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Local derivation on solvable Lie algebras with naturally graded filiform nilradicals

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RESUME

In this work, we study derivations and local derivations of finite-dimensional solvable Lie algebras whose nilradicals are naturally graded filiform. Specifically, the general form of the matrices of derivations and local derivations of these algebras is determined. We show that these algebras admit local derivations that are not derivations.

Key words: Lie algebra, derivation, solvable Lie algebra, local derivation.

1. Introduction

In recent developments, the study of local and 2-local maps has attracted considerable interest, particularly within the framework of certain non-associative algebraic systems such as Lie, Jordan, and Leibniz algebras. The idea of local derivations was first introduced in 1990 by Kadison [16], and independently by Larson and Sourour [21]. Later, in 1997, Šemrl expanded this area by defining the concepts of 2-local derivations and 2-local automorphisms related to algebras [22].

Investigation of local derivations on Lie algebras was initiated in [7] by Ayupov and Kudaybergenov. They proved that every local derivation on semisimple Lie algebras is a derivation and gave examples of nilpotent finite-dimensional Lie algebras with local derivations that are not derivations. In [4], local derivations of solvable Lie algebras are investigated, and it is shown that in the class of solvable Lie algebras there exist algebras that admit local derivations that are not derivations and also algebras for which every local derivation is a derivation. Several authors investigated local derivations for the finite or infinite dimensional Lie and Leibniz algebras [2,3,5,8,9,11,14,17,18,26,28]. It was proved that all local derivations of the following algebras are derivations: Borel subalgebras of finite-dimensional simple Lie algebras; Witt algebras; solvable Lie algebras of maximal rank; Cayley algebras; locally finite split simple Lie algebras; the Schrödinger algebras; conformal Galilei algebras.

Several papers have been devoted to similar notions and corresponding problems for 2-local derivations and automorphisms of Lie algebras [6,9,10,12,13,15,24,27]. Specifically, in [6] it is proved that every 2-local derivation on the semisimple Lie algebras is a derivation, whereas each finite-dimensional nilpotent Lie algebra, with dimension larger than two, admits 2-local derivation which is not a derivation. Let us present the list of Lie algebras for which all 2-local derivations are derivations: finite-dimensional semisimple Lie algebras; Witt algebras; locally finite split simple Lie algebras; Virasoro algebras; Virasoro-like algebra; the Schrödinger-Virasoro algebra; Jacobson-Witt algebras; planar Galilean conformal algebras.

The investigation of local and 2-local δ-derivations on Lie algebras was initiated in [19] by A. Khudoyberdiyev and B. Yusupov. Specifically, in [19], they introduced the concepts of local and 2-local δ-derivations and described local and 2-local $\frac{1}{2}$-derivations on finite-dimensional solvable Lie algebras with filiform, Heisenberg, and abelian nilradicals. Moreover, they provided descriptions of local $\frac{1}{2}$-derivations on oscillator Lie algebras, conformal perfect Lie algebras, and Schrödinger algebras. In a recent paper [29], B. Yusupov, V. Vaisova, and T. Madrakhimov obtained similar results regarding local $\frac{1}{2}$-derivations of naturally graded quasi-filiform Leibniz algebras of type I. They showed that such algebras, in general, admit local $\frac{1}{2}$-derivations which are not ordinary $\frac{1}{2}$-derivations. In another work [20], U. Mamadaliyev, A. Sattarov, and B. Yusupov studied local and 2-local $\frac{1}{2}$-derivations on solvable Leibniz algebras. They proved that any local $\frac{1}{2}$-derivation on solvable Leibniz algebras with model or abelian nilradicals, where the complementary space of maximal dimension exists, is a $\frac{1}{2}$-derivation. Furthermore, they showed that solvable Leibniz algebras with abelian nilradicals and one-dimensional complementary space also admit such derivations. Additionally, 2-local $\frac{1}{2}$-derivations were investigated for these types of algebras, and an example of a solvable Leibniz algebra was constructed for which every 2-local $\frac{1}{2}$-derivation is a true $\frac{1}{2}$-derivation. However, examples were also given of algebras that admit 2-local $\frac{1}{2}$-derivations which are not $\frac{1}{2}$-derivations.

In this work, we study derivations and local derivations of finite-dimensional solvable Lie algebras whose nilradicals are naturally graded filiform.Specifically, the common form of the matrices of derivations and local derivations of these algebras is determined. We show that these algebras admit local derivations that are not derivations.

