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H_{A_2}-WEAKLY PERIODIC P-ADIC GENERALIZED GIBBS MEASURES FOR THE ISING MODEL WITH AN EXTERNAL FIELD ON THE CAYLEY TREE OF ORDER TWO

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RESUME

In this work, we study H_{A_2}-weakly periodic p-adic generalized Gibbs measures for the Ising model with an external field on the Cayley tree of order two. In the case $|A| = 2$, we prove that at least one unbounded measure of this kind exists for the model. Moreover, we demonstrate that for any odd prime p, a phase transition occurs in the considered model.

Key words: p-adic number, Gibbs measure, Cayley tree, p-adic Ising model with an external field, H_{A_2}-weakly periodic p-adic generalized Gibbs measure, phase transition.

Introduction

It is widely known that the set of Gibbs measures of a given model is the primary focus of research in classical (rigorous) statistical mechanics based on the measure-theoretical scheme [1]. Such a method relies on Kolmogorov's probability theory axioms [2]. p-adic probability, on the other hand, represents a distinct probability-like structure that has recently emerged in theoretical physics [4,5]. These probabilities naturally arise in p-adic physical models, such as the p-adic string proposed by Volovich [22].

This paper investigates weakly periodic p-adic generalized Gibbs measures for the p-adic Ising model with an external field on a Cayley tree. This model is known to have wide-ranging applications in various fields of applied and theoretical sciences. However, this study uniquely focuses on the phase transition within weakly periodic (in particular, non-periodic) p-adic generalized Gibbs measures, an area that has not been previously explored. The existence of a phase transition in models on hierarchical trees is commonly analyzed using the renormalization group (RG) technique, which involves constructing hierarchical lattices and models governed by p-adic rational functions. For the p-adic Ising and Potts models, it has been shown in [8, 9, 11-13] that a significant relationship exists between the phase transition and the chaotic behavior of the RG transformation.

In this paper, we study weakly periodic p-adic generalized Gibbs measures (associated with normal subgroups of index two, specifically when $|A| = 2$) of the Ising model with an external field on the Cayley tree of order two. Furthermore, we prove the existence of a phase transition for the model.

Preliminaries *p-adic Numbers and p-adic Measure*

Let $\mathbb{Q}$ be the field of rational numbers. For a fixed prime number p, any nonzero rational number $x \in \mathbb{Q}$ can be represented in the form

$$x = p^r \frac{n}{m},$$

where $r \in \mathbb{Z}$, $n \in \mathbb{Z}$, $m \in \mathbb{N}$, and both n and m are integers not divisible by p. The *p-adic norm* of x is defined as

$$|x|_p = p^{-r}.$$

Additionally, the p-adic norm of zero is defined by

$$|0|_p = 0.$$

This norm is *non-Archimedean*, i.e., it satisfies the *strong triangle inequality*:

$$|x+y|_p \leq \max\{|x|_p, |y|_p\}, \quad \text{for all } x, y \in \mathbb{Q}.$$

The completion of $\mathbb{Q}$ with respect to the p-adic norm yields the *p-adic number field*, denoted by $\mathbb{Q}_p$.

Every nonzero element $x \in \mathbb{Q}_p$ has a unique canonical representation of the form:

$$x = p^{\gamma(x)}(x_0 + x_1 p + x_2 p^2 + \cdots),$$

where $\gamma(x) \in \mathbb{Z}$ and the digits $x_j \in \{0, 1, \ldots, p-1\}$, with $x_0 \neq 0$. In this case, the p-adic norm is given by

$$|x|_p = p^{-\gamma(x)}.$$

For $a \in \mathbb{Q}_p$ and $r > 0$, define the open p-adic ball as

$$B(a, r) = \{x \in \mathbb{Q}_p : |x-a|_p < r\}.$$

The p-adic exponential is also defined by

$$\exp_p(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!},$$

which converges for $x \in B(0, \frac{1}{2})$ if $p = 2$, and for $x \in B(0, 1)$ if $p \neq 2$.

