

TRAJECTORY ROUTES AND VELOCITY-SIGNATURE CLASSIFICATION OF TRANSITIVE
AND MIXED LOTKA-VOLTERRA TOURNAMENT OPERATORS ON S^3

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ABSTRACT. We give a complete velocity-signature classification of two discrete Lotka–Volterra operators generated by fully oriented tournaments on the three-dimensional simplex S^3 : the transitive tournament T_1 and the mixed tournament T_2 , in which one vertex is dominated by all others while the remaining three vertices form a directed 3-cycle. The two matrices coincide in their first row and column, so any difference in the resulting dynamics is attributable solely to the internal order of the three subordinate strategies. Using the identity $x'_k - x_k = x_k(Ax)_k$, we convert the sixteen formal sign patterns of Ax into systems of linear inequalities and classify all admissible ones for both models, obtaining $N(T_1) = 3$ and $N(T_2) = 7$. For each operator we give the explicit convex-hull domains (where tractable), the complete Jacobian spectrum at every vertex, a Lyapunov-type argument for global convergence, the fixed-point set, and the evolutionary-game reading. Although both operators eliminate the same dominated strategy, T_1 converges to a single dominant vertex, whereas the restriction of V_2 to the complementary invariant face coincides with the classical non-convergent three-cycle Volterra operator.

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Introduction

Quadratic stochastic operators form an important class of nonlinear mappings acting on finite-dimensional simplexes, originating in mathematical genetics and heredity theory [1, 2, 3] and later developed as discrete dynamical systems with associated fixed points, invariant faces, and Lyapunov functions [4]. Among them, *Lotka–Volterra operators* are governed by a skew-symmetric interaction matrix, combining algebraic, geometric, and combinatorial methods [5, 6, 7, 8], in the tradition of the classical continuous Lotka–Volterra equations [13, 14].

Let

$$S^{m-1} = \left\{ x = (x_1, \dots, x_m) \in R^m : x_i \geq 0, \sum_{i=1}^m x_i = 1 \right\}.$$

A discrete-time Lotka–Volterra operator $V : S^{m-1} \rightarrow S^{m-1}$ is defined by

$$x'_k = x_k \left(1 + \sum_{i=1}^m a_{ki} x_i \right), \quad A = (a_{ki}), \quad a_{ki} = -a_{ik}, \quad |a_{ki}| \leq 1.$$

Skew-symmetry gives $x^T A x = 0$, so mass is preserved, and $|a_{ki}| \leq 1$ keeps all coordinates non-negative, so V maps S^{m-1} into itself. In the *tournament* case $a_{ki} \in \{-1, 1\}$ ($k \neq i$), every pair of components has a strict

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dominance relation, with the standard convention $i \rightarrow j \iff a_{ij} = 1$ [11], giving a natural correspondence between Lotka–Volterra dynamics and zero-sum evolutionary games [12].

This paper gives a complete classification of two non-isomorphic tournaments on $m = 4$ vertices that share a common structural feature: one vertex is dominated by all three others, while the relative order among those three differs. In T_1 the dominance hierarchy is fully transitive ($4 \rightarrow 3 \rightarrow 2 \rightarrow 1$); in T_2 the same hub vertex is present (vertex 1, dominated by all others), but the remaining three vertices 2, 3, 4 form a directed 3-cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$ rather than a linear order. Since A_1 and A_2 coincide in their first row and column, every difference between the two classifications below is attributable to this single change in the subordinate order.

The main tool is the sign structure of the velocity vector $Vx - x$. Since $x'_k - x_k = x_k(Ax)_k$, the sign of $(Ax)_k$ decides growth or decline of strategy k in $\text{int}(S^3)$. Each $\sigma \in \{+, -\}^4$ defines a *velocity signature*; the admissible signatures determine a decomposition of S^3 into convex polyhedra [9, 10]. We give a complete classification for both T_1 and T_2 at the same level of detail: explicit domains, full Jacobian spectra at every vertex, the Lyapunov-type method of Ganikhodzhaev [5] for global convergence, the fixed-point set, and the evolutionary-game reading.

Our main results are

$$N(T_1) = 3, \quad N(T_2) = 7,$$

and although both operators eliminate the same hub strategy, the cyclic subtournament in T_2 more than doubles the number of admissible domains and replaces T_1 's convergence to a single vertex with the non-convergent dynamics of the classical 3-cycle Volterra operator studied by Zakharevich [3].

