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AN INVERSE PROBLEM FOR A LOADED DEGENERATE FRACTIONAL ORDER DIFFUSION EQUATION WITH INVOLUTION PERTURBATION

Abdullaev O. Kh.
Alfraganus University, Tashkent, Uzbekistan
obidjon.mth@gmail.com
Duysenbaev R. S.
Tashkent state technical university named after Islam Karimov, Tashkent, Uzbekistan
ruslanduysenbaev.0299@gmail.com

RESUME

This work deals to the study an inverse problem for the loaded degenerating fractional order diffusion equation with involution perturbation. The existence and uniqueness theorem of solutions to the formulated problem was proved.

Key words: Inverse problems; fractional-order loaded diffusion equation; involution perturbation; existence and uniqueness of solution; series expansion.

1. Introduction

In this paper, we consider an inverse problem for a degenerating diffusion equation with Rimann-Liouville type derivatives involving involution. This inverse problem is close to that investigated in [1, 2]. Together with the solution it is necessary to find an unknown right-hand side of the equation. Unlike the above the mentioned works, considered equation contains the fractional time derivative and an involution with respect to the spatial variable. The conditions for over determination are initial and final states. In work [3], by authors was considered a process that is so slow that it is described by an evolutionary equation with a fractional time derivative. Thus, this process is described by equation

$$t^{-\beta} D_t^{\alpha} \Phi(x,y)' \Phi_{xx}(x,t) + \varepsilon \Pi_{xx}(-x,t) = f(x),$$

in a rectangular domain. Here, $f(x)$ is the influence of an external source that does not change with time; $\alpha + \beta > 0$; $t = 0$ is an initial time point and $t = T$ is a final one; and the derivative D_t^{α} is a Caputo derivative.

Different problems for the equations with involutions have been studied by many authors, as A. Ashyralyev [4], M. A. Sadybekov [5], A.Andreev [6], M.Sh Burlutskayaa [7] and others. Concerning inverse problems for heat equations, some recent works have been implemented by M. Kirane, B. Samet, B.T. Torebek [8], N. Al-Salti, M. Kirane, B.T.Torebek [9], B.T. Torebek., R. Tapdigoglu [10], B. Ahmad , A. Alsaedi, M. Kirane., R.G. Tapdigoglu [11], B.Turmetov, B. J. Kadirkulov [12] and others.

As far as we know, direct and inverse problems for a degenerating loaded diffusion equations with involution, including fractional derivatives, have not been investigated before.

In this work, we state an inverse problem with the Dirichlet boundary conditions for a degenerating loaded diffusion equation with Rimann-Liouville type derivatives involving involution, in rectangular domain. We seek formal solution to this problem in a form of series expansions using orthogonal basis obtained by separation of variables and we also examine the convergence of the obtained series solution. The main result on existence and uniqueness are formulated in a theorem in the last section of this paper.

Let $J = [a, b], (-\infty < a < b < \infty)$ be a finite interval on the real axis $\mathbb{R}$.

Definition 1.[13] The *Rimann-Liouville fractional integral* $I_{a+}^{\alpha} f$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) > 0)$ is defined by

$$(I_{a+}^{\alpha} f)(x) := \frac{1}{\Gamma(\alpha)} \int_a^x \frac{f(t)dt}{(x-t)^{1-\alpha}}, \ \ (x > a; \quad \Re(\alpha) > 0).$$

Here $\Gamma(\alpha)$ is the Gamma function.
Definition 2.[13] *The Rimann-Liouville fractional derivative* $D^{\alpha}_{a+}y$ of order $\alpha \in \mathbb{C}$ $(\Re(\alpha) \geq 0)$ is defined by

$$(D^{\alpha}_{a+}y)(x) := \left(\frac{d}{dx}\right)^n (I^{n-\alpha}_{a+}y)(x) =$$

$$= \frac{1}{\Gamma(n-\alpha)}\left(\frac{d}{dx}\right)^n \int_a^x \frac{y(t)dt}{(x-t)^{\alpha-n+1}}, \ (n = [\Re(\alpha)] + 1; \quad x > a).$$

Definition 3.[13] A function of generalized Mittag-Leffler type is a function defined by the series

$$E_{\alpha,m,l}(z) = \sum_{k=0}^{\infty} c_k z^k$$

with

$$c_0 = 1 \quad and \quad c_k = \prod_{j=0}^{k-1} \frac{\Gamma[\alpha(jm+l)+1]}{\Gamma[\alpha(jm+l+1)+1]}, \quad (k \in \mathbb{N}), \quad \alpha, l \in \mathbb{C}, \quad m \in \mathbb{R}.$$