Lemma 1 ([7,21]). Let $x \in B(0, p^{-1/(p-1)})$. Then the following equalities hold:

$$|\exp_p(x)|_p = 1, \quad |\exp_p(x) - 1|_p = |x|_p < 1.$$

In [14], the authors introduced a new symbol "o" to facilitate the computation of p-adic norms. This symbol generalizes the notation $\equiv \pmod{p^k}$, without explicitly referencing the exponent k. Let us recall the main idea behind this notation. Given a p-adic number x, the expression $o[x]$ denotes a p-adic number with norm strictly less than $p^{-\gamma(x)}$, i.e.,

$$|o(x)|_p < |x|_p.$$

Example. If $x = 1 - p + p^2$, then one can write $o[1] = x - 1$ or $o[p] = x - 1 + p$. Thus, the symbol $o[\cdot]$ provides a convenient tool to simplify calculations involving p-adic numbers.

Define the following subset of $\mathbb{Q}_p$

$$\mathcal{E}_p = \left\{x \in \mathbb{Q}_p : |x-1|_p < p^{-1/(p-1)}\right\}.$$

We recall the definitions of p-adic integers and units:

$$\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}, \quad \mathbb{Z}_p^* = \{x \in \mathbb{Q}_p : |x|_p = 1\}.$$

The following result is a classical and widely used statement, known as Hensel's Lemma:

Lemma 2 (Hensel's Lemma [21]). *Let $f(x) = c_0 + c_1 x + \cdots + c_n x^n$ be a polynomial with coefficients in $\mathbb{Z}_p$. Let $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots + n c_n x^{n-1}$ denote its derivative. Suppose $a \in \mathbb{Z}_p$ satisfies the conditions*

$$f(a) \equiv 0 \pmod{p}, \quad f'(a) \not\equiv 0 \pmod{p}.$$

Then there exists a unique root $x \in \mathbb{Z}_p$ of $f(x)$ such that

$$x \equiv a \pmod{p}.$$

A more detailed exposition of p-adic analysis and its applications in mathematical physics can be found in [20,21].

Let $(X, \mathcal{B})$ be a measurable space, where $\mathcal{B}$ is an algebra of subsets of X. A function $\mu : \mathcal{B} \to \mathbb{Q}_p$ is called a *p-adic measure* if for any finite collection $A_1, A_2, \ldots, A_n \in \mathcal{B}$ of pairwise disjoint sets, i.e., $A_i \cap A_j = \emptyset$ for $i \neq j$, the following additivity condition holds:

$$\mu\left(\bigcup_{j=1}^{n} A_j\right) = \sum_{j=1}^{n} \mu(A_j).$$

A p-adic measure μ is called a *probability measure* if $\mu(X) = 1$. One important property is *boundedness*: a p-adic measure μ is said to be bounded if

$$\sup\{|\mu(A)|_p : A \in \mathcal{B}\} < \infty.$$

It is worth noting that p-adic probability measures are not necessarily bounded in general.

For more detailed information on p-adic measures, we refer the reader to [6,20].

Cayley Tree

The Cayley tree Γ^k of order $k \geq 1$ is an infinite tree in which each vertex is connected to exactly $k+1$ neighbors. A more detailed description of the Cayley tree Γ^k can be found in [19].

Fix a vertex $x_0 \in \Gamma^k$. Define the following sets (see [19]):

$$W_n = \{x \in V : d(x, x_0) = n\}, \quad V_n = \bigcup_{m=0}^{n} W_m, \quad L_n = \{\langle x, y\rangle \in L : x, y \in V_n\},$$

$$S(x) = \{y \in W_{n+1} : d(x, y) = 1\}, \quad x \in V_n,$$

$$S_1(x) = \{y \in V : d(x, y) = 1\}, \quad x_{\downarrow} = S_1(x) \setminus S(x).$$

p-adic Gibbs measures for the Ising model with an external field

We consider the Ising model with an external field on the Cayley tree. Let $\Phi = \{-1, 1\}$. A configuration σ on $A \subset V$ is defined by the function $x \in A \to \sigma(x) \in \Phi$. The set of all configurations on A is denoted by $\Omega_A = \Phi^A$, and $\Omega_V := \Omega$.

A p-adic Hamiltonian in V_n, i.e., a function $H_n : \Omega_{V_n} \to \mathbb{Q}_p$, for the Ising model with an external field is given by

$$H_n(\sigma_n) = J \sum_{\langle x,y\rangle \in L_n} \sigma(x)\sigma(y) + \alpha \sum_{x \in V_n} \sigma(x), \tag{1}$$

where $J, \alpha \in B(0, p^{-1/(p-1)}) \setminus \{0\}$.