Preliminaries

Write

$$S^3 = \left\{ x = (x_1, x_2, x_3, x_4) \in R^4 : x_i \geq 0, \sum_{i=1}^4 x_i = 1 \right\},$$

with vertices e_1, e_2, e_3, e_4 . Its edge midpoints, face barycenters, and overall barycenter are, respectively,

$$M_{ij} = \frac{e_i + e_j}{2}, \quad N_{ijk} = \frac{e_i + e_j + e_k}{3}, \quad K_{1234} = \frac{1}{4} \sum_{i=1}^4 e_i.$$

For V as above with $a_{ki} \in \{-1, 1\}$ ($k \neq i$), every face of S^3 is V -invariant, and the set of fixed points is denoted

$$\text{Fix}(V) = \{x \in S^3 : Vx = x\}.$$

Since $x'_k - x_k = x_k(Ax)_k$, the sign of the k -th coordinate of $Vx - x$ in the interior of the simplex is determined by the sign of $(Ax)_k$; this observation underlies the following definitions.

Definition 1. (Speed of the mapping) [5]. For V as above, the vector $Vx - x$ is called the *speed of the mapping at the point $x \in S^3$* .

Definition 2. (Velocity signature) [5, 9]. A *velocity signature* $\sigma = (\sigma_1, \sigma_2, \sigma_3, \sigma_4) \in \{+, -\}^4$ is assigned to a point $x \in \text{int}(S^3)$ according to the signs of $(Ax)_k$. For the classification of admissible signatures, we consider the strict systems

$$\sigma_k(Ax)_k > 0, \quad k = 1, \dots, 4.$$

Definition 3. (Admissible signature). A signature σ is *admissible* if the strict system $\sigma_k(Ax)_k > 0$ ($k = 1, \dots, 4$) has a solution in $\text{int } S^3$. The corresponding non-strict system $\sigma_k(Ax)_k \geq 0$ ($k = 1, \dots, 4$), together with the simplex constraints, then determines the associated convex polyhedron F_σ .

Definition 4. (Trajectory route) [9]. The sequence of admissible domains F_σ through which a trajectory $\{x^{(n)}\}$ passes, in order, is called its *trajectory route*.

For a skew-symmetric matrix A we also use the standard sets

$$P = \{x \in S^3 : Ax \geq 0\}, \quad Q = \{x \in S^3 : Ax \leq 0\},$$

which are closely related to the fixed points of the operator and describe possible repelling and attracting configurations [5]. The local character of a fixed point $x^* \in \text{Fix}(V)$ is studied by means of the Jacobian matrix

$$J(V)(x) = \left(\frac{\partial V_k(x)}{\partial x_i} \right)_{k,i=1}^4,$$

evaluated at $x = x^*$, together with its eigenvalues.

Two Tournament Models on S^3

Both tournaments below share the same hub vertex, which is dominated by all three others; they differ only in the internal orientation of the subtournament on the remaining three vertices, as shown in Figure 1.

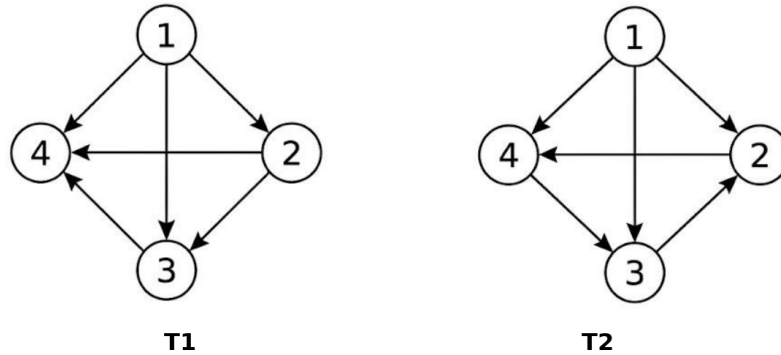


Figure 1: The two tournament representatives on four vertices. T_1 is transitive, with dominance order $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$. T_2 has the same hub structure (vertex 1 dominated by all others) but the subtournament on $\{2, 3, 4\}$ is a directed 3-cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$ instead of a linear order.

