

GEOMETRIC APPLICATION OF FUNCTORS OF PROBABILITY MEASURES P AND IDEMPOTENT PROBABILITY MEASURES I

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ABSTRACT. In the work defines the geometric properties of the space of idempotent probability measures. It also considers the construction of one-dimensional, two-dimensional, three-dimensional,..., $(n-1)$ -dimensional and infinite-dimensional simplices in the space of idempotent probability measures. In particular, for a compact of X it is established that if the space of $I_3(X) \setminus X$ is hereditary normally, then X is metrized; some subsets of space of idempotent probability measures which are, respectively, Z - sets, are allocated to max - plus - convex subsets.

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The theory of idempotent measures belongs to idempotent mathematics, i.e. the fields of the mathematics based on replacement of usual arithmetic operations idempotent (as, for example, $x \oplus y = \max\{x, y\}$). The idempotent mathematics intensively develops now (see, for example, [3], the review article [4] and the bibliography in it). Her communication with traditional mathematics is described by the informal principle [4] according to which there is a heuristic compliance between important, interesting and useful designs the last and similar results of idempotent mathematics.

In the present article we investigate a functor of I idempotent probability measures in category compact hausdorff spaces. In traditional mathematics to him there corresponds the functor of P probability measures. The concept of an idempotent measure (Maslov's measure) finds numerous applications in various fields of mathematics, mathematical physics and economy. In particular, such measures arise in problems of dynamic optimization [4]; analogy between Maslov's integration and optimization is noted also in [4].

Unlike a case of probability measures to which consideration extensive literature is devoted (see [5]) geometrical and topological properties of spaces of idempotent measures were practically not investigated.

We will consider the set $\mathbb{R}_{\max} = \mathbb{R} \cup \{-\infty\}$ with two algebraic operations: addition $\oplus$ and multiplication $\odot$ determined as follows $u \oplus v = \max\{u, v\}$ and $u \odot v = u + v$ where $\mathbb{R}$ a set of real numbers.

Let X be a Hausdorff compact space ($\equiv$ a compact), $C(X)$ be the algebra of continuous functions on X with the usual algebraic operations. For $C(X)$ operations $\oplus$ and $\odot$ we define as follows: $\varphi \oplus \psi = \max\{\varphi, \psi\}$ and $\varphi \odot \psi = \varphi + \psi$, $\varphi, \psi \in C(X)$.

The problem of constructing one-dimensional, two-dimensional, three-dimensional,..., $(n-1)$ - dimensional and infinite-dimensional simplices in the space of idempotent probability measures: Suppose we are given $\alpha, \beta \in \mathbb{R}_{\max}$ and $x, y \in L$. Using these, we can write their max -plus linear combination as follows $\alpha \odot x \oplus \beta \odot y$.

If $\alpha \oplus \beta = 1$, ($\max\{\alpha, \beta\} = 0$), and $-\infty \leq \alpha, \beta \leq 0$, then we call the expression $\alpha \odot x \oplus \beta \odot y$ is a max -plus-affine combination.

$$\sigma_{\oplus}^1 = [x, y]_{\oplus} = \{\alpha \odot x \oplus \beta \odot y : \max\{\alpha, \beta\} = 0, -\infty \leq \alpha, \beta \leq 0\}$$

is a one-dimensional max -plus simplex.

$$\sigma_{\oplus}^2 = [x, y, z]_{\oplus} = \{\alpha \odot x \oplus \beta \odot y \oplus \gamma \odot z : \max\{\alpha, \beta, \gamma\} = 0, -\infty \leq \alpha, \beta, \gamma \leq 0\}$$

is a two-dimensional max-plus simplex.

Similarly,

$$\sigma_{\oplus}^{(n-1)} = [x_1, x_2, \dots, x_n]_{\oplus} = \left\{ \bigoplus_{i=1}^n \alpha_i \odot x_i : -\infty \leq \alpha_i \leq 0, i = \overline{1, n}, \bigoplus_{i=1}^n \alpha_i = 0 \right\}$$

– $(n-1)$ is a dimensional max-plus simplex.

In this case $I(2) \simeq \sigma_{\oplus}^1, I(3) \simeq \sigma_{\oplus}^2, \dots, I(n) \simeq \sigma_{\oplus}^{(n-1)}$ s are appropriate.

If X is a topological space and $|X| = \aleph_0$, then a set of idempotent probability measures $I(X)$ is defined. This set is an infinite-dimensional max-plus simplex:

$$\sigma_{\oplus}^{\infty} = [x_1, x_2, \dots, x_n, \dots]_{\oplus} = \left\{ \bigoplus_{i=1}^{\infty} \alpha_i \odot x_i : -\infty \leq \alpha_i \leq 0, i = \overline{1, \infty}, \bigoplus_{i=1}^{\infty} \alpha_i = 0 \right\}$$

Definition 1. A functional $\mu: C(X) \rightarrow \mathbb{R}$ called an idempotent probability measure on X , if it satisfies the following conditions:

- (I1) μ is normed, i.e. $\mu(\lambda_X) = \lambda$ for all $\lambda \in \mathbb{R}$,
- (I2) μ is max-plus-homogeneous, i.e. $\mu(\lambda \odot \varphi) = \lambda \odot \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$,
- (I3) μ is max-plus-additive, i.e. $\mu(\varphi \oplus \psi) = \mu(\varphi) \oplus \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

The set of all idempotent probability measures on X denotes by $I(X)$ [1]. Consider $I(X)$ as a subspace of $\mathbb{R}^{C(X)}$.

For given compacts X, Y and a continuous mapping $f: X \rightarrow Y$ we can verify that the natural map $I(f): I(X) \rightarrow I(Y)$, defined by the formula $I(f)(\mu)(\psi) = \mu(\psi \circ f)$, is continuous. Moreover, the construction I is a normal functor.