We define a function $\widehat{h} : x \in V \to \widehat{h}_x \in \mathbb{Q}_p$, and consider p-adic probability measure $\mu_{\widehat{h}}^{(n)}$ on Ω_{V_n} defined by

$$\mu_{\widehat{h}}^{(n)}(\sigma_n) = \frac{1}{Z_n^{(\widehat{h})}} \exp_p\{H_n(\sigma_n)\} \prod_{x \in W_n} \widehat{h}_x^{\varphi(x)} \quad n = 1, 2, \ldots, \tag{2}$$

where $Z_n^{(\widehat{h})}$ is the normalizing constant

$$Z_n^{(\widehat{h})} = \sum_{\sigma_n(x) \in \Omega_{V_n}} \exp_p\{H_n(\sigma_n)\} \prod_{x \in W_n} \widehat{h}_x^{\varphi(x)}. \tag{3}$$

A p-adic probability measure $\mu_{\widehat{h}}^{(n)}$ is said to be compatable if for all $n \in \mathbb{N}$ and $\sigma_{n-1} \in \Omega_{V_{n-1}}$, we have

$$\sum_{\varphi^{(n)} \in \Omega_{W_n}} \mu_{\widehat{h}}^{(n)}(\sigma_{n-1} \vee \varphi^{(n)}) = \mu_{\widehat{h}}^{(n-1)}(\sigma_{n-1}). \tag{4}$$

In this case, by the p-adic analogue of the Kolmogorov theorem [15], there exists a unique splitting p-adic measure $\mu_{\widehat{h}}$ on the set Ω such that $\mu_h\left(\{\sigma|_{V_n}\equiv\sigma_n\}\right)=\mu_h^{(n)}\left(\sigma_n\right)$ for all $n\in\mathbb{N}$ and $\sigma_n\in\Omega_{V_n}$. Such a limiting p-adic measure generated by (2) is called a *p-adic generalized Gibbs measure*. We note that if $\widehat{h}_x\in\mathcal{E}_p$ for all $x\in V$, then the corresponding measure is called a *p-adic Gibbs measure* (see [8,13]). Clearly, every p-adic Gibbs measure is a p-adic generalized Gibbs measure; however, there exist p-adic generalized Gibbs measures which are not p-adic Gibbs measures.

If there exist at least two distinct p-adic Gibbs measures, one of which is bounded and the other unbounded, then a *phase transition* is said to occur. Furthermore, a *quasi phase transition* is said to occur if there exist two different functions $\mathbf{s}$ and $\mathbf{h}$ such that the corresponding measures $\mu_{\mathbf{s}}$ and $\mu_{\mathbf{h}}$ exist and are either both bounded or both unbounded (see [10]).

The following statement describes the condition on h guaranteeing compatibility of the sequence of probability distributions $\{\mu_h^{(n)}\}_{n\geq1}$.

Theorem 1. [16] *The sequence of p-adic probability distributions $\{\mu_h^{(n)}\}_{n\geq1}$, determined by formula (2), is consistent if and only if for any $x\in V\setminus\{x_0\}$, the following equation holds:*

$$\widehat{h}_x=\eta^{k+1}\prod_{y\in S(x)}\frac{\theta\widehat{h}_y+1}{\widehat{h}_y+\theta},\tag{5}$$

where $\theta=\exp_p(2J)$, $\eta=\exp_p\left(\frac{2\alpha}{k+1}\right)$, and $\widehat{h}_x=\eta\cdot\exp_p(2h_x)$.

Note that, based on this theorem, the task of characterizing p-adic Gibbs measures is simplified to finding the solutions of the functional equation (5).

H_A-weakly periodic p-adic generalized Gibbs measure

Let G_k^* be a normal subgroup with index $r\geq1$ of the group G_k and $G_k/G_k^*=\{H_0,H_1,...,H_{r-1}\}$ be a quotient group (see [19]).