The Transitive Tournament T_1 . Let T_1 be the transitive tournament with ordering $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$. The corresponding matrix and operator are

$$A_1 = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & -1 & -1 \\ 1 & 1 & 0 & -1 \\ 1 & 1 & 1 & 0 \end{pmatrix}, \quad V_1 : \begin{cases} x'_1 = x_1(1 - x_2 - x_3 - x_4) \\ x'_2 = x_2(1 + x_1 - x_3 - x_4) \\ x'_3 = x_3(1 + x_1 + x_2 - x_4) \\ x'_4 = x_4(1 + x_1 + x_2 + x_3) \end{cases}$$

$$(A_1x)_1 = -x_2 - x_3 - x_4, \quad (A_1x)_2 = x_1 - x_3 - x_4, \quad (A_1x)_3 = x_1 + x_2 - x_4, \quad (A_1x)_4 = x_1 + x_2 + x_3.$$

For every $x \in \text{int}(S^3)$,

$$(A_1x)_1 < 0 \quad \text{and} \quad (A_1x)_4 > 0.$$

The Mixed Tournament T_2 . Let T_2 be the mixed tournament in which vertex 1 is dominated by all other vertices, while 2, 3, 4 form the directed cycle $2 \rightarrow 3 \rightarrow 4 \rightarrow 2$. The corresponding matrix and operator are

$$A_2 = \begin{pmatrix} 0 & -1 & -1 & -1 \\ 1 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \\ 1 & 1 & -1 & 0 \end{pmatrix}, \quad V_2 : \begin{cases} x'_1 = x_1(1 - x_2 - x_3 - x_4) \\ x'_2 = x_2(1 + x_1 + x_3 - x_4) \\ x'_3 = x_3(1 + x_1 - x_2 + x_4) \\ x'_4 = x_4(1 + x_1 + x_2 - x_3) \end{cases}$$

$$(A_2x)_1 = -x_2 - x_3 - x_4, \quad (A_2x)_2 = x_1 + x_3 - x_4, \quad (A_2x)_3 = x_1 - x_2 + x_4, \quad (A_2x)_4 = x_1 + x_2 - x_3.$$

For every $x \in \text{int}(S^3)$, $(A_2x)_1 < 0$ – identical to T_1 's first coordinate, since A_1 and A_2 share their first row and column exactly. Unlike T_1 , none of $(A_2x)_2, (A_2x)_3, (A_2x)_4$ has a fixed sign, since the cyclic subtournament on $\{2, 3, 4\}$ admits no internal dominance order.

General Signature Decomposition Method

For V_r generated by A_r ($r = 1, 2$), $x'_k - x_k = x_k(A_r x)_k$, so on $\text{int}(S^3)$ the sign of $x'_k - x_k$ equals that of $(A_r x)_k$.

Proposition 1. *For every tournament T_r , each admissible set $F_\sigma^{(r)}$ is a convex polyhedron in S^3 .*

Proof. The simplex S^3 is a convex polytope. For a fixed signature σ , the conditions defining $F_\sigma^{(r)}$ are linear inequalities of the form $(A_r x)_k \geq 0$ or $(A_r x)_k \leq 0$. Therefore $F_\sigma^{(r)}$ is the intersection of S^3 with finitely many closed half-spaces. $\square$

Algorithm 1 (Velocity-signature feasibility verification). For a given tournament matrix A_r : (1) list all $\sigma \in \{+, -\}^4$; (2) for each σ , test the strict system $\sigma_k(A_r x)_k > 0$ ($k = 1, \dots, 4$) together with $x_i > 0$, $\sum_i x_i = 1$; (3) if feasible, σ is admissible, defining $F_\sigma^{(r)}$ by the corresponding non-strict system; (4) otherwise discard σ ; (5) enumerate the surviving domains $F_1^{(r)}, \dots, F_{N_r}^{(r)}$. For the present four-dimensional case, all feasibility tests are verified analytically (Theorems 1 and 3); the finite enumeration above is included only as a systematic organizational procedure.

Lemma 1. *Algorithm 1 terminates after finitely many steps and returns exactly the set of admissible signatures.*

Proof. The algorithm tests each of the $2^4 = 16$ elements of $\{+, -\}^4$ exactly once, so every possible sign pattern of $A_r x$ in the interior of the simplex is tested exactly once; feasible systems give admissible signatures, while infeasible systems are excluded. $\square$

Complete Analysis of the Transitive Tournament T_1

Admissible signatures are determined by the signs of $(A_1 x)_2 = x_1 - x_3 - x_4$ and $(A_1 x)_3 = x_1 + x_2 - x_4$. Moreover,

$$(A_1 x)_3 - (A_1 x)_2 = x_2 + x_3 > 0,$$

so $(A_1 x)_2 > 0$ together with $(A_1 x)_3 < 0$ is impossible.