Here $\Gamma(\alpha)$ is the Gamma function.
Proposition 1.[15] For every $\alpha \in (0,1]$, $m > 0$, $l > m - 1/\alpha$ and $x \geq 0$, one has

$$\frac{1}{1 + \frac{\Gamma(1+\alpha(l-m))}{\Gamma(1+\alpha(l-m+1))}x} \leq E_{\alpha,m,l}(-x) \leq \frac{1}{1 + \frac{\Gamma(1+\alpha l)}{\Gamma(1+\alpha(1+l))}x}.$$

We consider

$$(D^{\alpha}_{a+}u)(t) = \lambda(t-a)^{\beta}u(t) + \sum_{r=1}^{k} f_r(t-a)^{\mu_r}, \quad (a < t < b \leq \infty), \tag{1}$$

where $\lambda, \beta \in \mathbb{R}$ and $f_r, \mu_r \in \mathbb{R}$ $(r = 1, ..., k)$ are given real constants.
Theorem 1.[13] Let $n-1 < \alpha < n$ $(n \in \mathbb{N})$, $\beta > -\alpha$, $\mu_r \in \mathbb{R}$ $(r = 1, ..., k)$ be such that

$$\mu_r > -1-\alpha, \quad \mu_r \neq j \quad (j = 1, ..., [\alpha]+1; \quad r = 1, ..., k), \tag{2}$$

$$(j+1)(\alpha+\beta) + \mu_r \notin \mathbb{Z}^- \quad (r = 1, ..., k; \quad j \in \mathbb{N}_0), \tag{3}$$

(i) equation (1) is solvable in the space $I_{loc}(a,b)$ if there exists a $j \in \{1, ..., k\}$ such that $\mu_j < -\alpha$ and in the space $B_{loc}(a,b)$ if $\mu_r \geq -\alpha$ for all $r \in \{1, ..., k\}$. Furthermore, there is a particular solution of the form

$$u_0(t) = \sum_{r=1}^{k} \frac{\Gamma(\mu_r+1)f_r}{\Gamma(\mu_r+\alpha+1)}(t-a)^{\alpha+\mu_r} E_{\alpha,1+\beta/\alpha,1+(\beta+\mu_r)/\alpha}\left(\lambda(t-a)^{(\alpha+\beta)}\right), \tag{4}$$

(ii) if $\beta > -\{\alpha\}$, then the general solution to the question (1) is given by

$$u(t) = \sum_{j=1}^{n} c_j(t-a)^{\alpha-j} E_{\alpha,1+\beta/\alpha,1+(\beta-j)/\alpha}\left(\lambda(t-a)^{(\alpha+\beta)}\right) + u_0(t), \tag{5}$$

where c_j $(j = 1, ..., n)$ are arbitrary real constants.

2. Statement of problem

Consider the diffusion equation

$$D^{\alpha}_{0t}u(x,t) - a_1 t^{\beta}u_{xx}(x,t) + a_2 t^{\beta}u_{xx}(-x,t) + b_1 u(x,t_0) + b_2 u(-x,t_0) = f(x), \quad (x,t) \in \Omega \tag{6}$$

where D^{α}_{0t} is Rimann-Liouville derivative, α, β a_i, b_i $(i = 1,2)$ are nonzero real numbers such that, $0 < \alpha \leq 1$, $\alpha+\beta > 0$, $a_1 > 0$, $|a_2| < a_1$ and Ω is a rectangular domain given by $\Omega = \{(x,t) : -l < x < l, \ 0 < t < T\}$, t_0

is any fixed number, and that $t_0 \in (0,T)$. Our aim is to find a regular solution to the following inverse problem:
Problem IP. Find a pair of functions $u(x,t)$ and $f(x)$ in the domain Ω satisfying equation (6) and the conditions

$$\lim_{t\to 0} D_{0t}^{\alpha-1}u(x,t)=\varphi(x), \quad u(x,T)=\psi(x), \quad x\in[-l,l], \tag{7}$$

$$u(-l,t)=0, \quad u(l,t)=0, \quad t\in[0,T], \tag{8}$$

where $\varphi(x)$ and $\psi(x)$ are given functions, $\varphi(x),\psi(x)\in C^3[-l,l]$, and that

$$\varphi^i(-l)=\psi^i(l)=0, \ (i=0,2).$$

3. Solution method

Here, we seek solution to problem IP in a form of series expansion, using a set of functions that form orthogonal basis in $L_2(-l,l)$. To find the appropriate set of functions for the problem, we shall solve the homogeneous equation corresponding to equation (6) along with the associated boundary conditions using separation of variables.