Definition 1. *A set $h=\{h_x,x\in G_k\}$ of quantities is called G_k^*-periodic if $h_x=h_i$, for all $x\in H_i$. A G_k^*-periodic function is called translation-invariant.*

Definition 2. *A set of quantities $h=\{h_x,x\in G_k\}$ is called G_k^*-weakly periodic if $h_x=h_{ij}$, for any $x\in H_i$ and $x_\downarrow\in H_j$.*

Definition 3. *A p-adic generalized Gibbs measure μ_h is said to be G_k^*-(weakly) periodic if it corresponds to a G_k^*-(weakly) periodic h. A G_k-periodic p-adic generalized Gibbs measure is called a translation-invariant p-adic generalized Gibbs measure.*

Note that any normal divisor of index two of the group G_k has the following form

$$H_A=\{x\in G_k:\sum_{i\in A}\omega_x(a_i)\ \text{ is an even number}\},$$

where $\emptyset\neq A\subseteq N_k=\{1,2,3,...,k+1\}$, and $\omega_x(a_i)$ is the number of letters a_i in a word $x\in G_k$. The subgroup H_A is a normal subgroup of the group G_k (see [19]). It can be verified that the form of a weakly periodic Gibbs measure depends on the choice of the normal subgroup H_A of G_k. Note that when $|A|=k+1$, i.e., $A=N_k$, the notion of weakly periodicity coincides with that of the usual periodicity (here, $|A|$ denotes the cardinality of the set A). Therefore, we restrict our attention to the case where $A\subset N_k$ and $A\neq N_k$. Under this condition, an H_A-weakly periodic collection $\{h_x\}_{x\in G_k}$ takes the following form:

$$h_x=\begin{cases}h_{00}, & \text{if}\quad x\in H_0,\quad x_\downarrow\in H_0,\\ h_{01}, & \text{if}\quad x\in H_0,\quad x_\downarrow\in H_1,\\ h_{10}, & \text{if}\quad x\in H_1,\quad x_\downarrow\in H_0,\\ h_{11}, & \text{if}\quad x\in H_1,\quad x_\downarrow\in H_1,\end{cases}\tag{6}$$

i.e., $G_k/H_A=\{H_0,H_1\}$, where $H_0=H_A$ and $H_1=G_k\setminus H_A$.

In [16], translation-invariant and $G_2^{(2)}$-periodic p-adic generalized Gibbs measures for the Ising model with an external field on the Cayley tree were studied. It was shown that if $p \equiv 1 (\operatorname{mod} 4)$, then there exist three translation-invariant and two non-translation-invariant (i.e., $G_2^{(2)}$-periodic) p-adic generalized Gibbs measures. In contrast, if $p \equiv 3 \pmod 4$ with $p \neq 3$, then there exists a unique translation-invariant p-adic generalized Gibbs measure. In [17], we considered H_A-weakly periodic p-adic generalized Gibbs measures for the case $|A| = 1$, i.e., the normal subgroup of index two in the group G_k was of the following form:

$$H_{A_1} = \{x \in G_k : \omega_x(a_i) \equiv 0 \pmod 2\}, i \in \{1, 2, 3\}$$

that is, the number of occurrences of the letter a_i in the word x is even.

In the present paper, we aim to investigate H_{A_2}-weakly periodic (in particular, non-periodic) p-adic generalized Gibbs measures for the Ising model with an external field on the Cayley tree of order two, specifically focusing on the case $|A| = 2$. In this case, the normal subgroup of index two in the group G_k is given by:

$$H_{A_2} = \{x \in G_k : (\omega_x(a_i) + \omega_x(a_j)) \equiv 0 \pmod 2\}, i, j \in \{1, 2, 3\}, i \neq j,$$

that is, the total number of letters a_i and a_j in the word x is even. By (5), we have

$$\begin{cases} \widehat{h}_{00} = \eta^3 \left(\frac{\theta \widehat{h}_{10}+1}{\widehat{h}_{10}+\theta}\right)^2, \\ \widehat{h}_{01} = \eta^3 \frac{\theta \widehat{h}_{00}+1}{\widehat{h}_{00}+\theta} \cdot \frac{\theta \widehat{h}_{01}+1}{\widehat{h}_{01}+\theta}, \\ \widehat{h}_{10} = \eta^3 \frac{\theta \widehat{h}_{11}+1}{\widehat{h}_{11}+\theta} \cdot \frac{\theta \widehat{h}_{10}+1}{\widehat{h}_{10}+\theta}, \\ \widehat{h}_{11} = \eta^3 \left(\frac{\theta \widehat{h}_{01}+1}{\widehat{h}_{01}+\theta}\right)^2. \end{cases} \tag{7}$$