Theorem 1. *A signature σ is admissible for T_1 if and only if*

$$\sigma \in \{(-, -, -, +), (-, -, +, +), (-, +, +, +)\}.$$

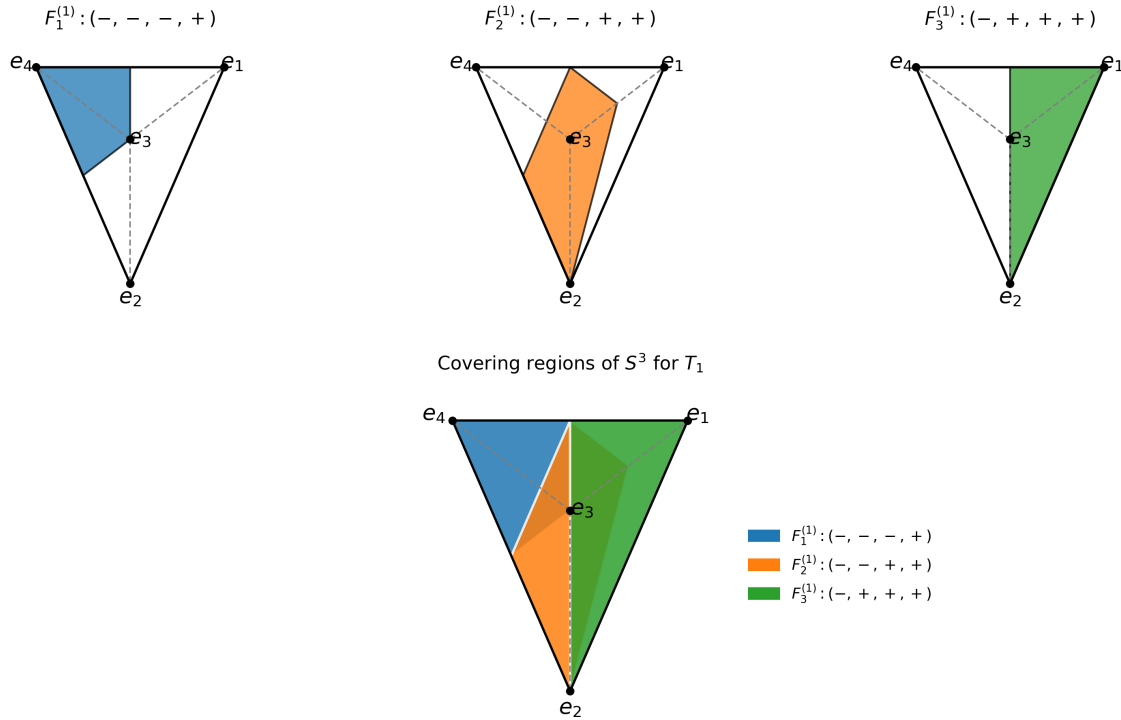
Consequently $N(T_1) = 3$.

Proof. Only σ_2, σ_3 can vary; as noted above, $(A_1 x)_3 - (A_1 x)_2 = x_2 + x_3 > 0$ throughout $\text{int}(S^3)$, so the pattern $\sigma_2 = +, \sigma_3 = -$ is excluded, leaving exactly the three signatures above. $\square$

The corresponding domains and their convex hulls are

$$F_1^{(1)} = \text{co}\{e_4, e_3, M_{24}, M_{14}\}, \quad F_2^{(1)} = \text{co}\{e_3, e_2, M_{24}, M_{14}, M_{13}\}, \quad F_3^{(1)} = \text{co}\{e_2, e_1, M_{14}, M_{13}\},$$

so that $S^3 = F_1^{(1)} \cup F_2^{(1)} \cup F_3^{(1)}$ (Figure 2).

Velocity-signature decomposition of S^3 for T_1 ($N = 3$)

 Figure 2: Velocity-signature decomposition of S^3 for T_1 : the three admissible domains $F_1^{(1)}$, $F_2^{(1)}$, $F_3^{(1)}$ and their union.

The sets are $P_1 = \{e_1\}$, $Q_1 = \{e_4\}$. The Jacobian of V_1 at each vertex has the following spectrum on S^3 (Table 1).

 Table 1: Jacobian spectra of V_1 at the four vertices.