2. Preliminaries

All the algebras below will be over the complex field, and all the linear maps will be $\mathbb{C}$-linear unless otherwise stated. Omitted products in the multiplication table of an algebra are assumed to be zero. Moreover, due to the anti-commutativity of Lie algebras, symmetric products for these algebras are also omitted.

A derivation on a Lie algebra $\mathcal{L}$ is a linear map $D : \mathcal{L} \to \mathcal{L}$ which satisfies the Leibniz rule:

$$D([x, y]) = [D(x), y] + [x, D(y)], \quad \text{for any} \quad x, y \in \mathcal{L}. \tag{1}$$

The set of all derivations of $\mathcal{L}$ is denoted by $\mathrm{Der}(\mathcal{L})$ and with respect to the commutation operation is a Lie algebra.

For any element $y \in \mathcal{L}$ the left multiplication operator $\mathrm{ad}_x : \mathcal{L} \to \mathcal{L}$, defined as $\mathrm{ad}_x(y) = [x, y]$ is a derivation, and derivations of this form are called inner derivations. The set of all inner derivations of $\mathcal{L}$, denoted by $\mathrm{Inn}(\mathcal{L})$, is an ideal in $\mathrm{Der}(\mathcal{L})$.

Definition 1. *A linear operator Δ is called a local derivation if for any $x \in \mathcal{L}$, there exists a derivation $D_x : \mathcal{L} \to \mathcal{L}$ (depending on x) such that $\Delta(x) = D_x(x)$. The set of all local derivations on $\mathcal{L}$ we denote by* $\mathrm{LocDer}(\mathcal{L})$.

For an arbitrary Lie algebra L we define the **derived** and **central series** as follows:

$$\mathfrak{L}^{[1]} = \mathfrak{L},\ \mathfrak{L}^{[s+1]} = [\mathfrak{L}^{[s]}, \mathfrak{L}^{[s]}],\ s \geq 1,$$

$$\mathfrak{L}^{1} = \mathfrak{L},\ \mathfrak{L}^{k+1} = [\mathfrak{L}^{k}, \mathfrak{L}],\ k \geq 1.$$

Definition 2. *An n-dimensional Lie algebra $\mathfrak{L}$ is called **solvable (nilpotent)** if there exists $s \in \mathbb{N}$ ($k \in \mathbb{N}$) such that $\mathfrak{L}^{[s]} = \{0\}$ ($\mathfrak{L}^{k} = \{0\}$). Such minimal number is called **index of solvability (nilpotency)**.*

The maximal nilpotent ideal of a Lie algebra is said to be the nilradical of the algebra.

Definition 3. *An n-dimensional Lie algebra $\mathfrak{L}$ is said to be **filiform** if $\dim \mathfrak{L}^i = n - i$, for $2 \leq i \leq n$.*

Now let us define a natural gradation for the nilpotent Lie algebras.

Definition 4. *Given a nilpotent Lie algebra $\mathfrak{L}$ with nilindex s, put $\mathfrak{L}_i = \mathfrak{L}^i/\mathfrak{L}^{i+1},\ 1 \leq i \leq s-1$, and $Gr(\mathfrak{L}) = \mathfrak{L}_1 \oplus \mathfrak{L}_2 \oplus \ldots \oplus \mathfrak{L}_{s-1}$. Define the product in the vector space $Gr(\mathfrak{L})$ as follows:*

$$[x + \mathfrak{L}^{i+1}, y + \mathfrak{L}^{j+1}] := [x, y] + \mathfrak{L}^{i+j+1},$$

*where $x \in \mathfrak{L}^i/\mathfrak{L}^{i+1}$, $y \in \mathfrak{L}^j/\mathfrak{L}^{j+1}$. Then $[\mathfrak{L}_i, \mathfrak{L}_j] \subseteq \mathfrak{L}_{i+j}$ and we obtain the graded algebra $Gr(\mathfrak{L})$. If $Gr(\mathfrak{L})$ and $\mathfrak{L}$ are isomorphic, then we say that the algebra $\mathfrak{L}$ is **naturally graded**.*

It is well known that there are two types of naturally graded filiform Lie algebras. In fact, the second type will appear only in the case when the dimension of the algebra is even.