Therefore, for an arbitrary idempotent probability measure $\mu \in I(X)$ one can define the concept of support:

$$\text{supp} \mu = \bigcap \{A \subset X : \overline{A} = A, \mu \in I(A)\}.$$

For a positive integer n , we define the following set

$$I_n(X) = \{\mu \in I(X) : |\text{supp} \mu| \leq n\}.$$

Set

$$I_{\omega}(X) = \bigcup_{n=1}^{\infty} I_n(X).$$

The set $I_{\omega}(X)$ is everywhere dense [1] in $I(X)$.

Idempotent probability measure $\mu \in I_{\omega}(X)$ is called an idempotent probability measure with finite support. Note that if μ is an idempotent probability measure with finite support $\text{supp} \mu = \{x_1, x_2, \dots, x_k\}$, then μ can be represented as $\mu = \lambda_1 \odot \delta_{x_1} \oplus \lambda_2 \odot \delta_{x_2} \oplus \dots \oplus \lambda_k \odot \delta_{x_k}$ the only way, where $-\infty < \lambda_i \leq 0, i = 1, 2, \dots, k, \lambda_1 \oplus \lambda_2 \oplus \dots \oplus \lambda_k = 0$. Here, as usual, for $x \in X$ by δ_x is denoted a functional on $C(X)$, defined by the formula $\delta_x(\varphi) = \varphi(x), \varphi \in C(X)$, and called the Dirac measure. It is concentrated at the point x .

Definition 2. A functional $\mu: C(X) \rightarrow \mathbb{R}$ called a probability measure on X , if it satisfies the following conditions:

- (P1) μ is normed, i.e. $\mu(\lambda_X) = \lambda$, here λ_X is a constant function such that $\lambda_X(x) = \lambda, \lambda \in \mathbb{R}$.
- (P2) μ is homogeneous, i.e. $\mu(\lambda \varphi) = \lambda \mu(\varphi)$ for all $\lambda \in \mathbb{R}$ and $\varphi \in C(X)$,
- (P3) μ is additive, i.e. $\mu(\varphi + \psi) = \mu(\varphi) + \mu(\psi)$ for all $\varphi, \psi \in C(X)$.

The set of all probability measures on X denotes by $P(X)$ [3]. Consider $P(X)$ as subspaces of $\mathbb{R}^{C(X)}$. In the induced topology the sets of the view

$$\langle \mu; \varphi_1, \dots, \varphi_k; \varepsilon \rangle = \{v \in F(X) : |\mu(\varphi_i) - \nu(\varphi_i)| < \varepsilon, i = 1, \dots, k\}$$

form a base of neighbourhoods of $\mu \in F(X)$ where $\varphi_i \in C(X), i = 1, \dots, k, \varepsilon > 0$ and $F = P$ or $F = I$.

The topology generated by this base coincide with the pointwise convergence topology on $F(X)$. For a compact Hausdorff space X the topological spaces $I(X)$ and $P(X)$ are compact Hausdorff spaces.

To show the difference between sets $P(X)$ and $I(X)$, consider a finite set X . Let X be a set consisting of 3 different points, and equipped with the discrete topology. Consider the following sets:

$$P(X) = \left\{ \sum_{i=1}^3 \mu_i \delta_{x_i} : \mu_i \geq 0, i = 1, 2, 3, \sum_{i=1}^3 \mu_i = 1 \right\},$$

$$I(X) = \left\{ \bigoplus_{i=1}^3 \mu_i \odot \delta_{x_i} : \mu_i \geq -\infty, i = 1, 2, 3, \bigoplus_{i=1}^3 \mu_i = 0 \right\}.$$

It is well-known that $P(X)$ represents the set of all probability measures on X , and $I(X)$ represents the set of all idempotent probability measures on X . Moreover, $P(X) \subset \mathbb{R}^n$ and $I(X) \subset \mathbb{R}_{\max}^n$.

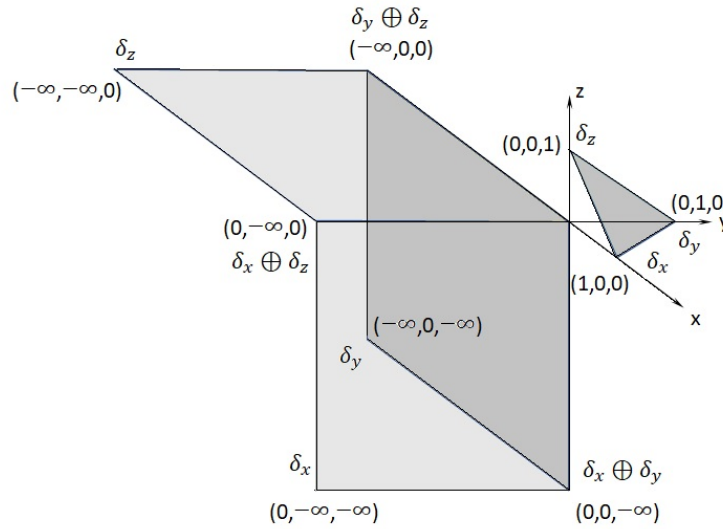


Figure 5: The images $I(X)$ (trihedron) and $P(X)$ (triangle) when X consists of three points.

For a given map $f: X \rightarrow Y$ of compact hausdorff spaces formula

$$F(f)(\mu)(\varphi) = \mu(\varphi \circ f), \quad \mu \in F(X),$$

defines a map $F(f): F(X) \rightarrow F(Y)$ where $\varphi \in C(Y)$, $F = P$ or $F = I$. In such cases they say that f induces $F(f)$.

If X consists of n points then $P(X)$ is a compact convex subset of $\mathbb{R}^n$, and $I(X)$ is a compact max-plus-convex subset of $\mathbb{R}_{\max}^n$

Theorem 1. For an arbitrary compact space X , the spaces $P_\omega(X)$ and $I_\omega(X)$ are homeomorphic.