3.1. Spectral problem

Separation of variables leads to the following spectral problem for IP

$$a_1X''(x)-a_2X''(-x)+\lambda X(x)=0, \quad X(-l)=X(l)=0. \tag{9}$$

The eigenvalue problem (9) are self-adjoint and hence they have real eigenvalues and its eigenfunctions form a complete orthogonal basis in $L_2(-l,l)$ [15]. Eigenvalues are given by

$$\lambda_{1k}=(a_1+a_2)\left(\frac{\pi k}{l}\right)^2, \quad \lambda_{2k}=(a_1-a_2)\left(\frac{\pi(2k-1)}{2l}\right)^2, \ k\in\mathbb{N}, \tag{10}$$

and the corresponding eigenfunctions are given by

$$X_{1k}=\sin\frac{\pi k}{l}x, \quad X_{2k}=\cos\frac{\pi(2k-1)}{2l}x, \ \ k\in\mathbb{N}. \tag{11}$$

Lemma 1. The system of functions (11) are complete and orthogonal in $L_2(-l,l)$. This lemma was proven by E.I. Moiseev [16].
Since the system of eigenfunctions (11) is complete and forms a basis in $L_2(-l,l)$, the solution pair $u(x,t)$ and $f(x)$ of the inverse problem can be expressed in a form of series expansion using the eigenfunctions.

3.2. Existence of solution

Here, we give a full proof of existence of solution to the Inverse Problem IP. Using the orthogonal system (11), the functions $u(x,t)$ and $f(x)$ can be represented as follows

$$u(x,t)=\sum_{k=1}^{\infty}u_{1k}(t)\sin\frac{\pi k}{l}x+\sum_{k=1}^{\infty}u_{2k}(t)\cos\frac{\pi(2k-1)}{2l}x, \tag{12}$$

$$f(x)=\sum_{k=1}^{\infty}f_{1k}\sin\frac{\pi k}{l}x+\sum_{k=1}^{\infty}f_{2k}\cos\frac{\pi(2k-1)}{2l}x, \tag{13}$$

where the coefficients $u_{1k}(t),u_{2k}(t),f_{1k}$ and f_{2k} are unknown. Substituting (12) and (13) into equation (6), we obtain the following equations relating the functions $u_{1k}(t),u_{2k}(t)$ and the constants f_{1k},f_{2k} :

$$D_{0t}^{\alpha}u_{1k}(t)+\lambda_{1k}t^{\beta}u_{1k}(t)=f_{1k}-(b_1-b_2)u_{1k}(t_0), \tag{14}$$

$$D_{0t}^{\alpha}u_{2k}(t)+\lambda_{2k}t^{\beta}u_{2k}(t)=f_{2k}-(b_1+b_2)u_{2k}(t_0). \tag{15}$$

Solving Cauchy problems for these equations with initial data

$$\lim_{t\to 0} D_{0t}^{\alpha-1}u_{1k}(t)=\varphi_{1k},$$

$$\lim_{t\to 0} D_{0t}^{\alpha-1}u_{2k}(t)=\varphi_{2k},$$

respectively, we get (see Theorem 1)

$$\begin{gathered} u_{1k}(t)=\frac{\varphi_{1k}t^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}t^{\alpha+\beta}\right)+ \\ +\frac{(f_{1k}-\varepsilon_1(b)u_{1k}(t_0))t^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t^{\alpha+\beta}\right), \end{gathered} \tag{16}$$

$$\begin{gathered} u_{2k}(t)=\frac{\varphi_{2k}t^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}t^{\alpha+\beta}\right)+ \\ +\frac{(f_{2k}-\varepsilon_2(b)u_{2k}(t_0))t^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t^{\alpha+\beta}\right). \end{gathered} \tag{17}$$

Here $\lambda_{1k}=(a_1+a_2)\left(\frac{\pi k}{l}\right)^2$, $\lambda_{2k}=(a_1-a_2)\left(\frac{\pi(2k-1)}{2l}\right)^2$, $\varepsilon_1(b)=b_1-b_2$, $\varepsilon_2(b)=b_1+b_2$ and the unknown constants f_{1k}, f_{2k} are to be determined using the second condition in (7), and $u_{1k}(t_0)$, $u_{2k}(t_0)$ are unknown constants which also required to determine.