Theorem 1.[25] *Any naturally graded filiform Lie algebra is isomorphic to one of the following non-isomorphic algebras:*

$$\mathfrak{n}_{n,1}: \quad [e_i, e_1] = e_{i+1}, \quad 2 \leq i \leq n-1;$$

$$\mathfrak{Q}_{2n}: \quad [e_i, e_1] = e_{i+1},\ 2 \leq i \leq 2n-2, \quad [e_i, e_{2n+1-i}] = (-1)^i e_{2n},\ 2 \leq i \leq n.$$

All solvable Lie algebras whose nilradical is the naturally graded filiform Lie algebra $\mathfrak{n}_{n,1}$ are classified in [23] ($n \geq 4$). Furthermore, solvable Lie algebras whose nilradical is the naturally graded filiform Lie algebra $\mathfrak{Q}_{2n}$ are classified in [1]. It is proved that the dimension of a solvable Lie algebra whose nilradical is isomorphic to an n-dimensional naturally graded filiform Lie algebra is not greater than $n+2$.

Here we give the list of such solvable Lie algebras. We denote by $\mathfrak{s}^i_{n,1}$ solvable Lie algebras with nilradical $\mathfrak{n}_{n,1}$ and codimension one, and by $\mathfrak{s}_{n,2}$ with codimension two:

$$\begin{aligned}
&\mathfrak{s}^1_{n,1}(\beta): && [e_i,e_1]=e_{i+1},\ 2\leq i\leq n-1, \quad [e_1,h]=e_1, \quad [e_i,h]=(i-2+\beta)e_i,\ 2\leq i\leq n;\\
&\mathfrak{s}^2_{n,1}: && [e_i,e_1]=e_{i+1},\ 2\leq i\leq n-1, \quad [e_i,h]=e_i,\ 2\leq i\leq n;\\
&\mathfrak{s}^3_{n,1}: && [e_i,e_1]=e_{i+1},\ 2\leq i\leq n-1, \quad [e_1,h]=e_1+e_2, \quad [e_i,h]=(i-1)e_i,\ 2\leq i\leq n;
\end{aligned}$$

$$\mathfrak{s}^4_{n,1}(\alpha_3,\alpha_4,\dots,\alpha_{n-1}): [e_i,e_1]=e_{i+1},\ 2\leq i\leq n-1, \quad [e_i,h]=e_i+\sum_{l=i+2}^{n}\alpha_{l+1-i}e_l,\ 2\leq i\leq n;$$

$$\mathfrak{s}_{n,2}\ : \begin{cases}[e_i,e_1]=e_{i+1},\ 2\leq i\leq n-1, \quad [e_1,h_1]=e_1,\\ [e_i,h_1]=(i-2)e_i,\ 3\leq i\leq n, \quad [e_i,h_2]=e_i, \quad 2\leq i\leq n.\end{cases}$$

Any solvable complex Lie algebra of dimension $2n+1$ with nilradical isomorphic to $\mathfrak{Q}_{2n}$ is isomorphic to one of the following algebras:

$$\mathfrak{r}_{2n+1}(\lambda): \begin{cases}[e_i,e_1]=e_{i+1},\ 2\leq i\leq 2n-2, \quad [e_i,e_{2n+1-i}]=(-1)^i e_{2n},\ 2\leq i\leq n,\\ [e_1,h]=e_1, \quad [e_i,x]=(i-2+\lambda)e_i,\ 2\leq i\leq 2n-1, \quad [e_{2n},h]=(2n-3+2\lambda)e_{2n};\end{cases}$$

$$\mathfrak{r}_{2n+1}(2-n,\varepsilon): \begin{cases}[e_i,e_1]=e_{i+1},\ 2\leq i\leq 2n-2, \quad [e_i,e_{2n+1-i}]=(-1)^i e_{2n}, \quad 2\leq i\leq n,\\ [e_1,h]=e_1+\varepsilon e_{2n},\ \varepsilon=-1,1, \quad [e_i,h]=(i-n)e_i, \quad 2\leq i\leq 2n-1,\\ [e_{2n},x]=e_{2n};\end{cases}$$

$$\mathfrak{r}_{2n+1}(\lambda_5,\dots,\lambda_{2n-1}): \begin{cases}[e_i,e_1]=e_{i+1},\ 2\leq i\leq 2n-2, \quad [e_i,e_{2n+1-i}]=(-1)^i e_{2n}, \quad 2\leq i\leq n,\\ [e_{2+i},h]=e_{2+i}+\sum\limits_{k=2}^{\left[\frac{2n-2-i}{2}\right]}\lambda_{2k+1}e_{2k+1+i},\ 0\leq i\leq 2n-6,\\ [e_{2n-i},h]=e_{2n-i},\ i=1,2,3, \quad [e_{2n},h]=2e_{2n}.\end{cases}$$

Moreover, the first nonvanishing parameter λ_{2k+1} can be normalized to 1.