It is easy to verify that the following sets are invariant under the mapping W, which is defined by the right-hand side of equation (7):

$$I_1 = \{(\widehat{h}_{00}, \widehat{h}_{01}, \widehat{h}_{10}, \widehat{h}_{11}) \in \mathbb{Q}_p^4 : \widehat{h}_{00} = \widehat{h}_{01} = \widehat{h}_{10} = \widehat{h}_{11}\},$$
$$I_2 = \{(\widehat{h}_{00}, \widehat{h}_{01}, \widehat{h}_{10}, \widehat{h}_{11}) \in \mathbb{Q}_p^4 : \widehat{h}_{00} = \widehat{h}_{11}, \widehat{h}_{01} = \widehat{h}_{10}\}.$$

On the set I_1, the solutions of equation (7) coincide with the translation-invariant solutions which are studied in [16].

On the set I_2, the system of equations (7) can be rewritten as follows:

$$\begin{cases} \widehat{h}_{00} = \eta^3 \left(\frac{\theta \widehat{h}_{01}+1}{\widehat{h}_{01}+\theta}\right)^2, \\ \widehat{h}_{01} = \eta^3 \frac{\theta \widehat{h}_{00}+1}{\widehat{h}_{00}+\theta} \cdot \frac{\theta \widehat{h}_{01}+1}{\widehat{h}_{01}+\theta}. \end{cases} \tag{8}$$

The system of equations, is similar to (8), was studied in [17]. In [17], it was shown that a similar system exists at least one H_{A_1}-weakly periodic (non-periodic) solution as follows:

$$\widehat{h}_x = \begin{cases} -\frac{\theta+\eta^3}{\theta\eta^3+1}, & \text{if} \quad x \in H_0, \quad x_\downarrow \in H_0, \\ \frac{1}{\eta^3}, & \text{if} \quad x \in H_0, \quad x_\downarrow \in H_1, \\ \frac{1}{\eta^3}, & \text{if} \quad x \in H_1, \quad x_\downarrow \in H_0, \\ -\frac{\theta+\eta^3}{\theta\eta^3+1}, & \text{if} \quad x \in H_1, \quad x_\downarrow \in H_1. \end{cases} \tag{9}$$

Using similar methods as in [17], we obtain the following H_{A_2}-weakly periodic (in particular, non-periodic) solution of (8):

$$\widehat{h}_x = \begin{cases} \frac{1}{\eta^3}, & \text{if} \quad x \in H_0, \quad x_\downarrow \in H_0, \\ -\frac{\theta+\eta^3}{\theta\eta^3+1}, & \text{if} \quad x \in H_0, \quad x_\downarrow \in H_1, \\ -\frac{\theta+\eta^3}{\theta\eta^3+1}, & \text{if} \quad x \in H_1, \quad x_\downarrow \in H_0, \\ \frac{1}{\eta^3}, & \text{if} \quad x \in H_1, \quad x_\downarrow \in H_1. \end{cases} \tag{10}$$

Remark 1. *We note that, according to Appendex A, the H_{A_1}-weakly periodic solution and the H_{A_2}-weakly periodic solution are different. Hence, these solutions determine different weakly periodic p-adic generalized Gibbs measures.*

Using this solution, we get the following theorem:

Theorem 2. *For the Ising model with an external field, there exist at least one H_{A_1}-weakly periodic (non-periodic) and one H_{A_2}-weakly periodic (non-periodic) p-adic generalized Gibbs measure on the Cayley tree of order two.*

Proof. The proof of the existence of at least one H_{A_1}-weakly periodic (non-periodic) p-adic generalized Gibbs measure is given in [17]. Therefore, we provide the proof for the existence of an H_{A_2}-weakly periodic (non-periodic) p-adic generalized Gibbs measure.

The solution given in (10) is an H_{A_2}-weakly periodic (non-periodic) solution of the system (7). According to Definition 3, this implies the existence of an H_{A_2}-weakly periodic (non-periodic) p-adic generalized Gibbs measure for the p-adic Ising model with an external field on the Cayley tree of order two.

The theorem is proved.

Remark 2.