Vertex	Eigenvalues	Character
e_1	1, 2, 2, 2	repelling
e_2	0, 1, 2, 2	saddle
e_3	0, 0, 1, 2	saddle
e_4	0, 0, 0, 1	attracting

The global attracting property is obtained by the Lyapunov-function method of Ganikhodzhaev [5]. Ganikhodzhaev's Theorem 1 associates a monomial Lyapunov function

$$\varphi(x) = x_1^{p_1} x_2^{p_2} x_3^{p_3} x_4^{p_4}$$

with every solution $p = (p_1, p_2, p_3, p_4)$ of $A_1 y \geq 0$ in S^3 , i.e., with every $p \in P_1$; since $P_1 = \{e_1\}$, taking $p = e_1$ gives the degenerate monomial $\varphi(x) = x_1$. This is complemented by the block-sum construction of the same paper (§3), applied to $Q_1 = \{e_4\}$: the sum of the coordinates *outside* Q_1 ,

$$\psi(x) = x_1 + x_2 + x_3 = 1 - x_4.$$

Lemma 2. (Lyapunov functions for T_1) [5]. *For every $x \in S^3$,*

$$\varphi(V_1 x) = \varphi(x)^2, \quad \psi(V_1 x) = \psi(x)^2.$$

Consequently $0 \leq \varphi(V_1x) \leq \varphi(x) \leq 1$ and $0 \leq \psi(V_1x) \leq \psi(x) \leq 1$, with strict inequality on $\text{int}(S^3)$, and

$$\lim_{n \rightarrow \infty} \varphi(x^{(n)}) = \lim_{n \rightarrow \infty} \psi(x^{(n)}) = 0 \quad \text{for every } x \in \text{int}(S^3).$$

Proof. Since $x_1 + x_2 + x_3 + x_4 = 1$,

$$\varphi(V_1x) = x'_1 = x_1(1 - x_2 - x_3 - x_4) = x_1(1 - (1 - x_1)) = x_1^2 = \varphi(x)^2,$$

$$\psi(V_1x) = 1 - x'_4 = 1 - x_4(1 + x_1 + x_2 + x_3) = 1 - x_4(2 - x_4) = (1 - x_4)^2 = \psi(x)^2.$$

Since $0 \leq \varphi(x) \leq 1$ and $0 \leq \psi(x) \leq 1$ on S^3 , squaring gives $\varphi(V_1x) \leq \varphi(x)$ and $\psi(V_1x) \leq \psi(x)$, with equality only at the respective endpoints $\{0, 1\}$; both inequalities are strict on $\text{int}(S^3)$. Iterating, $\varphi(x^{(n)}) = \varphi(x)^{2^n} \rightarrow 0$ and $\psi(x^{(n)}) = \psi(x)^{2^n} \rightarrow 0$. $\square$

Thus $x_1 \rightarrow 0$ and $x_4 \rightarrow 1$ strictly monotonically along every interior trajectory.

Theorem 2. For every $x \in \text{int}(S^3)$, $\lim_{n \rightarrow \infty} V_1^n x = e_4$.

Proof. By Lemma 2, $\psi(x^{(n)}) = 1 - x_4^{(n)} \rightarrow 0$, so $x_4^{(n)} \rightarrow 1$. Since $\sum_{k=1}^4 x_k^{(n)} = 1$ with all coordinates non-negative, the remaining coordinates tend to 0. $\square$

The trajectory route (Definition 4) is $P_1 \rightarrow F_3^{(1)} \rightarrow F_2^{(1)} \rightarrow F_1^{(1)} \rightarrow Q_1$, from the weakest to the dominant pure strategy.

Complete Analysis of the Mixed Tournament T_2

Define

$$g_2(x) = (A_2x)_2 = x_1 + x_3 - x_4,$$

$$g_3(x) = (A_2x)_3 = x_1 - x_2 + x_4,$$

$$g_4(x) = (A_2x)_4 = x_1 + x_2 - x_3.$$

As noted above, $(A_2x)_1 < 0$ throughout $\text{int}(S^3)$, and $g_2 + g_3 + g_4 = 3x_1 > 0$, so the all-negative pattern for (g_2, g_3, g_4) is impossible.

Theorem 3. A signature σ is admissible for T_2 if and only if

$$\sigma_1 = - \quad \text{and} \quad (\sigma_2, \sigma_3, \sigma_4) \neq (-, -, -).$$

Consequently, $N(T_2) = 7$.