Finally, for any $n\geq 3$ there is only one solvable Lie algebra $\mathfrak{r}_{2n+2}$ of dimension $2n+2$ having a nilradical isomorphic to $\mathfrak{Q}_{2n}$:

$$\mathfrak{r}_{2n+2}: \begin{cases}[e_i,e_1]=e_{i+1},\ 2\leq i\leq 2n-2, \quad [e_i,e_{2n+1-i}]=(-1)^i e_{2n},\ 2\leq i\leq n,\\ [e_i,h_1]=ie_i,\ 1\leq i\leq 2n-1, \quad [e_{2n},h_1]=(2n+1)e_{2n},\\ [e_i,h_2]=e_i,\ 2\leq i\leq 2n-1, \quad [e_{2n},h_2]=2e_{2n}.\end{cases}$$

Now we describe derivation of solvable Lie algebras with filiform nilradical algebras.

Theorem 2. *Any derivation of the algebras $\mathfrak{s}^1_{n,1}(\beta)$, $\mathfrak{s}^2_{n,1}$, $\mathfrak{s}^3_{n,1}$, $\mathfrak{s}^4_{n,1}(\alpha_3,\dots,\alpha_{n-1})$, $\mathfrak{r}_{2n+1}(\lambda)$, $\mathfrak{r}_{2n+1}(2-n,\varepsilon)$ and $\mathfrak{r}_{2n+1}(\lambda_5,\lambda_6,\dots,\lambda_{2n-1})$ has the following form:*

- for the algebra $\mathfrak{s}^1_{n,1}(\beta)$:

$$\varphi(h)=-\beta_3e_1+\sum_{i=2}^{n-1}\alpha_{i+1}(i-2+\beta)e_i+\delta_ne_n, \quad \varphi(e_1)=\sum_{i=1}^{n}\alpha_ie_i, \quad \varphi(e_2)=\beta_2e_2+\beta_3e_3,$$
$$\varphi(e_i)=(\beta_2+(i-2)\alpha_1)e_i+\beta_3e_{i+1}, \quad 3\leq i\leq n-1, \quad \varphi(e_n)=((n-2)\alpha_1+\beta_2)e_n,$$

  with the restriction $(\beta-1)\alpha_2=0$.

- for the algebra $\mathfrak{s}^2_{n,1}$:

$$\varphi(h)=\sum_{i=2}^{n-1}\alpha_{i+1}e_i+\delta_ne_n,\quad \varphi(e_1)=\alpha_1e_1+\sum_{i=3}^{n}\alpha_ie_i,\quad \varphi(e_2)=\sum_{i=2}^{n}\beta_ie_i,$$
$$\varphi(e_i)=((i-2)\alpha_1+\beta_2)e_i+\sum_{j=i+1}^{n}\beta_{j-i+2}e_j,\quad 3\le i\le n.$$

- for the algebras $\mathfrak{s}^3_{n,1}$:

$$\varphi(h)=-\beta_3e_1+(\alpha_3-\beta_3)e_2+\sum_{i=3}^{n-1}(i-1)\alpha_{i+1}e_i+\delta_ne_n,\quad \varphi(e_1)=\sum_{i=1}^{n}\alpha_ie_i,$$
$$\varphi(e_i)=((i-2)\alpha_1+\beta_2)e_i+\beta_3e_{i+1},\quad 2\le i\le n.$$

- for the algebra $\mathfrak{s}^4_{n,1}(\alpha_3,\alpha_4,\dots,\alpha_{n-1})$:

$$\varphi(h)=\sum_{i=2}^{n-1}(\beta_{i+1}+\sum_{j=3}^{i-1}\beta_j\alpha_{i-j+2})e_i+\delta_ne_n,\quad \varphi(e_1)=\sum_{i=3}^{n}\beta_ie_i,$$
$$\varphi(e_i)=\mu_2e_i+\sum_{j=i+1}^{n}\mu_{j-i+2}e_j,\quad 2\le i\le n.$$

- for the algebra $\mathfrak{r}_{2n+1}(\lambda)$:

$$\varphi(h)=\sum_{i=2}^{2n-1}\alpha_{i+1}(i-2+\lambda)e_i+\delta_{2n}e_{2n},$$
$$\varphi(e_1)=\alpha_1e_1+\sum_{i=3}^{2n-1}\alpha_ie_i,\quad \varphi(e_2)=\beta_2e_2+\beta_{2n}e_{2n},$$
$$\varphi(e_i)=((i-2)\alpha_1+\beta_2)e_i+(-1)^{i-1}\alpha_{2n-i+2}e_{2n},\quad 3\le i\le 2n-1,$$
$$\varphi(e_{2n})=((2n-3)\alpha_1+2\beta_2)e_{2n}.$$