The sets $P(X)$ of all probability measures and $I(X)$ of all idempotent probability measures are endowed with the topology of pointwise convergence. The topological spaces $P(X)$ and $I(X)$ are compact Hausdorff spaces.

Theorem 2. For an arbitrary finite compact set X , the spaces $P(X)$ and $I(X)$ are homeomorphic.

Proof. Consider the mapping

$$z_I^P : P(X) \longrightarrow I(X),$$

given by equality

$$z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) = \bigoplus_{i=1}^n \left(\left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right),$$

and

$$z_P^I : I(X) \longrightarrow P(X),$$

rule defined

$$z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) = \sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \cdot \delta_{x_i}.$$

We show that z_I^P and z_P^I – are mutually inverse mappings.

1) For each probability measure $\sum_{i=1}^n \alpha_i \delta_{x_i} \in P(X)$ the following equalities hold

$$\begin{aligned} z_P^I \left(z_I^P \left(\sum_{i=1}^n \alpha_i \delta_{x_i} \right) \right) &= z_P^I \left(\bigoplus_{i=1}^n \left(\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j \right) \odot \delta_{x_i} \right) = \\ &= \sum_{i=1}^n \frac{e^{\ln \alpha_i - \bigoplus_{j=1}^n \ln \alpha_j}}{\sum_{l=1}^n e^{\ln \alpha_l - \bigoplus_{j=1}^n \ln \alpha_j}} \cdot \delta_{x_i} = \sum_{i=1}^n \frac{e^{\ln \alpha_i} : e^{\bigoplus_{j=1}^n \ln \alpha_j}}{\left(\sum_{l=1}^n e^{\ln \alpha_l} \right) : e^{\bigoplus_{j=1}^n \ln \alpha_j}} \cdot \delta_{x_i} = \\ &= \sum_{i=1}^n \frac{\alpha_i \delta_{x_i}}{\sum_{l=1}^n \alpha_l} = \sum_{i=1}^n \alpha_i \delta_{x_i}; \end{aligned}$$

2) For each idempotent probability measure $\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \in I(X)$ we have

$$\begin{aligned} z_I^P \left(z_P^I \left(\bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i} \right) \right) &= z_I^P \left(\sum_{i=1}^n \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} \delta_{x_i} \right) = \\ &= \bigoplus_{i=1}^n \left(\ln \frac{e^{\lambda_i}}{\sum_{j=1}^n e^{\lambda_j}} - \bigoplus_{l=1}^n \ln \frac{e^{\lambda_l}}{\sum_{j=1}^n e^{\lambda_j}} \right) \odot \delta_{x_i} = \\ &= \bigoplus_{i=1}^n \left(\ln e^{\lambda_i} - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \left(\ln e^{\lambda_l} - \ln \sum_{j=1}^n e^{\lambda_j} \right) \right) \odot \delta_{x_i} = \\ &= \bigoplus_{i=1}^n \left(\lambda_i - \ln \sum_{j=1}^n e^{\lambda_j} - \bigoplus_{l=1}^n \lambda_l + \ln \sum_{j=1}^n e^{\lambda_j} \right) \odot \delta_{x_i} = \bigoplus_{i=1}^n \lambda_i \odot \delta_{x_i}. \end{aligned}$$

Consequently, $z_I^P z_P^I : I(X) \rightarrow I(X)$ and $z_P^I z_I^P : P(X) \rightarrow P(X)$ – identical mappings.

We now show that the mapping z_I^P and z_P^I are continuous. Since they are mutually inverse mappings between compacta, it suffices to show the continuity of only one of them.

We show that the mapping $z_I^P : P(X) \rightarrow I(X)$ is continuous. Let $\mu_0 \in P(X)$ – be a probability measure, $\{\mu_t\}_{t=1}^\infty \subset P(X)$ – sequence converging to μ_0 in topology of pointwise convergence (symbolically $\lim_{t \rightarrow \infty} \mu_t = \mu_0$).

Since X is finite, without loss of generality, we can assume that $\text{supp} \mu_t = \text{supp} \mu_0 = \{x_1, x_2, \dots, x_n\}$, $n \in \mathbb{N}$, for all $t = 1, 2, \dots$

Then we have $\lim_{t \rightarrow \infty} \alpha_t^i = \alpha_0^i$, где $\mu_t = \sum_{i=1}^n \alpha_t^i \delta_{x_i}$, $t = 0, 1, 2, \dots$. On the other hand, according to the continuity of $\ln$ and the operation *oplus*, the equality $\lim_{t \rightarrow \infty} \ln \alpha_t^i = \ln \alpha_0^i$ follows. It attracts her t

$$\lim_{t \rightarrow \infty} \left(\ln \alpha_t^i - \bigoplus_{j=1}^n \ln \alpha_t^j \right) = \ln \alpha_0^i - \bigoplus_{j=1}^n \ln \alpha_0^j.$$

Therefore, $\lim_{t \rightarrow \infty} z_I^P(\mu_t) = z_I^P(\mu_0)$, i.e. the mapping z_I^P is continuous.

Theorem 1 is proved.

Corollary 1. For an arbitrary metrizable compact set X , the spaces $P(X)$ and $I(X)$ are homomorphic. The proof of the following assertion is established using the max-plus variant of the Hahn-Banach theorem [9].

We call a subset L of the space $C(X)$ max-plus-a linear subspace in $C(X)$, if:

- 1) $c_X \in L$ for each $c \in \mathbb{R}$;
- 2) $\lambda \odot \varphi \in L$, for any $\lambda \in \mathbb{R}$ and $\varphi \in L$;
- 3) $\varphi \oplus \psi \in L$, for each $\varphi, \psi \in L$.