Proof. By Definition 2, $\sigma_k = \text{sign}((A_2x)_k)$ for $k = 1, 2, 3, 4$; in particular $\sigma_k = \text{sign}(g_k(x))$ for $k = 2, 3, 4$. As shown above, $(A_2x)_1 < 0$ throughout $\text{int}(S^3)$, so

$$\sigma_1 = \text{sign}((A_2x)_1) = -$$

for every admissible signature. Among the eight sign patterns of $(\sigma_2, \sigma_3, \sigma_4) \in \{+, -\}^3$, exactly one is infeasible:

$$(\sigma_2, \sigma_3, \sigma_4) = (-, -, -) \implies g_2, g_3, g_4 < 0 \implies g_2 + g_3 + g_4 < 0,$$

which contradicts the identity

$$g_2 + g_3 + g_4 = 3x_1 > 0.$$

It remains to show that each of the other seven patterns is genuinely realized by a point of $\text{int}(S^3)$; this is verified directly from the formulas for g_2, g_3, g_4 at the points below.

$(\sigma_2, \sigma_3, \sigma_4)$	$x = (x_1, x_2, x_3, x_4)$
$(-, -, +)$	$(0.1, 0.5, 0.1, 0.3)$
$(-, +, -)$	$(0.1, 0.1, 0.3, 0.5)$
$(-, +, +)$	$(1/6, 1/6, 1/6, 1/2)$
$(+, -, -)$	$(0.1, 0.3, 0.5, 0.1)$
$(+, -, +)$	$(1/6, 1/2, 1/6, 1/6)$
$(+, +, -)$	$(1/6, 1/6, 1/2, 1/6)$
$(+, +, +)$	$(1/4, 1/4, 1/4, 1/4)$

Each point has strictly positive coordinates summing to 1, and a direct substitution into $g_2 = x_1 + x_3 - x_4$, $g_3 = x_1 - x_2 + x_4$, $g_4 = x_1 + x_2 - x_3$ confirms the stated signs in every row. Hence all seven patterns are admissible, and $N(T_2) = 7$. $\square$

The seven domains are listed in Table 2, together with their explicit vertex sets. Unlike T_1 's three domains, most of T_2 's domains are polytopes with vertices beyond the standard points e_i, M_{ij}, N_{ijk} ; each vertex set below is obtained by the same half-space intersection method.

Table 2: The seven admissible domains of T_2 : signatures, defining inequalities, and vertices.

Domain	Signature	Conditions	Vertices
$F_1^{(2)}$	$(-, -, -, +)$	$g_2 \leq 0, g_3 \leq 0, g_4 \geq 0$	$\text{co}\{N_{234}, M_{24}, e_2, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4})\}$
$F_2^{(2)}$	$(-, -, +, -)$	$g_2 \leq 0, g_3 \geq 0, g_4 \leq 0$	$\text{co}\{e_4, M_{34}, (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), N_{234}\}$
$F_3^{(2)}$	$(-, -, +, +)$	$g_2 \leq 0, g_3 \geq 0, g_4 \geq 0$	$\text{co}\{M_{14}, (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), e_4, M_{24}, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}), N_{234}\}$
$F_4^{(2)}$	$(-, +, -, -)$	$g_2 \geq 0, g_3 \leq 0, g_4 \leq 0$	$\text{co}\{e_3, N_{234}, M_{23}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)\}$
$F_5^{(2)}$	$(-, +, -, +)$	$g_2 \geq 0, g_3 \leq 0, g_4 \geq 0$	$\text{co}\{M_{12}, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}), e_2, M_{23}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0), N_{234}\}$
$F_6^{(2)}$	$(-, +, +, -)$	$g_2 \geq 0, g_3 \geq 0, g_4 \leq 0$	$\text{co}\{M_{13}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0), e_3, M_{34}, (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), N_{234}\}$
$F_7^{(2)}$	$(-, +, +, +)$	$g_2 \geq 0, g_3 \geq 0, g_4 \geq 0$	$\text{co}\{e_1, M_{12}, M_{14}, (\frac{1}{4}, \frac{1}{2}, 0, \frac{1}{4}), (\frac{1}{4}, 0, \frac{1}{4}, \frac{1}{2}), N_{234}, M_{13}, (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0)\}$

Each vertex set was obtained by intersecting the corresponding half-spaces $\sigma_k g_k \geq 0$ with S^3 and computing the extreme points of the resulting polytope. The barycenter K_{1234} lies in the strict part of $F_7^{(2)}$, since

$$g_2(K_{1234}) = g_3(K_{1234}) = g_4(K_{1234}) = \frac{1}{4}.$$

Since each $F_\sigma^{(2)}$ is a full three-dimensional sub-polytope of S^3 , no single planar picture can display all seven domains without some overlap once they are projected onto the page; Figure 3 therefore shows each domain on its own set of axes.