Let $\varphi_{ik}, \psi_{ik}, i=1,2$ be the coefficients of the series expansions of $\varphi(x)$ and $\psi(x)$, respectively, i.e.,

$$\varphi_{1k}=\frac{1}{l}\int_{-l}^{l}\varphi(x)\sin\frac{\pi k}{l}xdx, \quad \varphi_{2k}=\frac{1}{l}\int_{-l}^{l}\varphi(x)\cos\frac{\pi(2k-1)}{2l}xdx,$$

$$\psi_{1k}=\frac{1}{l}\int_{-l}^{l}\psi(x)\sin\frac{\pi k}{l}xdx, \quad \psi_{2k}=\frac{1}{l}\int_{-l}^{l}\psi(x)\cos\frac{\pi(2k-1)}{2l}xdx.$$

Then, the second conditions in (7) leads to

$$\begin{gathered} (f_{1k}-\varepsilon_1(b)u_{1k}(t_0))\frac{T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)= \\ =\psi_{1k}-\varphi_{1k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right), \end{gathered} \tag{18}$$

$$\begin{gathered} (f_{2k}-\varepsilon_2(b)u_{2k}(t_0))\frac{T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)= \\ =\psi_{2k}-\varphi_{2k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right). \end{gathered} \tag{19}$$

Considering, $E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{ik}T^{\alpha+\beta}\right)\geq\delta>\varepsilon$, $i=1,2$ (see Proposition 1), from (18) and (19), respectively we obtain

$$f_{1k}-\varepsilon_1(b)u_{1k}(t_0)=\frac{\psi_{1k}-\varphi_{1k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)}{\frac{T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)}, \tag{20}$$

$$f_{2k}-\varepsilon_2(b)u_{2k}(t_0)=\frac{\psi_{2k}-\varphi_{2k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)}{\frac{T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)}. \tag{21}$$

On the other hand, from (16) and (17) we find:

$$f_{1k}-\varepsilon_1(b)u_{1k}(t_0)=\frac{u_{1k}(t_0)-\varphi_{1k}\frac{t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)}{\frac{t_0^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)}, \tag{22}$$

$$f_{2k}-\varepsilon_2(b)u_{2k}(t_0)=\frac{u_{2k}(t_0)-\varphi_{2k}\frac{t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)}{\frac{t_0^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)}. \tag{23}$$

Solving these set of algebraic equations (20)-(23), we get

$$u_{1k}(t_0)=\frac{\varphi_{1k}t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)+$$

$$+\frac{\psi_{1k}-\varphi_{1k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}((-\lambda_{1k}T^{\alpha+\beta})}{T^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)}\cdot t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right), \tag{24}$$

$$u_{2k}(t_0)=\frac{\varphi_{2k}t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)+$$

$$+\frac{\psi_{2k}-\varphi_{2k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}((-\lambda_{2k}T^{\alpha+\beta})}{T^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)}\cdot t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right). \tag{25}$$

Hence, substituting $u_{1k}(t_0)$ and $u_{2k}(t_0)$ into (20) and (21), we find

$$f_{1k}=\frac{\psi_{1k}-\varphi_{1k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)}{\frac{T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)}\times$$

$$\times\left(1+\frac{\varepsilon_1(b)t_0^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)\right)+\frac{\varphi_{1k}\varepsilon_1(b)t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right) \tag{26}$$

and

$$f_{2k}=\frac{\psi_{2k}-\varphi_{2k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)}{\frac{T^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)}\times$$

$$\times\left(1+\frac{\varepsilon_2(b)t_0^{\alpha}}{\Gamma(\alpha+1)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)\right)+\frac{\varphi_{2k}\varepsilon_2(b)t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right). \tag{27}$$

Further, taking (16),(17) and (20),(21) from (12) we get

$$u(x,t)=\sum_{k=1}^{\infty}\frac{\varphi_{1k}t^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}t^{\alpha+\beta}\right)sin\frac{\pi k}{l}x+$$

$$+\sum_{k=1}^{\infty}C_{1k}t^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t^{\alpha+\beta}\right)sin\frac{\pi k}{l}x+$$

$$+\sum_{k=1}^{\infty}\frac{\varphi_{2k}t^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}t^{\alpha+\beta}\right)cos\frac{\pi(2k-1)}{2l}x+$$

$$+\sum_{k=1}^{\infty}C_{2k}t^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t^{\alpha+\beta}\right)cos\frac{\pi(2k-1)}{2l}x \tag{28}$$

and from (13),(26) and (27) we get

$$f(x)=\sum_{k=1}^{\infty}F_{1k}sin\frac{\pi k}{l}x+\sum_{k=1}^{\infty}F_{1k}^{*}sin\frac{\pi k}{l}x+\sum_{k=1}^{\infty}F_{2k}cos\frac{\pi(2k-1)}{2l}x+\sum_{k=1}^{\infty}F_{2k}^{*}cos\frac{\pi(2k-1)}{2l}x, \tag{29}$$

where

$$C_{1k}=\frac{\psi_{1k}-\varphi_{1k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)}{T^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)},$$

$$C_{2k} = \frac{\psi_{2k} - \varphi_{2k}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)}{T^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)},$$

and

$$F_{1k} = C_{1k}\left(\Gamma(\alpha+1) + \varepsilon_1(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)\right),$$

$$F_{1k}^{*} = \frac{\varphi_{1k}\varepsilon_1(b)t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)$$

$$F_{2k} = C_{2k}\left(\Gamma(\alpha+1) + \varepsilon_2(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)\right),$$

$$F_{2k}^{*} = \frac{\varphi_{2k}\varepsilon_2(b)t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right).$$