- for the algebra $\mathfrak{r}_{2n+1}(2-n,\varepsilon)$:

$$\varphi(h)=\sum_{i=1}^{2n}\delta_ie_i,\quad \varphi(e_1)=\sum_{i=1}^{2n}\alpha_ie_i,\quad \varphi(e_2)=\beta_2e_2+\beta_3e_3+\beta_{2n}e_{2n},$$
$$\varphi(e_i)=((i-2)\alpha_1+\beta_2)e_i+\beta_3e_{i+1}+(-1)^{i-1}\alpha_{2n+2-i}e_{2n},\quad 3\le i\le 2n-1,$$
$$\varphi(e_{2n})=((2n-3)\alpha_1+2\beta_2)e_2n.$$

- for the algebra $\mathfrak{r}_{2n+1}(\lambda_5,\dots,\lambda_{2n-1})$:

$$\varphi(h)=\sum_{i=2}^{2n-1}(\alpha_{i+1}+\sum_{k=2}^{[\frac{i-1}{2}]}\lambda_{2k+1}\alpha_{i-2k+2})e_i+\delta_{2n}e_{2n},\quad \varphi(e_1)=\alpha_1e_1+\sum_{i=3}^{2n-1}\alpha_ie_i,$$
$$\varphi(e_2)=\beta_2e_2+\beta_{2n}e_{2n},\quad \varphi(e_i)=((i-2)\alpha_1+\beta_2)e_i+(-1)^{i-1}\alpha_{2n-i+2}e_{2n},\quad 3\le i\le 2n,$$
$$\varphi(e_{2n})=((2n-3)\alpha_1+2\beta_2)e_{2n}.$$

Proof. The proof follows from straightforward calculations and the definition of derivation.

3. Local derivation on solvable Lie algebras with naturally graded filiform nilradical

In this section, we study local derivations on solvable Lie algebras with naturally graded filiform nilradicals $\mathfrak{n}_{n,1}$ and $\mathfrak{Q}_{2n}$. In the following theorem, we describe local derivations of solvable Lie algebras with naturally graded filiform nilradicals $\mathfrak{n}_{n,1}$ and $\mathfrak{Q}_{2n}$.

Theorem 3. *Any local derivation of the algebras* $\mathfrak{s}^1_{n,1}(\beta)$, $\mathfrak{s}^2_{n,1}$, $\mathfrak{s}^3_{n,1}$, $\mathfrak{s}^4_{n,1}(\alpha_3,\dots,\alpha_{n-1})$, $\mathfrak{r}_{2n+1}(\lambda)$, $\mathfrak{r}_{2n+1}(2-n,\varepsilon)$ *and* $\mathfrak{r}_{2n+1}(\lambda_5,\lambda_6,\dots,\lambda_{2n-1})$ *has the following form:*

- for the algebra $\mathfrak{s}^1_{n,1}(\beta)$:

$$\Delta(h)=\sum_{i=1}^{n} b_{i,0}e_i,\quad \Delta(e_1)=\sum_{i=1}^{n} b_{i,1}e_i,$$
$$\Delta(e_i)=b_{i,i}e_i+b_{i+1,i}e_{i+1},\quad 2\le i\le n,$$

  with the restriction $(\beta-1)b_{2,1}=0$.

- for the algebra $\mathfrak{s}^2_{n,1}$:

$$\Delta(h)=\sum_{i=2}^{n} b_{i,0}e_i,\quad \Delta(e_1)=b_{1,1}e_1+\sum_{i=3}^{n} b_{i,1}e_i,$$
$$\Delta(e_2)=\sum_{i=2}^{n} b_{i,2}e_i,\quad \Delta(e_i)=\sum_{j=i}^{n} b_{j,i}e_j,\quad 3\le i\le n.$$

- for the algebras $\mathfrak{s}^3_{n,1}$:

$$\Delta(h)=\sum_{i=1}^{n} b_{i,0}e_i,\quad \Delta(e_1)=\sum_{i=1}^{n} b_{i,1}e_i,$$
$$\Delta(e_i)=b_{i,i}e_i+b_{i+1,i}e_{i+1},\quad 2\le i\le n;$$

- for the algebra $\mathfrak{s}^4_{n,1}(\alpha_3,\alpha_4,\dots,\alpha_{n-1})$:

$$\Delta(h)=\sum_{i=2}^{n} b_{i,0}e_i,\quad \Delta(e_1)=\sum_{i=3}^{n} b_{i,1}e_i,$$
$$\Delta(e_i)=\sum_{j=i}^{n} b_{j,i}e_j,\quad 2\le i\le n.$$