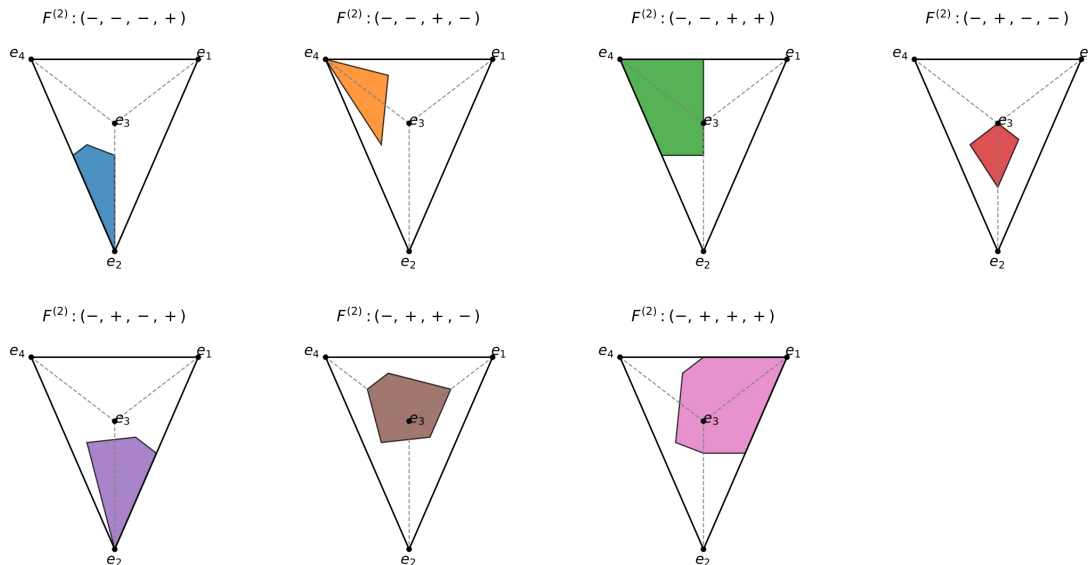


Figure 3: The seven admissible domains of T_2 , shown individually within the simplex S^3 .

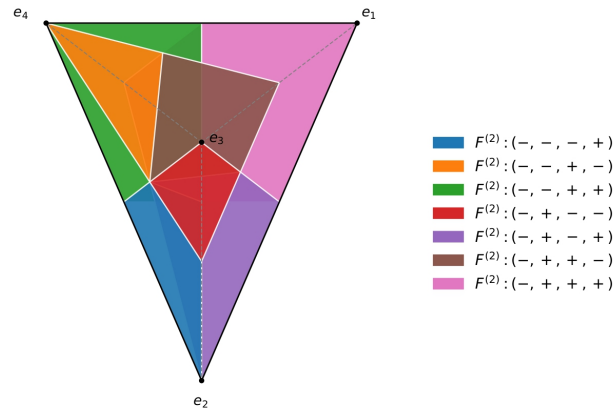


Figure 4: Velocity-signature decomposition of S^3 for T_2 : the seven domains of Figure 3 superimposed. Several projected domains overlap in this two-dimensional visualization, although they remain distinct subsets of the three-dimensional simplex.

The sets are $P_2 = \{e_1\}$, $Q_2 = \{N_{234}\}$, where $N_{234} = (0, 1/3, 1/3, 1/3)$.

Table 3: Jacobian spectra of V_2 at the four vertices.

Vertex	Eigenvalues	Character
e_1	1, 2, 2, 2	repelling
e_2, e_3, e_4	0, 0, 1, 2	saddle

Unlike T_1 , where the four vertices have four distinct spectra, in T_2 the three subordinate vertices e_2, e_3, e_4 share a single spectrum, a direct consequence of the cyclic symmetry among them.

Proposition 2. $\text{Fix}(V_2) = \{e_1, e_2, e_3, e_4\} \cup \{N_{234}\}$, and V_2 has no fixed point in $\text{int}(S^3)$.

Proof. Applying the support-based fixed-point method of [6] to A_2 : fixed points with $|S| \leq 2$ reduce to vertices; the unique 3-cycle $\{2, 3, 4\}$ gives N_{234} ; and $|S| = 4$ is excluded since $\det A_2 = 9 \neq 0$. $\square$

Theorem 4. For every $x \in \text{int}(S^3)$, $x_1^{(n)} \rightarrow 0$ and $V_2^n x$ approaches the face $\text{co}\{e_2, e_3, e_4\}$.

