(t) = k_m(0)$ is valid. That is

$$\frac{d\lambda_m(t)}{dt} = 0. \quad (24)$$

Now, we calculate the time evolution of the normalizing constants corresponding to the eigenvalues. Therefore, using relation (16), we obtain the following equality:

$$\frac{\partial g(x, k, t)}{\partial t} = 8\alpha(t) \left(2g_{xxxxx} - 5ug_{xxx} + \frac{15}{4}uu_xg - \frac{25}{4}u_{xx}g_x - \frac{15}{8}gu_{xxx} + \frac{15}{4}u^2g_x \right) + 60k^2\alpha(t)u_xg +$$

$$+ 16ik^5\alpha(t)g + \beta(t) (-u_x + 4ik^3)g + \beta(t) (2u + 4k^2)g_x - \frac{g(x, k, t)}{2ik} \int_{-\infty}^x G(s, t)g(s, k, t)g(s, -k, t)ds +$$

$$+ \frac{g(x, -k, t)}{2ik} \int_{-\infty}^x G(s, t)g^2(s, k, t)ds = -ikg(x, k, t) - g'_x(x, k, t) +$$

$$+ 4 \sum_{m=1}^N \alpha_m(t) \left[g(x, -k, t) \int_{-\infty}^x \frac{\partial |\varphi_m(s, t)|^2}{\partial s} g^2(s, k, t)ds + g(x, k, t) \int_{-\infty}^x \frac{\partial |\varphi_m(s, t)|^2}{\partial s} g(s, k, t)g(s, -k, t)ds \right].$$

If we use the following equality $g(x, k_m(t), t) = B_m(t)f_m(x, k_m(t), t)$ in the last expression and take the limit in $x \rightarrow +\infty$ and equate the terms before $e^{ik_n x}$, then we get the following equality

$$\frac{dB_m(t)}{dt} = (32ik_m^5 \alpha(t) + 8ik_m^3 \beta(t) - 2ik_m \gamma(t) + 2\omega_m(t)A_m(t)) B_m(t), \quad m = \overline{1, N}. \quad (25)$$

Finally, we were able to prove this theorem.

Theorem 2. If functions $u(x, t)$, $\varphi_m(x, t)$, $m = \overline{1, N}$, $x \in \mathbb{R}$, $t > 0$ are solutions to problem (1)-(5), then the potential $u(x, t)$ and the operator $L(t)$ given by the scattering theory $\{r^+(k, t), \lambda_m(t), B_m(t), m = \overline{1, N}\}$ satisfy the following differential equations (23),(24),(25).

Let's present an algorithm for solving problems (1)-(5):

- First of all, we find the givens of the scattering theory of the Sturm-Liouville operator whose potential $u_0(x)$ satisfies the condition (3), that is, the following summation

$$\left\{ r^+(k, 0) = -\frac{b(-k, 0)}{a(k, 0)}, \quad \lambda_m(0), \quad B_m(0), \quad m = 1, 2, \dots, N \right\}.$$

- Using the solutions of the differential equations in Theorem 2, we find the scattering data of the operator $L(t)$, $t > 0$:

$$\left\{ r^+(k, t) = -\frac{b(-k, t)}{a(k, t)}, \quad \lambda_m(t), \quad B_m(t), \quad m = 1, 2, \dots, N \right\}.$$

- Using the obtained scattering data for $t > 0$, the Gelfand-Levitan-Marchenko integral equation is solved, resulting in the determination of the unique solution $u(x, t)$, as established in theorem 1.
- Then, we solve the correct problem for the family of operators $L(t)$, $t > 0$ with potential $u(x, t)$ and find its Yost solutions and the functions $\varphi_m = \varphi_m(x, t)$, $m = 1, N$ satisfying condition (2).

Remark. The derived relations completely determine the evolution of the scattering data, thereby enabling the application of the inverse scattering transform method to solve the initial-value problem (1)-(5).

Example 1. Solve the following Cauchy problem

$$\begin{cases} u_t + t(u_{xxxxx} - 20u_x u_{xx} - 10uu_{xxx} + 30u^2 u_x) + 3t^2(u_{xxx} - 6uu_x) + \frac{1}{1+t^2}u_x = \\ = e^{2 \arctg t} (32t - 24t^2) \frac{\partial}{\partial x} |\varphi_1(x, t)|^2, \\ -\varphi_1'' + u(x, t)\varphi_1 = -\varphi_1 \end{cases}$$

$$u|_{t=0} = -\frac{2}{ch^2 x}, \quad \int_{-\infty}^{+\infty} |\varphi_1(x, t)|^2 dx = \frac{1}{2} e^{-2 \arctg t}.$$

The solution to this Cauchy problem is as follows:

$$u(x, t) = \frac{-2}{ch^2(x - \arctg t)}, \quad \varphi_1(x, t) = \frac{e^{-x}}{1 + e^{-2x+2 \arctg t}}.$$

Integration of the loaded and extended fifth-order Korteweg-de Vries equation with a self-consistent source

The most general definition of loaded equations and a detailed classification of various loaded equations are given in the works of A.M. Nakhushev [20]. Among the works devoted to loaded Korteweg-de Vries equations, the works [21-24] should be especially noted.