- for the algebra $\mathfrak{r}_{2n+1}(\lambda)$:

$$\Delta(h)=\sum_{i=2}^{2n} b_{i,0}e_i,\quad \Delta(e_1)=b_{1,1}e_1+\sum_{i=3}^{2n-1} b_{i,1}e_i,\quad \Delta(e_2)=b_{2,2}e_2+b_{2n,2}e_{2n},$$
$$\Delta(e_i)=b_{i,i}e_i+b_{i,2n}e_{2n},\quad 3\le i\le 2n-1,\quad \Delta(e_{2n})=b_{2n,2n}e_{2n}.$$

- for the algebra $\mathfrak{r}_{2n+1}(2-n,\varepsilon)$:

$$\Delta(h)=\sum_{i=1}^{2n} b_{i,0}e_i,\quad \Delta(e_1)=\sum_{i=1}^{2n} b_{i,1}e_i,\quad \Delta(e_2)=b_{2,2}e_2+b_{3,2}e_3+b_{2n,2}e_{2n},$$
$$\Delta(e_i)=b_{i,i}e_i+b_{i+1,i}e_{i+1}+b_{i,2n}e_{2n},\quad 3\le i\le 2n-1,\quad \Delta(e_{2n})=b_{2n,2n}e_2n.$$

- for the algebra $\mathfrak{r}_{2n+1}(\lambda_5,\dots,\lambda_{2n-1})$:

$$\Delta(h)=\sum_{i=2}^{2n} b_{i,0}e_i,\quad \Delta(e_1)=b_{1,1}e_1+\sum_{i=3}^{2n-1} b_{i,1}e_i,$$
$$\Delta(e_2)=b_{2,2}e_2+b_{2n,2}e_{2n},\quad \Delta(e_i)=b_{i,i}e_i+b_{i,2n}e_{2n},\quad 3\le i\le 2n,\quad \Delta(e_{2n})=b_{2n,2n}e_{2n}.$$

Proof. We prove the theorem for the algebra $\mathfrak{s}^2_{n,1}$, and for the algebras $\mathfrak{s}^1_{n,1}(\beta)$, $\mathfrak{s}^3_{n,1}$, $\mathfrak{s}^4_{n,1}(\alpha_3, \alpha_4, \dots, \alpha_{n-1})$, $\mathfrak{r}_{2n+1}(\lambda)$, $\mathfrak{r}_{2n+1}(2-n, \varepsilon)$ and $\mathfrak{r}_{2n+1}(\lambda_5, \lambda_6, \dots, \lambda_{2n-1})$ the proofs are similar. Let Δ be an arbitrary local derivation on $\mathfrak{s}^2_{n,1}$ and let $\mathfrak{B}$ be the matrix of Δ :

$$\mathfrak{B} = \begin{pmatrix} b_{0,0} & b_{0,1} & \cdots & b_{0,n-1} & b_{0,n} \\ b_{1,0} & b_{1,1} & \cdots & b_{1,n-1} & b_{1,n} \\ b_{2,0} & b_{2,1} & \cdots & b_{2,n-1} & b_{2,n} \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-1,0} & b_{n-1,1} & \cdots & b_{n-1,n-1} & b_{n-1,n} \\ b_{n,0} & b_{n,1} & \cdots & b_{n,n-1} & b_{n,n} \end{pmatrix}.$$

By the definition for all $x = x_0h + \sum_{i=1}^{n} x_ie_i \in \mathfrak{s}^2_{n,1}$ there exists a derivation D_x on $\mathfrak{s}^2_{n,1}$ such that

$$\Delta(x) = D_x(x).$$

By Theorem [2], D_x has the following matrix representation:

$$\mathfrak{B}_x = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & \alpha_1^x & 0 & 0 & 0 & \cdots & 0 & 0 \\ \alpha_3^x & 0 & \beta_2^x & 0 & 0 & \cdots & 0 & 0 \\ \alpha_4^x & \alpha_3^x & \beta_3^x & \alpha_1^x + \beta_2^x & 0 & \cdots & 0 & 0 \\ \alpha_5^x & \alpha_4^x & \beta_4^x & \beta_3^x & \alpha_1^x + 2\beta_2^x & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \alpha_{n-1}^x & \alpha_{n-2}^x & \beta_{n-2}^x & \beta_{n-3}^x & \beta_{n-4}^x & \cdots & 0 & 0 \\ \alpha_n^x & \alpha_{n-1}^x & \beta_{n-1}^x & \beta_{n-2}^x & \beta_{n-3}^x & \cdots & \alpha_1^x + (n-3)\beta_2^x & 0 \\ \delta_n^x & \alpha_n^x & \beta_n^x & \beta_{n-1}^x & \beta_{n-2}^x & \cdots & \beta_3^x & \alpha_1^x + (n-2)\beta_2^x \end{pmatrix}.$$