Proof. Since $(A_2 x)_1 = -x_2 - x_3 - x_4 = -(1 - x_1)$ on S^3 , we get $x'_1 = x_1(1 - (1 - x_1)) = x_1^2$, so $x_1^{(n)} = (x_1^{(0)})^{2^n} \rightarrow 0$. $\square$

Unlike T_1 , whose three domains are totally ordered by a single inequality and therefore admit the linear trajectory route $P_1 \rightarrow F_3^{(1)} \rightarrow F_2^{(1)} \rightarrow F_1^{(1)} \rightarrow Q_1$, the seven domains of T_2 carry no such total order: (g_2, g_3, g_4) vary independently subject only to $g_2 + g_3 + g_4 = 3x_1$, so a trajectory's passage among $F_1^{(2)}, \dots, F_7^{(2)}$ depends on the initial point, and the only structural invariant along every route is the monotone elimination $x_1^{(n)} \rightarrow 0$.

Remark 1. Restricted to the invariant face $x_1 = 0$, V_2 reduces exactly to the classical 3-cycle Volterra operator on $\{e_2, e_3, e_4\}$ studied by Zakharevich [3], whose trajectories exhibit non-convergent (“historic”) behavior: the time spent near each vertex grows without bound, and no interior trajectory converges. Thus, although $x_1 \rightarrow 0$ always, the limiting behavior of (x_2, x_3, x_4) is generally *not* convergence to N_{234} or to any vertex.

Comparative Discussion

Table 4 summarizes the classification. Both operators share the same repelling vertex e_1 and the same first-coordinate identity $(A_r x)_1 < 0$; the difference in the number of admissible domains traces to the internal cyclic structure on $\{2, 3, 4\}$ alone.

Table 4: Comparative summary of T_1 and T_2 .

	T_1 (transitive)	T_2 (mixed)
Domains N	3	7
P_r	$\{e_1\}$	$\{e_1\}$
Q_r	$\{e_4\}$	$\{N_{234}\}$
Domain ordering	Total order (single route)	No total order; route depends on initial data
Asymptotics	$V_1^n x \rightarrow e_4$ for all $x \in \text{int}(S^3)$	$x_1 \rightarrow 0$; residual dynamics on $\text{co}\{e_2, e_3, e_4\}$
Spectra at subordinate vertices	Four distinct saddle types	Single saddle type, repeated 3 times

In particular, $N(T_1) \neq N(T_2)$: the velocity-signature count distinguishes the transitive and mixed structures even though both tournaments share an identical hub vertex and an identical first coordinate $(A_r x)_1$.

Evolutionary Game-Theoretic Interpretation

Both A_1, A_2 are payoff matrices of zero-sum games on four pure strategies [12]. In T_1 , the strict hierarchy $4 \rightarrow 3 \rightarrow 2 \rightarrow 1$ makes e_4 the globally attracting pure strategy for interior initial conditions: Theorem 2 shows every interior population converges to it, consistent with the evolutionary-game reading of e_4 as the unique stable outcome. In T_2 , the hub strategy 1 is still eliminated (Theorem 4), but the cyclic subgame on $\{2, 3, 4\}$ has no such stable pure outcome: by Remark 1, the residual dynamics is the classical non-convergent rock-paper-scissors-type cycle, so the population perpetually cycles among e_2, e_3, e_4 rather than settling on any one of them. This is the qualitative dynamical consequence of replacing a single transitive edge-order among the three subordinate strategies with a 3-cycle, while leaving the dominance relation to the hub vertex unchanged.

Conclusion

We gave a complete velocity-signature classification of two Lotka–Volterra tournament operators on S^3 sharing a common dominance hub: $N(T_1) = 3$ for the fully transitive case and $N(T_2) = 7$ for the mixed case with a cyclic subordinate triple. For T_1 , explicit convex-hull domains, full Jacobian spectra, and a Lyapunov-type argument establish global convergence to e_4 . For T_2 , the analogous machinery shows elimination of the same hub strategy, but the residual dynamics has the classical non-convergent behavior of the Zakharevich three-cycle Volterra operator [3], which does not converge to any pure strategy. The two classifications differ only through the internal order of the three subordinate vertices, so this order alone controls both the size of the velocity-signature decomposition and whether the population ultimately settles on a pure strategy or exhibits this non-convergent cyclic behavior.

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