3.3. Convergence of series

In order to justify that the obtained formal solution is indeed a true solution, we need to show that the series appeared in $t^{1-\alpha}u(x,t)$ and $f(x)$ as well as the corresponding series representations of $t^{1-\alpha}u_{xx}(x,t)$ and $D_{0t}^{\alpha}u(x,t)$ converge uniformly in Ω. For this purpose, let

$$\varphi^{(i)}(-l) = \varphi^{(i)}(l) = 0, \quad i = 0, 2,$$

$$\psi^{(i)}(-l) = \psi^{(i)}(l) = 0, \quad i = 0, 2,$$

Hence, on integration by parts, φ_{ik}, ψ_{ik} and C_{ik}, F_{ik} $(i = 1, 2)$ can now be rewritten as

$$\varphi_{1k} = \frac{l^4}{(\pi k)^4}\varphi_{1k}^{(4)}, \quad \varphi_{2k} = \frac{16l^4}{(\pi(2k-1))^4}\varphi_{2k}^{(4)};$$

$$\psi_{1k} = \frac{l^4}{(\pi k)^4}\psi_{1k}^{(4)}, \quad \psi_{2k} = \frac{16l^4}{(\pi(2k-1))^4}\psi_{2k}^{(4)};$$

where

$$\varphi_{1k}^{(4)} = \frac{1}{l}\int_{-l}^{l}\varphi^{(4)}(x)sin\frac{\pi k}{l}xdx, \quad \varphi_{2k}^{(4)} = \frac{1}{l}\int_{-l}^{l}\varphi^{(4)}(x)cos\frac{\pi(2k-1)}{2l}xdx,$$

$$\psi_{1k}^{(4)} = \frac{1}{l}\int_{-l}^{l}\psi^{(4)}(x)sin\frac{\pi k}{l}xdx, \quad \psi_{2k}^{(4)} = \frac{1}{l}\int_{-l}^{l}\psi^{(4)}(x)cos\frac{\pi(2k-1)}{2l}xdx.$$

Easy to prove that, the series representations of $t^{1-\alpha}u(x,t)$ and $f(x)$ converge absolutely and uniformly in the region Ω. Actually, considering

$$C_{1k} = \frac{l^4}{(\pi k)^4}\cdot\frac{\psi_{1k}^{(4)} - \varphi_{1k}^{(4)}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)}{T^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}T^{\alpha+\beta}\right)},$$

$$C_{2k} = \frac{16l^4}{(\pi(2k+1))^4}\cdot\frac{\psi_{2k}^{(4)} - \varphi_{2k}^{(4)}\frac{T^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)}{T^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}T^{\alpha+\beta}\right)},$$

$$F_{1k} = \frac{l^4}{(\pi k)^4}\cdot C_{1k}\left(\Gamma(\alpha+1) + \varepsilon_1(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)\right),$$

$$F_{1k}^{*} = \frac{l^4}{(\pi k)^4}\cdot\frac{\varphi_{1k}^{(4)}\varepsilon_1(b)t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right),$$

$$F_{2k} = \frac{l^4}{(\pi k)^4}\cdot C_{2k}\left(\Gamma(\alpha+1) + \varepsilon_2(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)\right),$$

$$F_{2k}^{*}=\frac{16l^4}{(\pi(2k+1))^4}\cdot\frac{\varphi_{2k}^{(4)}\varepsilon_2(b)t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right).$$

and estimations

$$|C_{1k}|\le\frac{l^4\left(c_1\cdot\left|\psi_{1k}^{(4)}\right|+c_1^{*}\cdot\left|\varphi_{1k}^{(4)}\right|\right)}{(\pi k)^4},$$

$$|C_{2k}|\le\frac{16l^4\left(c_2\cdot\left|\psi_{2k}^{(4)}\right|+c_2^{*}\cdot\left|\varphi_{2k}^{(4)}\right|\right)}{(\pi(2k-1))^4},$$

$$|F_{1k}|\le\frac{l^4}{(\pi k)^4}\cdot|C_{1k}|\left|\Gamma(\alpha+1)+\varepsilon_1(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)\right|,$$