Let $\mathfrak{B}$ be the matrix of Δ then by choosing subsequently $x = h, x = e_1, ..., x = e_n$ and using $\Delta(x) = D_x(x)$, i.e. $\mathfrak{B}\overline{x} = D_x(\overline{x})$, where $\overline{x}$ is the vector corresponding to x, which implies

$$\mathfrak{B} = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & b_{1,1} & 0 & 0 & 0 & \cdots & 0 & 0 \\ b_{2,0} & 0 & b_{2,2} & 0 & 0 & \cdots & 0 & 0 \\ b_{3,0} & b_{3,1} & b_{3,2} & b_{3,3} & 0 & \cdots & 0 & 0 \\ b_{4,0} & b_{4,1} & b_{4,2} & b_{4,3} & b_{4,4} & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ b_{n-2,0} & b_{n-2,1} & b_{n-2,2} & b_{n-2,3} & b_{n-2,4} & \cdots & 0 & 0 \\ b_{n-1,0} & b_{n-1,1} & b_{n-1,2} & b_{n-1,3} & b_{n-1,4} & \cdots & b_{n-1,n-1} & 0 \\ b_{n,0} & b_{n,1} & b_{n,2} & b_{n,3} & b_{n,4} & \cdots & b_{n,n-1} & b_{n,n} \end{pmatrix}.$$

Using again $\Delta(x) = D_x(x)$, i.e. $\mathfrak{B}\overline{x} = \mathfrak{B}_x(\overline{x})$, where $\overline{x}$ is the vector corresponding to $x = x_0h + \sum_{i=1}^{n} x_ie_i$, we obtain the next system of equalities

$$\begin{aligned} b_{1,1}x_1 &= \alpha_1^x x_1, \\ b_{2,0}x_0 + b_{2,2}x_2 &= \alpha_3^x x_0 + \beta_2^x x_2, \\ \sum_{j=0}^{i} b_{i,j}x_j &= \alpha_{i+1}^x x_0 + \alpha_i^x x_1 + \sum_{j=3}^{i} \beta_{i-j+3}^x x_{j-1} + (\alpha_1^x + (i-2)\beta_2^x)x_i, \quad 3 \le i \le n-1, \\ \sum_{j=0}^{n} b_{n,j}x_j &= \delta_n^x x_0 + \alpha_n^x x_1 + \sum_{j=2}^{n-1} \beta_{n+2-j}^x x_j + (\alpha_1^x + (n-2)\beta_2^x)x_n. \end{aligned}$$

Let us consider the next cases:

Case 1: If $x_0 \neq 0$, then

$$\alpha_3^x = b_{2,0} + \frac{(b_{2,2} - \beta_2^x)x_2}{x_0},$$

$$\alpha_{i+1}^x = \frac{\sum\limits_{j=0}^{i} b_{i,j}x_j - \alpha_i^x x_1 - \sum\limits_{j=3}^{i} \beta_{i-j+3}^x x_{j-1} - (\alpha_1^x + (i-2)\beta_2^x)x_i}{x_0}, \quad 3 \le i \le n-1,$$

$$\delta_n^x = \frac{\sum\limits_{j=0}^{n} b_{n,j}x_j - \alpha_n^x x_1 - \sum\limits_{j=2}^{n-1} \beta_{n+2-j}^x x_j - (\alpha_1^x + (n-2)\beta_2^x)x_n}{x_0},$$

where $\alpha_1^x, \beta_2^x, \beta_3^x, \cdots, \beta_n^x$ defined arbitrary.

Case 2: If $x_0 = 0$ and $x_1 \neq 0$, then $\alpha_1^x = b_{1,1}$,

$$\alpha_i^x = \frac{\sum\limits_{j=1}^{i} b_{i,j}x_j - \sum\limits_{j=3}^{i} \beta_{i-j+3}^x x_{j-1} - (\alpha_1^x + (i-2)\beta_2^x)x_i}{x_1}, \quad 3 \le i \le n-1,$$

$$\alpha_n^x = \frac{\sum\limits_{j=1}^{n} b_{n,j}x_j - \sum\limits_{j=2}^{n-1} \beta_{n+2-j}^x x_j - (\alpha_1^x + (n-2)\beta_2^x)x_n}{x_1},$$

where $\alpha_n^x, \beta_2^x, \beta_3^x, \cdots, \beta_n^x, \delta_n^x$ defined arbitrary.