Let us now turn to the consideration of a special case of equation (1), namely, we will consider the loaded Korteweg-de Vries equation with a source:

$$\begin{aligned} u_t + P(u(x_0, t))(u_{xxxxx} - 20u_x u_{xx} - 10uu_{xxx} + 30u^2 u_x) + Q(u(x_1, t))(u_{xxx} - 6uu_x) + \\ + Z(u(x_2, t))u_x = 4 \sum_{m=1}^N F_m(u(x_3, t)) \frac{\partial}{\partial x} |\varphi_m|^2. \end{aligned} \quad (26)$$

Where $P(y)$, $Q(z)$, $Z(s)$ and $F_m(t)$, $m = \overline{1, N}$ are polynomials from y , z , s and t and $x_0, x_1, x_2, x_3 \in \mathbb{R}$ respectively. Equation (26) is not a special case of equation (1), since in equation (26) the coefficients depend on the value of the solution on a manifold of lower dimension. If in problem (26)+(2)-(5) instead of equation (1) we consider equation (26), then the following theorem is valid.

Theorem 3. If the functions $u(x, t)$, $\varphi_m(x, t)$, $m = \overline{1, N}$, $x \in \mathbb{R}$, $t > 0$ are a solution to the problem (26)+(2)-(5), then the scattering data $\{r^+(k, t), \lambda_m(t), B_m(t), m = \overline{1, N}\}$ of the operator $L(t)$ with potential $u(x, t)$ satisfy the following differential equations:

$$\frac{\partial r^+(k, t)}{\partial t} = (32ik^5 P(u(x_0, t)) + 8ik^3 Q(u(x_1, t)) - 2ikZ(u(x_2, t))) r^+(k, t), \quad \frac{\partial \lambda_m(t)}{\partial t} = 0, \quad m = \overline{1, N},$$

$$\frac{dB_m(t)}{dt} = (32ik_m^5 P(u(x_0, t)) + 8ik_m^3 Q(u(x_1, t))) B_m(t) +$$

$$+ (-2ik_m Z(u(x_2, t)) + 2F_m(u(x_3, t)) A_m(t)) B_m(t), \quad m = \overline{1, N}.$$

Example 2. Solve the following Cauchy problem

$$\begin{cases} u_t + \alpha(t)u(0, t) (u_{xxxx} - 20u_x u_{xx} - 10u u_{xxx} + 30u^2 u_x) + \beta(t)u(\ln 2, t) (u_{xxx} - 6u u_x) + \\ + \gamma(t)u(1, t)u_x = (t+1)u(2, t) \frac{\partial}{\partial x} |\varphi_1(x, t)|^2, \\ -\varphi_1'' + u(x, t)\varphi_1 = -\varphi_1. \end{cases}$$

$$u|_{t=0} = -\frac{2}{ch^2 x}, \quad \int_{-\infty}^{+\infty} |\varphi_1(x, t)|^2 dx = \frac{1}{2(t+1)^2}.$$

Where

$$\alpha(t) = \frac{(-24t^3 - 23t^2 + t + 2)(t^2 + 2t + 2)^2}{256(t+1)^3}, \quad \beta(t) = \frac{-3t^2(t^2 + 2t + 5)^2}{32(t+1)^2},$$

$$\gamma(t) = \frac{e^2 \left(t(e^2 + e^{-2}(t+1)^2)^2 - 8(t+1) \right) (e^2 + (t+1)^2)^2}{16(t+1)^2 (e^4 + (t+1)^2)^2}.$$

The solution to this Cauchy problem is as follows:

$$u(x, t) = \frac{-2}{ch^2(x - \ln(t+1))}, \quad \varphi_1(x, t) = \frac{e^{-x}}{1 + (t+1)^2 e^{-2x}}.$$

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Hoitmetiv U., Sobirov Sh. *Integration of the extended fifth-order Korteweg-de Vries equation with time-dependent coefficients and a self-consistent source in the class of rapidly decreasing functions*, **Acta NUUZ**, 2026, 2/3, pp. 92-101.