$$|F_{2k}|\le\frac{16l^4}{(\pi(2k-1))^4}\cdot|C_{2k}|\left|\Gamma(\alpha+1)+\varepsilon_2(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)\right|,$$

$$|F_{1k}^{*}|\le c_3\cdot\frac{l^4\left|\varphi_{1k}^{(4)}\right|}{(\pi k)^4},\qquad |F_{2k}^{*}|\le c_4\cdot\frac{16l^4\left|\varphi_{2k}^{(4)}\right|}{(\pi(2k-1))^4},$$

owing to $\left|E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{ik}t^{\alpha+\beta}\right)\right|\le\frac{1}{1+\frac{\Gamma(\alpha+\beta)}{\Gamma(2\alpha+\beta)}\cdot\lambda_{ik}t^{\alpha+\beta}}\le c_{0_i},\quad i=1,2$ (see Proposition 1), from (28) and (29), we find

$$\left|t^{1-\alpha}u(x,t)\right|\le\sum_{k=1}^{\infty}\frac{l^4\left(c_1\cdot\left|\psi_{1k}^{(4)}\right|+(c_{0_1}+c_1^{*})\cdot\left|\varphi_{1k}^{(4)}\right|\right)}{(\pi k)^4}+\sum_{k=1}^{\infty}\frac{16l^4\left(c_2\cdot\left|\psi_{2k}^{(4)}\right|+(c_{0_2}+c_2^{*})\cdot\left|\varphi_{2k}^{(4)}\right|\right)}{(\pi(2k-1))^4}$$

and

$$|f(x)|\le\sum_{k=1}^{\infty}\frac{l^4\left(|C_{1k}|\left|\Gamma(\alpha+1)+\varepsilon_1(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k}t_0^{\alpha+\beta}\right)\right|+c_3\left|\varphi_{1k}^{(4)}\right|\right)}{(\pi k)^4}+$$

$$+\sum_{k=1}^{\infty}\frac{16l^4\left(|C_{2k}|\left|\Gamma(\alpha+1)+\varepsilon_2(b)t_0^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k}t_0^{\alpha+\beta}\right)\right|+c_4\left|\varphi_{1k}^{(4)}\right|\right)}{(\pi(2k-1))^4},$$

where

$$c_i=\left|\frac{1}{T^{\alpha}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{ik}T^{\alpha+\beta}\right)}\right|,\quad c_i^{*}=\left|\frac{E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{ik}T^{\alpha+\beta}\right)}{T\Gamma(\alpha)E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{ik}T^{\alpha+\beta}\right)}\right|,$$

$$c_{i+2}=\left|\frac{\varepsilon_i(b)t_0^{\alpha-1}}{\Gamma(\alpha)}E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{ik}t_0^{\alpha+\beta}\right)\right|,\quad i=1,2.$$

The convergence of the series representations of $t^{1-\alpha}u_{xx}(x,t)$ which are obtained by term-wise differentiation of the series representation of $t^{1-\alpha}u(x,t)$ can be shown in a similar way:

$$\left|t^{1-\alpha}u_{xx}(x,t)\right|\le\sum_{k=1}^{\infty}\frac{l^2\left(c_1\cdot\left|\psi_{1k}^{(4)}\right|+(c_{0_1}+c_1^{*})\cdot\left|\varphi_{1k}^{(4)}\right|\right)}{(\pi k)^2}\left|\sin\frac{\pi k}{l}x\right|+$$

$$+\sum_{k=1}^{\infty}\frac{4l^2\left(c_2\cdot\left|\psi_{2k}^{(4)}\right|+(c_{0_2}+c_2^{*})\cdot\left|\varphi_{2k}^{(4)}\right|\right)}{(\pi(2k-1))^2}\left|\cos\frac{\pi(2k-1)}{2l}x\right|<\infty.$$

Finally, owing to class of functions $t^{1-\alpha}u(x,t)$ and $f(x)$, from the absolutely and uniformly convergence of the series representations of $u(x,t)$, $f(x)$ and $t^{1-\alpha}u_{xx}(x,t)$, in the domain Ω, we conclude that a series representations of $D_{0t}^{\alpha}u$ is absolutely and uniformly converge also, in the domain Ω.