Lemma 1 [7]. Let L – max-plus- be a linear subspace in $C(X)$. Let $\mu : L \rightarrow \mathbb{R}$ – be a functional satisfying the conditions of normalization, homogeneity and additivity (with $C(X)$ replaced by L). For an arbitrary $\varphi_0 \in C(X) \setminus L$ there exists a continuation of the mapping μ satisfying the conditions of normalization, homogeneity and additivity for the minimal max-plus- is a linear subspace of L' , containing $L \cup \{\varphi_0\}$ (with $C(X)$ replaced by L').

Consider the following subset in $C(X \times Y)$:

$$C_0 = \left\{ \bigoplus_{i=1}^n \varphi_i \odot \psi_i : \varphi_i \in C(X) \text{ and } \psi_i \in C(Y), i = 1, 2, \dots, n, n \in \mathbb{N} \right\}.$$

It is obvious that C_0 – max-plus-is a linear subspace in $C(X)$.

For every pair $(\mu, \nu) \in I(X) \times I(Y)$ we set

$$(\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_i \odot \psi_i \right) = \bigoplus_{i=1}^n \mu(\varphi_i) \odot \nu(\psi_i).$$

Proposition 1. $\mu \widetilde{\otimes} \nu$ is an idempotent probability measure on C_0 .

Proof. Each $c \in \mathbb{R}$ can be represented as $c_{X \times Y} = a_X \odot b_Y$, where $a, b \in \mathbb{R}$ and $a + b = c$. Therefore, $(\mu \widetilde{\otimes} \nu)(c_{X \times Y}) = (\mu \otimes \nu)(a_X \odot b_Y) = \mu(a) \odot \nu(b) = a \odot b = c$.

Let $\lambda \in \mathbb{R}_{\max}$ and $\bigoplus_{i=1}^n \varphi_i \odot \psi_i \in C_0$. Then $(\mu \widetilde{\otimes} \nu)(\lambda \odot \bigoplus_{i=1}^n \varphi_i \odot \psi_i) = (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n (\lambda \odot \varphi_i) \odot \psi_i \right) = \bigoplus_{i=1}^n \mu(\lambda \odot \varphi_i) \odot \nu(\psi_i) = \bigoplus_{i=1}^n \lambda \odot \mu(\varphi_i) \odot \nu(\psi_i) = \lambda \odot \bigoplus_{i=1}^n \mu(\varphi_i) \odot \nu(\psi_i) = \lambda \odot (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_i \odot \psi_i \right)$.

Finally, let $\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \in C_0$ and $\bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \in C_0$. Then fair $(\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \oplus \bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \right) = (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{k=1}^n \varphi_{kl} \odot \psi_{kl} \right) = \bigoplus_{k=1}^n \mu(\varphi_{kl}) \odot \nu(\psi_{kl}) = \bigoplus_{i=1}^n \mu(\varphi_{1i}) \odot \nu(\psi_{1i}) \oplus \bigoplus_{j=1}^m \mu(\varphi_{2j}) \odot \nu(\psi_{2j}) = (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{i=1}^n \varphi_{1i} \odot \psi_{1i} \right) \oplus (\mu \widetilde{\otimes} \nu) \left(\bigoplus_{j=1}^m \varphi_{2j} \odot \psi_{2j} \right)$. Proposition 1 is proved.

Since C_0 – max-plus-is a linear subspace in $C(X \times Y)$, according to Lemma 1, there is the extension of the idempotent probability measure $\mu \widetilde{\otimes} \nu$ на $C(X \times Y)$.

Set

$$\mu \otimes \nu = \bigoplus \{ \xi \in I(X \times Y) : \xi|_{C_0} = \mu \widetilde{\otimes} \nu \}.$$

Thus, we have proved the following max-plus-a variant of the Fubini theorem.

Theorem 3. For every pair $(\mu, \nu) \in I(X) \times I(Y)$ there exists a unique idempotent probability measure $\mu \otimes \nu \in I(X \times Y)$ such that $(\mu \otimes \nu)(\varphi \odot \psi) = \mu(\varphi) \odot \nu(\psi)$, $\varphi \in C(X)$, $\psi \in C(Y)$.

Although the spaces $I(X)$ and $P(X)$ are homeomorphic for an arbitrary metrizable compact set X , however, as the following example shows, the P and I constructions form different functors from each other.

To construct a corresponding example in which the functors P and I are not isomorphic, we need some concepts from the categorical algebra.

Let $F, G : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ – functors from the category $\mathfrak{R}_1$ to the category $\mathfrak{R}_2$.

The natural transformation $\alpha : F \rightarrow G$ of the functor F into the functor G is called [6] such a function $\alpha : ob\mathfrak{R}_1 \rightarrow Mor\mathfrak{R}_2$, such that $\alpha_A := \alpha(A) : F(A) \rightarrow G(A)$, (α_A , which means the value of the function α in the object A , is a morphism of the category $\mathfrak{R}_2$ from the object $F(A)$ to $G(A)$) and for any $\varphi : A \rightarrow B$, (from the object A to the object B) $B \in ob\mathfrak{R}_1$, in the category $\mathfrak{R}_2$ a commutative diagram

$$\begin{array}{ccc} F(A) & \xrightarrow{\alpha_A} & G(A) \\ F(\varphi) \downarrow & & \downarrow G(\varphi) \\ F(B) & \xrightarrow{\alpha_B} & G(B), \end{array}$$

i.e.

$$G(\varphi) \circ \alpha_A = \alpha_B \circ F(\varphi)$$

The morphisms, $\alpha_A, A \in ob\mathfrak{R}_1$ are called the components of the natural transformation α .