Case 3: If $x_0 = x_1 = 0$ and $x_2 \neq 0$, then $\beta_2^x = b_{2,2}$,

$$\beta_i^x = \frac{\sum\limits_{j=2}^{i} b_{i,j}x_j - \sum\limits_{j=4}^{i} \beta_{i-j+3}^x x_{j-1} - (\alpha_1^x + (i-2)\beta_2^x)x_i}{x_2} \quad 3 \le i \le n,$$

where $\alpha_1^x, \alpha_3^x, \cdots, \alpha_n^x, \delta_n^x$ defined arbitrary.

Case 4: If $x_0 = x_1 = ... = x_{t-1} = 0$ and $x_t \neq 0,\ 3 \le t \le n$, then $\beta_2^x = b_{t,t} - (t-2)\alpha_1^x$,

$$\beta_{i-t+2}^x = b_{i,t} + \frac{\sum\limits_{j=t+1}^{i} b_{i,j}x_j - \sum\limits_{j=t+2}^{i} \beta_{i-j+3}^x x_{j-1} - (\alpha_1^x + (i-2)\beta_2^x)x_i}{x_t}, \qquad t+1 \le i \le n,$$

where $\alpha_1^x, \alpha_3^x, \cdots, \alpha_n^x, \delta_n^x$ defined arbitrary.

This completes the proof.

In the following table, we give the dimensions of the spaces of derivations and local derivations of solvable Lie algebras with naturally graded filiform nilradical:

Algebra	dim Der	dim LocDer
$\mathfrak{s}_{n,1}^1(\beta)$	$n+3$	$6n-2$
$\mathfrak{s}_{n,1}^2$	$2n-1$	$\frac{(n-1)(n+4)}{2}$
$\mathfrak{s}_{n,1}^3$	$n+3$	$4n-2$
$\mathfrak{s}_{n,1}^4(\alpha_3, \alpha_4, \ldots, \alpha_{n-1})$	$2n-2$	$\frac{n^2+3n-6}{2}$
$\mathfrak{r}_{2n+1}(\lambda)$	$2n+1$	$8n-6$
$\mathfrak{r}_{2n+1}(2-n, \varepsilon)$	$4n+3$	$10n-5$
$\mathfrak{r}_{2n+1}(\lambda_5, \ldots, \lambda_{2n-1})$	$2n+1$	$8n-3$

Corollary. *Solvable Lie algebras* $\mathfrak{s}^1_{n,1}(\beta)$, $\mathfrak{s}^2_{n,1}$, $\mathfrak{s}^3_{n,1}$, $\mathfrak{s}^4_{n,1}(\alpha_3, \dots, \alpha_{n-1})$, $\mathfrak{r}_{2n+1}(\lambda)$, $\mathfrak{r}_{2n+1}(2-n, \varepsilon)$ *and* $\mathfrak{r}_{2n+1}(\lambda_5, \lambda_6, \dots, \lambda_{2n-1})$ *admit a local derivation which is not a derivation.*

Remark.

In this paper, we have considered only those solvable Lie algebras, whose complementary space is one dimensional. Since the classification of local derivations on solvable Lie algebras of maximal rank was obtained in [18], we have restricted our investigation here to the case $\dim(\textit{complementary space}) = 1$.

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REZYUME

Ushbu ishda biz nilradikali tabiiy gradurlangan filiform bo'lgan chekli o'lchamli yechiluvchan Li algebralarining differensiallashlari va lokal differensiallashlarini o'rganilgan. Aniq qilib aytganda, bu algebralarning differensiallashlari va lokal differensiallashlarining umumiy matritsaviy ko'rinishlari keltirilgan. Bu algebralarda differensiallash bo'lmagan lokal differensiallashlar mavjudligini ko'rsatilgan.

Kalit so'zlar: Lie algebrasi, differensiallash, yechiluvchan Li algebrasi, lokal differensiallash.

РЕЗЮМЕ

В настоящей работе мы исследуем дифференцирования и локальные дифференцирования конечномерных разрешимых алгебр Ли, нильрадикалы, являющиеся естественно градуированными филиформными нильрадикалами. В частности устанавливается общий вид матриц, задающих дифференцирования и локальные дифференцирования этих алгебр. Мы показываем, что в указанных алгебрах существуют локальные дифференцирования, не сводящиеся к дифференцированиям.

Ключевые слова: алгебра Ли, дифференцирования, разрешимые алгебра Ли, локальные дифференцирования.