1. We note that the Gibbs measures depend on the choice of the normal subgroup H_A of G_k. This implies that selecting different values of $|A|$ corresponds to choosing different subgroups H_A, which in turn leads to different systems of equations and, consequently, to different types of H_A-weakly periodic Gibbs measures. The set of weakly periodic Gibbs measures also contains the set of periodic Gibbs measures, including, in particular, the translation-invariant Gibbs measures.

2. Translation-invariant p-adic generalized Gibbs measures for the Ising model on the Cayley tree of order two were studied in [16].

3. In particular, when $|A| = 1$, the corresponding H_{A_1}-weakly periodic Gibbs measures have been studied in [17]. In [17], we establish the existence of at least one H_{A_1}-weakly periodic p-adic generalized Gibbs measure for the Ising model on the Cayley tree of order two.

4. If $|A| = k+1$, then the $H_{A_{k+1}}$-weakly periodic Gibbs measure coincides with the $G_k^{(2)}$-periodic Gibbs measure. In [16], the authors studied $G_2^{(2)}$-periodic Gibbs measures for the Ising model on the Cayley tree of order two.

Now we turn to the investigation of the boundedness of the above-established H_{A_2}-weakly periodic p-adic generalized Gibbs measure.

Theorem 3. *Let $p \geq 3$. For the Ising model with an external field on the Cayley tree of order two, there exists at least one unbounded H_{A_1}-weakly periodic (non-periodic) and at least one unbounded H_{A_2}-weakly periodic (non-periodic) p-adic generalized Gibbs measure μ_{h_x}.*

Proof Assume $p \geq$. In [17], we proved the existence of at least one unbounded H_{A_1}-weakly periodic (non-periodic) p-adic generalized Gibbs measure. Hence, we will show at least one unbounded H_{A_2}-weakly periodic (non-periodic) p-adic generalized Gibbs measured. Using (2), we get

$$|\mu^{(n)}_{\widehat{h}_x}(\sigma_n)|_p = \left|\frac{1}{Z_n^{(\widehat{h}_x)}}\right|_p \cdot \left|\exp_p\{H_n(\sigma_n)\} \prod_{x \in W_n} \widehat{h}_x^{\varphi^{(n)}(x)}\right|_p .$$

According to Lemma 1, we have $\left|\exp_p\{H_n(\sigma_n)\}\right|_p = 1$. Since $\theta \in \mathcal{E}_p$ and $\eta \in \mathcal{E}_p$, we obtain $\left|\frac{1}{\eta^3}\right|_p = \left|-\frac{\theta+\eta^3}{\theta\eta^3+1}\right|_p = 1$. Using these results and the equality (22) in [17] we rewrite the p-adic norm of the measure as follow:

$$|\mu^{(n)}_{\widehat{h}_x}(\sigma_n)|_p = \left|\frac{(\theta\widehat{h}_{00}+1)(\widehat{h}_{00}+\theta)}{\theta\widehat{h}_{00}}\right|_p^{2-2^n} \cdot \left|\frac{(\theta\widehat{h}_{01}+1)(\widehat{h}_{01}+\theta)}{\theta\widehat{h}_{01}}\right|_p^{1-2^{n-1}} \cdot \left|\frac{1}{Z_1^{(\widehat{h}_x)}}\right|_p . \quad (11)$$

It is easy to check that

$$0 < \left|Z_1^{(\widehat{h}_x)}\right|_p \leq 1,$$

$$|\theta\widehat{h}_{01}+1|_p=\left|\frac{1-\theta^2}{\theta\eta^3+1}\right|_p<1,$$

$$|\widehat{h}_{01}+\theta|_p=\left|\frac{\eta^3(\theta^2-1)}{\theta\eta^3+1}\right|_p<1, \quad |\theta\widehat{h}_{01}|_p=1,$$

$$|\theta\widehat{h}_{00}+1|_p=\left|\frac{\theta+\eta^3}{\eta^3}\right|_p\leq 1,$$

$$|\widehat{h}_{00}+\theta|_p=\left|\frac{\theta\eta^3+1}{\eta^3}\right|_p\leq 1, \quad |\theta\widehat{h}_{00}|_p=1.$$

From last results we get

$$\lim_{n\longrightarrow\infty}|\mu^{(n)}_{\widehat{h}_x}(\sigma_n)|_p=\infty. \tag{12}$$

This completes the proof.