Thus, $\alpha = \{\alpha_A : A \in ob\mathfrak{R}_1\}$. The natural transformation $\varepsilon : F \rightarrow G$ of the functors $F, G : \mathfrak{R}_1 \rightarrow \mathfrak{R}_2$ is called [6] (natural) isomorphism if $\varepsilon_A : F(A) \rightarrow G(A)$ – an isomorphism of the category $\mathfrak{R}_2$, that is, in $\mathfrak{R}_2$ there exists a morphism $\eta_A : G(A) \rightarrow F(A)$, such that $\eta_A \circ \varepsilon_A = id_{F(A)}$ and $\varepsilon_A \circ \eta_A = id_{G(A)}$ for any object $A \in ob\mathfrak{R}_1$.

We now give the construction of an example in which the functors P and I are not isomorphic.

Example 1. Consider the sets $X = \{a, b, c\}$, $Y = \{a, b\}$, $Z = \{a, c\}$, where a, b, c – are different points (these sets are supplied with discrete topologies). Define the following mappings:

$$\begin{aligned} f : X &\rightarrow Y, & f(a) &= f(c) = a, & f(b) &= b \\ g : X &\rightarrow Z, & g(a) &= g(b) = a, & g(c) &= c. \end{aligned}$$

We show that there is no natural transformation of the functor P to the functor I and vice versa. For this purpose, consider the objects $X, Y \times Z$ from the category $Comp$ and the morphism $(f, g) : X \rightarrow Y \times Z$ from the category $Comp$.

It suffices to show that the mapping $P((f, g))$ has a property that the mapping $I((f, g))$ does not possess.

The mapping

$$X \xrightarrow{(f, g)} Y \times Z$$

under the influence of the functor P goes to the map

$$P(X) \xrightarrow{P((f, g))} P(Y \times Z),$$

and under the influence of the I functor –

$$I(X) \xrightarrow{I((f, g))} I(Y \times Z).$$

The morphisms $P((f, g))$ and $I((f, g))$ are determined pointwise [6], i.e.

$$P((f, g)) = (P(f), P(g))$$

and

$$I((f, g)) = (I(f), I(g)).$$

Note that, by the Fubini theorem [5], the correspondence $(P(Y) \times P(Z)) \ni (\mu, \nu) \mapsto \mu \otimes \nu \in P(Y \times Z)$ sets up an attachment

$$P(Y) \times P(Z) \subset \rightarrow P(Y \times Z).$$

By max-plus to the variant of the Fubini theorem (Theorem 2), $(I(Y) \times I(Z)) \ni (\mu, \nu) \mapsto \mu \otimes \nu \in I(Y \times Z)$ sets the embedding

$$I(Y) \times I(Z) \hookrightarrow I(Y \times Z).$$

The morphisms $(P(f), P(g))$ and $(I(f), I(g))$ are determined by the formulas

$$(P(f), P(g))(\mu)(\varphi \cdot \psi) = P(f)(\mu)(\varphi) \cdot P(g)(\mu)(\psi)$$

and

$$(I(f), I(g))(\mu)(\varphi \odot \psi) = I(f)(\mu)(\varphi) \odot I(g)(\mu)(\psi),$$

where $\varphi \in C(Y)$, $\psi \in C(Z)$, $\mu \in P(X)$ or $\mu \in I(X)$.

We show that the mapping $(P(f), P(g)) : P(X) \longrightarrow P(Y) \times P(Z)$ – is an embedding.

In fact, for any

$$\mu = \alpha_1 \delta_a + \alpha_2 \delta_b + \alpha_3 \delta_c,$$

$$\nu = \beta_1 \delta_a + \beta_2 \delta_b + \beta_3 \delta_c,$$

with positive $\alpha_1, \alpha_2, \alpha_3, \beta_1, \beta_2, \beta_3$, $\alpha_1 + \alpha_2 + \alpha_3 = 1$, $\beta_1 + \beta_2 + \beta_3 = 1$, equality takes place

$$P(f)(\mu) = (\alpha_1 + \alpha_3) \delta_a + \alpha_2 \delta_b,$$

$$P(g)(\mu) = (\alpha_1 + \alpha_2) \delta_a + \alpha_3 \delta_c,$$

$$P(f)(\nu) = (\beta_1 + \beta_3) \delta_a + \beta_2 \delta_b,$$

$$P(g)(\nu) = (\beta_1 + \beta_2) \delta_a + \beta_3 \delta_c.$$

Therefore, $(P(f), P(g))(\mu) = (P(f), P(g))(\nu)$ if and only if

$$\begin{cases} \alpha_1 + \alpha_3 = \beta_1 + \beta_3, \\ \alpha_2 = \beta_2, \\ \alpha_1 + \alpha_2 = \beta_1 + \beta_2, \\ \alpha_3 = \beta_3. \end{cases}$$

This system has a unique solution $\alpha_1 = \beta_1, \alpha_2 = \beta_2, \alpha_3 = \beta_3$. Hence, $\mu = \nu$. Thus, $(P(f), P(g))(\mu) = (P(f), P(g))(\nu)$ if and only if $\mu = \nu$, i.e., $(P(f), P(g)) : P(X) \rightarrow P(Y) \times P(Z)$ – is an embedding.