4. Uniqueness of solution

Suppose that there are two solution sets $\{u_1(x,t), f_1(x)\}$ and $\{u_2(x,t), f_2(x)\}$ to the Inverse Problem IP. Denote

$$u(x,t) = u_1(x,t) - u_2(x,t),$$

and

$$f(x) = f_1(x) - f_2(x).$$

Then, the functions $u(x,t)$ and $f(x)$ clearly satisfy equation (6), the boundary conditions in (7) and the homogeneous conditions

$$\lim_{t\to+0} D_{0t}^{\alpha-1} u(x,t) = 0, \quad u(x,T) = 0, \quad x \in [-l,l]. \tag{30}$$

If $\varphi(x) \equiv \psi(x) \equiv 0$, then due to completeness of functions $sin\frac{\pi k}{l}x$ and $cos\frac{\pi(2k-1)}{2l}x$, we get $\varphi_{ik} = \psi_{ik} = 0,\ (i = 1,2)$. Hence, from (24) and (25) we infer, that $u_{ik}(t_0) = 0$, respectively. Similarly, from (26) and (27) we obtain $f_{ik} = 0,\ (i = 1,2)$.
Let us now introduce the following:

$$\Omega_\varepsilon = \{-l+\varepsilon < x < l-\varepsilon, \varepsilon < t < T-\varepsilon\},\ \varepsilon > 0,$$

$$u_{1k}(t) = \frac{1}{l}\int_{-l+\varepsilon}^{l-\varepsilon} u(x,t)\sin\frac{\pi k}{l}x dx, \quad k \in \mathbb{N}, \tag{31}$$

$$u_{2k}(t) = \frac{1}{l}\int_{-l+\varepsilon}^{l-\varepsilon} u(x,t)\cos\frac{\pi(2k-1)}{2l}x dx, \quad k \in \mathbb{N}, \tag{32}$$

$$f_{1k} = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{\pi k}{l}x dx, \quad k \in \mathbb{N}, \tag{33}$$

$$f_{2k} = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{\pi(2k-1)}{2l}x dx, \quad k \in \mathbb{N}. \tag{34}$$

Note that the homogeneous conditions in (30) lead to

$$\lim_{t\to+0} D_{0t}^{\alpha-1} u_{ik}(t) = 0, \quad u_{ik}(T) = 0, \quad i = 1,2.$$

Assume that $u_{xx} \in C(\Omega_\epsilon)$, and

$$\lim_{x\to-l} u_x(x,t)\sin\frac{\pi k}{l}x = \lim_{x\to l} u_x(x,t)\sin\frac{\pi k}{l}x = \lim_{x\to l} u_x(-x,t)\sin\frac{\pi k}{l}x = 0,$$

applying the Rimann-Liouville differentiate operator to equation (31), we get

$$D_{0t}^{\alpha} u_{1k}(t) =$$

$$= \frac{1}{l}\int_{-l+\varepsilon}^{l-\varepsilon} \left(a_1 t^\beta u_{xx}(x,t) - a_2 t^\beta u_{xx}(-x,t) - b_1 u(x,t_0) - b_2 u(-x,t_0)\right)\sin\frac{\pi k}{l}x dx + f_{1k},$$

which on integrating by parts and using the conditions in (7), at $\varepsilon \to 0$ reduces to

$$D_{0t}^{\alpha} u_{1k}(t) = \lambda_{1k} u_{1k}(t) - \varepsilon_1(b) u_{1k}(t_0) + f_{1k}, \qquad 0 < t \le T.$$

One can then easily show that this equation together with the conditions $\lim\limits_{t\to+0} D_{0t}^{\alpha-1} u_{1k}(t) = u_{1k}(T) = 0$ and $u_{1k}(t_0) = 0, \quad f_{1k} = 0$, imply that $u_{1k}(t) \equiv 0$, for $t \in (0,T]$.
Similarly, for u_{2k} and f_{2k} as given in (32) and (34), respectively, one can show that $\lim\limits_{t\to+0} D_{0t}^{\alpha-1} u_{2k}(t) \equiv 0$, for $t \in (0,T]$.
Therefore, due to the completeness of the system of eigenfunctions (11) in $L_2(-l,l)$, we must have

$$f(t) \equiv 0, \quad t^{1-\alpha} u(x,t) \equiv 0, \quad (x,t) \in \bar{\Omega}.$$

This ends the proof of uniqueness of solution to the Inverse Problem IP.