We remark that the following statements are known for the Ising model with an external field:

1. In [16], the author showed that for any odd prime p, there exists a unique bounded translation-invariant p-adic generalized Gibbs measure.

2. In [17], it was shown that for any prime p, there exists at least one unbounded H_{A_1}-weakly periodic p-adic generalized Gibbs measure.

3. According to Theorem 3, there exists at least one unbounded H_{A_2}-weakly periodic p-adic generalized Gibbs measure.

Using these facts we get the following result:

Theorem 4. *For any odd number p, a phase transition occurs in the Ising model with an external field on the Cayley tree of order two.*

Appendix A

In this section, we prove that the measures defined by (9) and (10) are distinct.

Preposition 1. *The measures corresponding to (10) and (9) are distinct.*

Proof. To this end, let $\mu_n^{(1)}$ and $\mu_n^{(2)}$ denote the sequences of p-adic probability distributions corresponding to (10) and (9), respectively. Assume that $\mu_n^{(1)}\to\mu$ as $n\to\infty$, which implies $\lim_{n\to\infty}|\mu_n^{(1)}-\mu|_p\to 0$. Our objective is to demonstrate that $\lim_{n\to\infty}|\mu_n^{(2)}-\mu|_p\nrightarrow 0$. By the strong triangle inequality, we have:

$$|\mu_n^{(2)}-\mu|_p=|\mu_n^{(2)}-\mu_n^{(1)}+\mu_n^{(1)}-\mu|_p\leq\max\{|\mu_n^{(2)}-\mu_n^{(1)}|_p,|\mu_n^{(1)}-\mu|_p\}.$$

Since $\lim_{n\to\infty}|\mu_n^{(1)}-\mu|_p\to 0$, the proof reduces to showing that $\lim_{n\to\infty}|\mu_n^{(2)}-\mu_n^{(1)}|_p\neq 0$.

Let

$$h_0=-\frac{\theta+\eta^3}{\theta\eta^3+1} \quad \text{and} \quad h_1=\frac{1}{\eta^3}. \tag{13}$$

According to equation (11) and equation (22) in [17], the sequences of p-adic probability distributions corresponding to (10) and (9) can be expressed as:

$$\mu_n^{(1)}=\left(\frac{(\theta h_0+1)(h_0+\theta)}{\theta h_0}\right)^{2-2^n}\cdot\left(\frac{(\theta h_1+1)(h_1+\theta)}{\theta h_1}\right)^{1-2^{n-1}}\cdot$$

$$\cdot\frac{1}{Z_1^{(h_x)}}\exp_p\{H_n(\sigma_n)\}(h_0)^{\alpha_n+\delta_n}(h_1)^{\beta_n+\gamma_n},$$

$$\mu_n^{(2)}=\left(\frac{(\theta h_1+1)(h_1+\theta)}{\theta h_1}\right)^{2-2^n}\cdot\left(\frac{(\theta h_0+1)(h_0+\theta)}{\theta h_0}\right)^{1-2^{n-1}}\cdot$$

$$\cdot \frac{1}{Z_1^{(h_x)}} \exp_p\{H_n(\sigma_n)\}(h_1)^{\alpha_n+\delta_n}(h_1)^{\beta_n+\gamma_n},$$

where $\alpha_n, \beta_n, \gamma_n, \delta_n$ represent the counts of $h_{00}, h_{01}, h_{10}, h_{11}$ on W_n, respectively. From Lemma 2.15 in [18], we have $\alpha_n + \delta_n = 2^n$ and $\beta_n + \gamma_n = 2^{n-1}$. The p-adic norm of the difference between $\mu_n^{(2)}$ and $\mu_n^{(1)}$ is given by:

$$|\mu_n^{(2)} - \mu_n^{(1)}|_p = \left|\frac{(\theta h_1+1)(h_1+\theta)}{\theta h_1}\right|_p^{1-2^{n-1}} \cdot \left|\frac{(\theta h_0+1)(h_0+\theta)}{\theta h_0}\right|_p^{2-2^n} \frac{|h_0|_p^{2^n}|h_1|_p^{2^{n-1}}|\exp_p\{H_n(\sigma_n)\}|_p}{|Z_1^{(h_x)}|_p} \cdot$$

$$\left|\left(\frac{(\theta h_0+1)(h_0+\theta)}{(\theta h_1+1)(h_1+\theta)}\right)^{2^{n-1}-1} \cdot \left(\frac{h_1}{h_0}\right)^{2^n-1} - 1\right|_p .$$