We show that the mapping $(I(f), I(g)) : I(X) \rightarrow I(Y) \times I(Z)$ is not an embedding. In fact, for idempotent probability measures

$$\mu = \lambda_1 \odot \delta_a \oplus \lambda_2 \odot \delta_b \oplus \lambda_3 \odot \delta_c,$$

$$\nu = \gamma_1 \odot \delta_a \oplus \gamma_2 \odot \delta_b \oplus \gamma_3 \odot \delta_c,$$

to $-\infty < \lambda_1, \lambda_2, \lambda_3, \gamma_1, \gamma_2, \gamma_3 \leq 0$ и $\lambda_1 \oplus \lambda_2 \oplus \lambda_3 = \gamma_1 \oplus \gamma_2 \oplus \gamma_3 = 0$, there are equalities

$$I(f)(\mu) = (\lambda_1 \oplus \lambda_3) \odot \delta_a \oplus \lambda_2 \odot \delta_b,$$

$$I(g)(\mu) = (\lambda_1 \oplus \lambda_2) \odot \delta_a \oplus \lambda_3 \odot \delta_c,$$

$$I(f)(\nu) = (\gamma_1 \oplus \gamma_3) \odot \delta_a \oplus \gamma_2 \odot \delta_b,$$

$$I(g)(\nu) = (\gamma_1 \oplus \gamma_2) \odot \delta_a \oplus \gamma_3 \odot \delta_c.$$

The equality $(I(f), I(g))(\mu) = (I(f), I(g))(\nu)$ holds if and only if

$$\begin{cases} \lambda_1 \oplus \lambda_3 = \gamma_1 \oplus \gamma_3, \\ \lambda_2 = \gamma_2, \\ \lambda_1 \oplus \lambda_2 = \gamma_1 \oplus \gamma_2, \\ \lambda_3 = \gamma_3. \end{cases}$$

This, the system has a number of solutions. For example, with $-\infty < \lambda_1 \leq 0$, $-\infty < \gamma_1 \leq 0$ set $(\lambda_1, \gamma_1, 0, 0, 0, 0)$ is her solution.

The equality $(I(f), I(g))(\mu) = (I(f), I(g))(\nu)$ is true, although $\lambda_1 \neq \gamma_1$ for this set. This means that the mapping $(I(f), I(g))$ is not an embedding. Thus, there is no natural transformation of the functor P to the functor I , since the morphism $I((f, g)) = (I(f), I(g))$ is not an embedding.

In other words, there is no natural transformation $\alpha = \{\alpha_X : X \in \text{Comp}\}$ such that the diagram

$$\begin{array}{ccc} I(X) & \xrightarrow{I((f,g))} & I(Y \times Z) \\ \alpha_X \downarrow & & \downarrow \alpha_{Y \times Z} \\ P(X) & \xrightarrow{P((f,g))} & P(Y \times Z) \end{array}$$

would be commutative.

The set of all nonempty closed subsets of the topological space X is denoted by $\exp X$. For the subsets $U_1, \dots, U_n \subset X$ we set

$$\begin{aligned} O\langle U_1, \dots, U_n \rangle &= \{F : F \in \exp X, F \subset \bigcup_{i=1}^n U_n, F \cap U_1 \neq \emptyset, \dots, F \cap U_n \neq \emptyset\} = \\ &= \{F : F \in \exp X, F \subset \bigcup_{i=1}^n U_n\} \cap (\bigcap_{i=1}^n F : F \in \exp X, F \cap U_i \neq \emptyset) \end{aligned}$$

If the sets $U_1, \dots, U_n$ are open, then the sets

$$\{F : F \in \exp X, F \subset \bigcup_{i=1}^n U_n\} = \exp(\bigcup_{i=1}^n U_i, X)$$

$$\{F : F \in \exp X, F \cap U_i \neq \emptyset\} = \exp X \setminus \exp(X \setminus U_i, X)$$

open in the space of closed subsets by definition of the Vietoris topology. Therefore, the set $O\langle U_1, \dots, U_n \rangle$ is open. On the other hand, this kind They also have elements of the prebase of the Vietoris topology: $\exp(U, X) = O\langle U \rangle$, $\exp X \setminus \exp(X \setminus U, X) = O\langle U, X \rangle$. Thus, the family of all sets of the form $O\langle U_1, \dots, U_n \rangle$, where the sets $U_1, \dots, U_n$ are open in the space X , is the base of Vietoris topology. The resulting topological space $\exp X$ is called a hyperspace of X . For the compact X , the hyperspace $\exp X$ is compact.

For the compact X , the natural number n , the functor F we put

$$F_n = \{a \in F(X) : |\text{supp} a| \leq n\},$$

$$F_n^0 = F_n(X) \setminus F_{n-1}(X).$$

Proposition 2. If τ – is an uncountable cardinal, then the space $\exp_3^0 \alpha N_\tau$ is not normal.

Proof. Since the cardinal τ is infinite and the set N_τ is infinite, there are F_1 and F_2 spaces of N_τ , such that F_1 is uncountable, F_2 is countable and $F_1 \cap F_2 = \emptyset$. Take the point $x_0 \in F_1 \cup F_2$. Choose the subsets A_1 and A_2 of the space $\exp_3^0 \alpha N_\tau$, assuming

$$A_1 = \{\{p, x, x_0\} : x \in F_1 \setminus \{x_0\}\}, \quad A_2 = \{\{p, x', x_0\} : x' \in F_2 \setminus \{x_0\}\},$$

where $p \in \alpha N_\tau \setminus N_\tau$. Obviously, $A_1 \cap A_2 = \emptyset$.

Let $F = \{x_1, x_2, x_3\} \in \exp_3^0 \alpha N_\tau \setminus A_1$. The set $O(\{x_1\}, \{x_2\}, \{x_3\})$ is an open neighborhood of the set F , that does not intersect A_1 . Therefore, the set A_1 is closed in $\exp_3^0(\alpha N_\tau)$. The closure of the set A_2 is proved in a similar way.

For each $x \in N_\tau$ we put $U_x = O(\alpha N_\tau \setminus \{x_0, x\}, \{x_0\}, \{x\}) \cap \exp_3^0(\alpha N_\tau)$. It is easy to see that the smallest neighborhoods by inclusion of the sets A_1 and A_2 in $\exp_3^0(\alpha N_\tau)$ are the sets $OA_1 = \bigcup_{x \in F_1} U_x$ and $OA_2 = \bigcup_{x \in F_2} U_x$, respectively. For the set $\{a, b, x_0\}$, where $a \in F_1$, $b \in F_2$, we have

$$\{a, b, x_0\} \in O(\alpha N_\tau \setminus \{x_0, a\}, \{x_0\}, \{a\}) \subset OA_1,$$

$$\{a, b, x_0\} \in O(\alpha N_\tau \setminus \{x_0, b\}, \{x_0\}, \{b\}) \subset OA_2.$$

This means, $OA_1 \cap OA_2 \neq \emptyset$. Proposition 2 is proved.