5. Main result

The main result for the Inverse Problem IP can be summarized in the following theorem: **Theorem 2.** *Let* $\varphi(x), \psi(x) \in C^3[-l,l]$, $\varphi^{(4)}(x), \psi^{(4)}(x) \in L_1(-l,l)$ *and* $\varphi^{(i)}(-l) = \varphi^{(i)}(l) = \psi^{(i)}(-l) = \psi^{(i)}(l) = 0, \quad i = 0, 2.$ *Then, a unique solution to the inverse problem IP, exists and it can be written in the form*

$$u(x,t) = \sum_{k=1}^{\infty} \frac{l^4 \varphi_{1k}^{(4)} t^{\alpha-1}}{(\pi k)^4 \Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k} t^{\alpha+\beta}\right) sin\frac{\pi k}{l}x +$$

$$+\sum_{k=1}^{\infty} \frac{l^4\left(\psi_{1k}^{(4)} - \varphi_{1k}^{(4)} \frac{T^{\alpha-1}}{\Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k} T^{\alpha+\beta}\right)\right)}{(\pi k)^4 T^\alpha E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k} T^{\alpha+\beta}\right)} t^\alpha E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k} t^{\alpha+\beta}\right) sin\frac{\pi k}{l}x +$$

$$+\sum_{k=1}^{\infty} \frac{16 l^4 \varphi_{2k}^{(4)} t^{\alpha-1}}{(\pi(2k-1))^4 \Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k} t^{\alpha+\beta}\right) cos\frac{\pi(2k-1)}{2l}x +$$

$$+\sum_{k=1}^{\infty} \frac{16 l^4\left(\psi_{2k}^{(4)} - \varphi_{2k}^{(4)} \frac{T^{\alpha-1}}{\Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k} T^{\alpha+\beta}\right)\right)}{(\pi(2k-1))^4 T^\alpha E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k} T^{\alpha+\beta}\right)} t^\alpha E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k} t^{\alpha+\beta}\right) cos\frac{\pi(2k-1)}{2l}x$$

and

$$f(x) = \sum_{k=1}^{\infty} \frac{l^4\left(\psi_{1k}^{(4)} - \varphi_{1k}^{(4)} \frac{T^{\alpha-1}}{\Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k} T^{\alpha+\beta}\right)\right)\left(1 + \frac{\varepsilon_1(b) t_0^\alpha}{\Gamma(\alpha+1)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k} t_0^{\alpha+\beta}\right)\right)}{(\pi k)^4 \frac{T^\alpha}{\Gamma(\alpha+1)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{1k} T^{\alpha+\beta}\right)} \cdot sin\frac{\pi k}{l}x +$$

$$+\sum_{k=1}^{\infty} \frac{l^4 \varphi_{1k}^{(4)} \varepsilon_1(b) t_0^{\alpha-1}}{(\pi k)^4 \Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{1k} t_0^{\alpha+\beta}\right) sin\frac{\pi k}{l}x +$$

$$+\sum_{k=1}^{\infty} \frac{16 l^4\left(\psi_{2k}^{(4)} - \varphi_{2k}^{(4)} \frac{T^{\alpha-1}}{\Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k} T^{\alpha+\beta}\right)\right)\left(1 + \frac{\varepsilon_2(b) t_0^\alpha}{\Gamma(\alpha+1)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k} t_0^{\alpha+\beta}\right)\right)}{(\pi(2k-1))^4 \frac{T^\alpha}{\Gamma(\alpha+1)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta}{\alpha}}\left(-\lambda_{2k} T^{\alpha+\beta}\right)} \times$$

$$\times cos\frac{\pi(2k-1)}{2l}x +$$

$$+\sum_{k=1}^{\infty} \frac{16 l^4 \varphi_{2k}^{(4)} \varepsilon_2(b) t_0^{\alpha-1}}{(\pi(2k-1))^4 \Gamma(\alpha)} E_{\alpha,1+\frac{\beta}{\alpha},1+\frac{\beta-1}{\alpha}}\left(-\lambda_{2k} t_0^{\alpha+\beta}\right) cos\frac{\pi(2k-1)}{2l}x,$$

where $\lambda_{1k} = (a_1 + a_2)\left(\frac{\pi k}{l}\right)^2, \quad \lambda_{2k} = (a_1 - a_2)\left(\frac{\pi(2k-1)}{2l}\right)^2.$

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REZYUME

Ushbu ish involyutsiya bilan buzilgan kasr tartibli diffuziya tenglamasi uchun teskari masalani o‘rganishga bag‘ishlangan. Qo‘yilgan masala uchun yechimning mavjudligi va yagonaligi teoremasi isbotlandi.

Kalit so‘zlar: Teskari masala, buzilgan kasr tartibli diffuziya tenglamasi, involyutsiya, yechimning mavjudligi va yagonaligi, qatorga yoyish.

РЕЗЮМЕ

Данная работа посвящена изучению обратной задачи для нагруженного вырождающегося диффузионного уравнения дробного порядка с возмущением инволютивного типа. Доказана теорема существования и единственности решения сформулированной задачи.

Ключевые слова: Обратные задачи, нагруженное диффузионное уравнение дробного порядка, возмущение инволютивного типа, существование и единственность решения, разложение в ряд.