It follows from the results in the proof of Theorem 3 that

$$\lim_{n\to\infty} \left|\frac{(\theta h_1+1)(h_1+\theta)}{\theta h_1}\right|_p^{1-2^{n-1}} \cdot \left|\frac{(\theta h_0+1)(h_0+\theta)}{\theta h_0}\right|_p^{2-2^n} \cdot$$

$$\cdot \frac{|h_0|_p^{2^n}|h_1|_p^{2^{n-1}}|\exp_p\{H_n(\sigma_n)\}|_p}{|Z_1^{(h_x)}|_p} = \infty. \tag{14}$$

Applying the strong triangle inequality, we deduce that

$$\lim_{n\to\infty} \left|\left(\frac{(\theta h_0+1)(h_0+\theta)}{(\theta h_1+1)(h_1+\theta)}\right)^{2^{n-1}-1} \cdot \left(\frac{h_1}{h_0}\right)^{2^n-1} - 1\right|_p \leq$$

$$\leq \max\left\{\lim_{n\to\infty} \left|\left(\frac{(\theta h_0+1)(h_0+\theta)}{(\theta h_1+1)(h_1+\theta)}\right)^{2^{n-1}-1} \cdot \left(\frac{h_1}{h_0}\right)^{2^n-1}\right|_p, 1\right\} = 1. \tag{15}$$

Since $\eta, \theta \in \mathcal{E}_p$ and by (13) we have

$$\left|\frac{(\theta h_0+1)(h_0+\theta)}{(\theta h_1+1)(h_1+\theta)}\right|_p^{2^{n-1}-1} \cdot \left|\frac{h_1}{h_0}\right|_p^{2^n-1} = \left|\frac{\eta^9(\theta^2-1)^2}{(\theta\eta^3+1)^3(\theta+\eta^3)}\right|_p^{2^{n-1}-1}.$$

Consequently, we obtain

$$\lim_{n\to\infty} \left|\frac{(\theta h_0+1)(h_0+\theta)}{(\theta h_1+1)(h_1+\theta)}\right|_p^{2^{n-1}-1} \cdot \left|\frac{h_1}{h_0}\right|_p^{2^n-1} = 0.$$

Equations (14) and (15) together imply that

$$|\mu_n^{(2)} - \mu_n^{(1)}|_p = \infty.$$

This result shows that $\mu_n^{(2)} \nrightarrow \mu$. Therefore, the measures corresponding to (10) and (9) do not coincide; that is, they are distinct measures. This concludes the proof.

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REZYUME

Ushbu ishda biz ikkinchi tartibli Keli daraxtidagi tashqi maydonli Izing modeli uchun H_{A_2}-kuchsiz davriy p-adik umumlashgan Gibbs o‘lchovlarini o‘rganamiz. $|A| = 2$ bo‘lgan holatda, ushbu model uchun hech bo‘lmaganda bitta chegaralanmagan H_{A_2}-kuchsiz davriy p-adik umumlashgan Gibbs o‘lchovi mavjudligini isbotlaymiz. Bundan tashqari, ushbu ishda qaralayotgan modelda ixtiyoriy p

toq tub son uchun faza o‘tishi sodir bo‘lishini ko‘rsatamiz.

Kalit so‘zlar: p-adik son, Gibbs o‘lchovi, Keli daraxti, tashqi maydonli Ising modeli, H_{A_2}-kuchsiz davriy p-adik umumlashgan Gibbs o‘lchovi.

РЕЗЮМЕ

В данной работе мы изучаем H_{A_2}-слабо периодические p-адические обобщённые меры Гиббса для модели Изинга с внешним полем на дереве Кэли второго порядка. В случае $|A| = 2$ доказывается, что для данной модели существует по крайней мере одна неограниченная мера такого типа. Кроме того, мы показываем, что при любом нечётном простом числе p в рассматриваемой модели происходит фазовый переход.

Ключевые слова: p-адическое число, мера Гиббса, дерево Кэли, модель Изинга с внешним полем, H_{A_2}-слабо периодическая p-адическая обобщённая мера Гиббса, фазовый переход.