Proposition 3. If τ is an uncountable cardinal, then the $I_3^0(\alpha N_\tau)$ space is not normal.

Proof. Since $\exp_3^0(X)$ is closed in $I_3^0(X)$ for the compact X , and since normality is a hereditary property for closed subsets of the space, it follows that $I_3^0(\alpha N_\tau)$ is not normal. Proposition 3 is proven.

Theorem 4. Let X – be a compact. If the space $I_3(X) \setminus X$ is hereditarily normal, then X is metrizable.

Proof. Suppose that the compact, X is non-metrizable. If X has one nonisolated point, then X is homeomorphic to $\alpha \mathbb{N}_\tau$ для $\tau = |X| > \omega$. Proposition 3 implies that $I_3^0(X)$ is not normal. But, according to the condition $I_3^0(X)$ must be normal as a subset of the hereditarily normal space $I_3(X) \setminus X$. Got a contradiction.

Now let a and b – be different nonisolated points of the compact X . There are open neighborhoods of U and V of points a and b , such that $\overline{U} \cap \overline{V} = \emptyset$.

Consider the set $Z = \overline{U} \times \exp_2 \overline{V}$, and the formula $\lambda(x_0, x, y) = 0 \odot \delta_x \oplus 0 \odot \delta_y \oplus 0 \odot \delta_z$ we define the topological embedding $\lambda_0 : Z \rightarrow I_3(X) \setminus X$. The result of M. Katetov [11] (which asserts the perfect normality of the factor X under the condition of hereditary normality of the product of $X \times Y$ compact sets, $|Y| \geq \omega$) implies that the factor $\exp_2 \overline{V}$ product $Z = \overline{U} \times \exp_2 \overline{V}$ is perfectly normal. Further, applying the result of V. V. Fedorchuk [9] (which asserts the metrizability of the compact X , if for a normal functor F of degree at least 2 the space $F(X)$ is completely normal) we conclude that $\overline{V}$ is metrizable. Similarly, it can be shown that $\overline{U}$ is metrizable. Therefore, each nonisolated point of the compact X has a metrizable closed neighborhood. Hence, the compact X is locally metrizable. Therefore, it is metrizable. Theorem 4 is proved.

Corollary 2. Let X be a compact set and $n \geq 3$. If the space $I_n X \setminus X$ is hereditarily normal, then the compact X is metrizable.

A closed set A of a space X is called a Z -set in X , if the identity map id_X can be arbitrarily closely approximated by $f : X \rightarrow X \setminus A$. A countable union of Z -sets in X is called a σ - Z -set in X .

Theorem 5. For an arbitrary non-empty closed subset A of a compact X , where $A \neq X$, the subspace $I(A)$ is a Z -set in $I(X)$.

Proof. Fix $x_0 \in X \setminus A$, and for an arbitrary $\varepsilon > 0$, $\varepsilon \leq 1$ we define a continuous mapping $f_\varepsilon : I(X) \rightarrow I(X)$, assuming

$$f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \delta_{x_0}, \mu \in I(X).$$

The mapping is defined correctly. In addition, $f_\varepsilon(\mu) \notin I(A)$ for any $\mu \in I(X)$, because $x_0 \in \text{supp} f_\varepsilon(\mu)$, i.e. $(X \setminus A) \cap \text{supp} f_\varepsilon(\mu) \neq \emptyset$. Therefore, the mapping f_ε acts from $I(X)$ to $I(X) \setminus I(A)$. The distance between points x, y max-plus-the segment $[-\infty, 0]$ is determined by the rule $\rho(x, y) = |e^y - e^x|$. The distance between the points $\alpha_1 \odot x \oplus \beta_1 \odot y$ and $\alpha_2 \odot x \oplus \beta_2 \odot y$ max-plus-the segment $[x, y] = \{\alpha \odot x \oplus \beta \odot y : -\infty \leq \alpha \leq 0, -\infty \leq \beta \leq 0, \alpha \oplus \beta = 0\}$ we define by the formula

$$\rho(\alpha_1 \odot x \oplus \beta_1 \odot y, \alpha_2 \odot x \oplus \beta_2 \odot y) = |e^{\alpha_2} - e^{\alpha_1}| + |e^{\beta_2} - e^{\beta_1}|.$$

Consider the max-plus-segment

$$[\mu, \delta_{x_0}] = \{(\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \delta_{x_0} : 0 \leq \varepsilon \leq 1\}.$$

Calculate the distance between the points $\mu = 0 \odot \mu \oplus (-\infty) \odot \delta_{x_0}$ and $f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \delta_{x_0}$ max-plus-segment $[\mu, \delta_{x_0}]$:

$$\begin{aligned} \rho(\mu, f_\varepsilon(\mu)) &= |e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - e^0| + |e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon} - e^{-\infty}| = \\ &= |e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon}. \end{aligned}$$

Two cases are possible.

1-case. $0 < \varepsilon \leq \frac{1}{2}$. Then

$$|e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon} = 0 + e^{\ln\frac{\varepsilon}{1-\varepsilon}} = \frac{\varepsilon}{1-\varepsilon}.$$

2-case. $\frac{1}{2} < \varepsilon \leq 1$. Then

$$|e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon} = e^{\ln\frac{1-\varepsilon}{\varepsilon}} + 1 = \frac{1-\varepsilon}{\varepsilon} + 1 = \frac{1}{\varepsilon}.$$

So, for $0 < \varepsilon \leq \frac{1}{2}$ the equality $\rho(\mu, f_\varepsilon(\mu)) = \frac{\varepsilon}{1-\varepsilon}$ and idempotent probabilistic measures of $f_\varepsilon(\mu)$ approach μ in the metric of ρ with $\varepsilon \rightarrow 0$. This, in particular, implies that for $\varepsilon \rightarrow 0$ the measures $f_\varepsilon(\mu)$ converge to μ with respect to the topology of the space $I(X)$. Indeed, for an arbitrary function $\varphi \in C(X)$ with $0 < \varepsilon < \frac{1}{2}$ we have $f_\varepsilon(\mu)(\varphi) = 0 \odot \mu(\varphi) \oplus \ln \frac{\varepsilon}{1-\varepsilon} \odot \delta_{x_0}(\varphi) = \mu(\varphi) \oplus (\ln \frac{\varepsilon}{1-\varepsilon} + \varphi(x_0))$. For $0 < \varepsilon < \frac{1}{1+e^{2\|\varphi\|}}$ we have $f_\varepsilon(\mu)(\varphi) = \mu(\varphi)$. Therefore, $0 = |f_\varepsilon(\mu)(\varphi) - \mu(\varphi)| < \varepsilon$. In other words, the identity mapping $id_{I(X)}$ is approximated by the mappings $f_\varepsilon : I(X) \rightarrow I(X) \setminus I(A)$. Theorem 5 is proved.

Theorem 6. For an arbitrary compact X and every natural $n \in \mathbb{N}$, $n < |X|$, the subspace $I_n(X)$ is Z -set in $I(X)$. Therefore, $I_\omega(X)$ is a σ - Z -set.

Proof. Fix $\nu \in I(X) \setminus I_n(X)$. Then $|\text{supp}\nu| \geq n + 1$. For an arbitrary $\varepsilon > 0$, $\varepsilon \leq 1$ we define the mapping $f_\varepsilon : I(X) \rightarrow I(X)$ by the equality

$$f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \nu, \quad \mu \in I(X).$$

The mapping f_ε is defined correctly, and it is continuous. In addition, $f_\varepsilon(\mu) \notin I_n(X)$ for any $\mu \in I(X)$, because $\text{supp}f_\varepsilon(\mu) \supset \text{supp}\mu \cup \text{supp}\nu$, therefore, $|\text{supp}f_\varepsilon(\mu)| \geq n + 1$. Therefore, the mapping f_ε acts from $I(X)$ to $I(X) \setminus I_n(X)$.

Consider the max-plus-segment

$$[\mu, \nu] = \{(\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \nu : 0 \leq \varepsilon \leq 1\}.$$

Calculate the distance between the points $\mu = 0 \odot \mu \oplus (-\infty) \odot \nu$ and $f_\varepsilon(\mu) = (\ln(1 - \varepsilon) - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \mu \oplus (\ln \varepsilon - \ln(1 - \varepsilon) \oplus \ln \varepsilon) \odot \nu$ max-plus-segment $[\mu, \nu]$:

$$\rho(\mu, f_\varepsilon(\mu)) = |e^{\ln(1-\varepsilon)-\ln(1-\varepsilon)\oplus\ln\varepsilon} - 1| + e^{\ln\varepsilon-\ln(1-\varepsilon)\oplus\ln\varepsilon}.$$

For $0 < \varepsilon \leq \frac{1}{2}$ we have

$$\rho(\mu, f_\varepsilon(\mu)) = \frac{\varepsilon}{1-\varepsilon}.$$

When $\frac{1}{2} < \varepsilon \leq 1$ holds

$$\rho(\mu, f_\varepsilon(\mu)) = \frac{1}{\varepsilon}.$$

So, for $0 < \varepsilon \leq \frac{1}{2}$ the equality $\rho(\mu, f_\varepsilon(\mu)) = \frac{\varepsilon}{1-\varepsilon}$ and idempotent probabilistic measures of $f_\varepsilon(\mu)$ approach μ in the metric of ρ with $\varepsilon \rightarrow 0$. This, in particular, implies that for $\varepsilon \rightarrow 0$ the measures $f_\varepsilon(\mu)$ converge to μ with respect to the topology of the space $I(X)$. Indeed, for an arbitrary function $\varphi \in C(X)$ with $0 < \varepsilon < \frac{1}{2}$ we have $f_\varepsilon(\mu)(\varphi) = 0 \odot \mu(\varphi) \oplus \ln \frac{\varepsilon}{1-\varepsilon} \odot \nu(\varphi) = \mu(\varphi) \oplus (\ln \frac{\varepsilon}{1-\varepsilon} + \nu(\varphi))$. For $0 < \varepsilon < \frac{1}{1+e^{2\|\varphi\|}}$ we have $f_\varepsilon(\mu)(\varphi) = \mu(\varphi)$. Therefore $0 = |f_\varepsilon(\mu)(\varphi) - \mu(\varphi)| < \varepsilon$. In other words, the identity mapping $id_{I(X)}$ is approximated by the

mappings $f_\varepsilon : I(X) \rightarrow I(X) \setminus I_n(X)$. Therefore, $I_n(X)$ is a Z -set in $I(X)$. Since $I_\omega(X) = \bigcup_{i=1}^\infty I_n(X)$, it therefore becomes a σ - Z -set. Theorem 6 is proved.

For the functor I , the compact X , the nonempty set $A \subset X$ we set

$$S_I(A) = \{\mu \in I(X) : \text{supp } \mu \cap A \neq \emptyset\}.$$

It is clear that $S_I(\emptyset) = \emptyset$, $S_I(X) = I(X)$. By construction, the inclusion of $A \subset B$ implies $S_I(A) \subset S_I(B)$.

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