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An abstract characterization of trace class ideal $\mathcal{C}_1$

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RESUME

This paper has been devoted to establishing abstract characterizations of the sequence space l^1 and the Schatten ideal $\mathcal{C}^1$, demonstrating that both are uniquely determined within their respective classes by natural axiomatic properties.

Key words: non-increasing rearrangement, symmetric sequence space, compact operator, Banach ideals of compact operators, Calkin's construction.

Let l^∞ (respectively, c_0) be the Banach space of bounded (respectively, converging to zero) sequences $\{\xi_n\}_{n=1}^\infty$ of complex numbers equipped with the uniform norm $\|\{\xi_n\}\|_\infty = \sup\limits_{n\in\mathbb{N}} |\xi_n|$, where $\mathbb{N}$ is the set of natural numbers.

In l^∞ we consider the natural partial order

$$\{\xi_n\} \le \{\eta_n\} \iff \xi_n \le \eta_n \text{ for all } \quad n \in \mathbb{N}.$$

If $\xi = \{\xi_n\}_{n=1}^\infty \in l^\infty$, then the *non-increasing rearrangement* $\xi^* : (0,\infty) \to (0,\infty)$ of ξ is defined by

$$\xi^*(t) = \inf\{\lambda : \mu\{|\xi| > \lambda\} \le t\}, \quad t > 0,$$

(see, for example, [1, Ch. 2, Definition 1.5]). As such, the non-increasing rearrangement of a sequence $\{\xi_n\}_{n=1}^\infty \in l^\infty$ can be identified with the sequence $\xi^* = \{\xi_n^*\}_{n=1}^\infty$, where

$$\xi_n^* = \inf\left\{\sup_{n\notin F} |\xi_n| : F \subset \mathbb{N},\ |F| < n\right\}.$$

If $\{\xi_n\} \in c_0$, then $\xi_n^* \downarrow 0$; in this case there exists a bijection $\pi : \mathbb{N} \to \mathbb{N}$ such that $|\xi_{\pi(n)}| = \xi_n^*$, $n \in \mathbb{N}$.

Hardy-Littlewood-Polya partial order in the space l^∞ is defined as follows:

$$\xi = \{\xi_n\} \prec\prec \eta = \{\eta_n\} \iff \sum_{n=1}^m \xi_n^* \le \sum_{n=1}^m \eta_n^* \quad \text{for all} \quad m \in \mathbb{N}.$$

A non-zero linear subspace $E \subset l^\infty$ with a Banach norm $\|\cdot\|_E$ is called a *symmetric* (*fully symmetric*) sequence space if

$$\eta \in E,\ \xi \in l^\infty,\ \xi^* \le \eta^* \text{ (resp., } \xi^* \prec\prec \eta^*) \implies \xi \in E \text{ and } \|\xi\|_E \le \|\eta\|_E.$$

Every fully symmetric sequence space is a symmetric sequence space. The converse is not true in general. At the same time, any separable symmetric sequence space is a fully symmetric space.

If $(E, \|\cdot\|_E)$ is a symmetric sequence space, then

$$\|\xi\|_E = \|\,|\xi|\,\|_E = \|\xi^*\|_E \quad \text{for all } \xi \in E.$$

Besides, $(E_h, \|\cdot\|_E)$ is a Banach lattice with respect to the partial order induced from l^∞.

We say that the norm in a symmetric sequence space $(E, \|\cdot\|_E)$ is said to have the *Fatou property* if from the conditions

$$0 \le x^{(n)} \le x^{(n+1)} \in E, \quad n \in \mathbb{N}, \quad \sup_n \|x^{(n)}\|_E < \infty,$$

it follows that

$$\sup_{n\ge 1} x^{(n)} \in E \quad \text{and} \quad \|x\|_E = \sup_{n\ge 1} \|x^{(n)}\|_E.$$

It is known [1, Chapter II, §2.4, Theorem 2.4.2, pp. 44-46] that the norm of every fully symmetric sequence space $((E, \|\cdot\|_E)$ has the Fatou property. But general symmetric sequence spaces (not fully symmetric) do not always have the Fatou property.

Immediate examples of fully symmetric sequence spaces are $(l^\infty, \|\cdot\|_\infty)$, $(c_0, \|\cdot\|_\infty)$ and the Banach spaces

$$l^p = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in l^\infty : \ \|\xi\|_p = \left(\sum_{n=1}^\infty |\xi_n|^p \right)^{1/p} < \infty \right\}, \ 1 \le p < \infty.$$

For any symmetric sequence space $(E, \|\cdot\|_E)$ the following continuous embeddings hold [1, Ch. 2, § 6, Theorem 6.6]: $(l^1, \|\cdot\|_1) \subset (E, \|\cdot\|_E) \subset (l^\infty, \|\cdot\|_\infty)$. Besides, $\|\xi\|_E \le \|\xi\|_1$ for all $\xi \in l^1$ and $\|\xi\|_\infty \le \|\xi\|_E$ for all $\xi \in E$.

If there is $\xi \in E \setminus c_0$, then $\xi^* \ge \alpha\mathbf{1}$ for some $\alpha > 0$, where $\mathbf{1} = \{1, 1, ...\}$. Consequently, $\mathbf{1} \in E$ and $E = l^\infty$. Therefore, either $E \subset c_0$ or $E = l^\infty$.

Now, let $(\mathcal{H}, (\cdot,\cdot))$ be an infinite-dimensional separabel Hilbert space over $\mathbb{C}$, and let $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ be the $C^\star$–algebra of all bounded linear operators in $\mathcal{H}$. Denote by $\mathcal{K}(\mathcal{H})$ $(\mathcal{F}(\mathcal{H}))$ the two-sided ideal of compact (respectively, finite rank) linear operators in $\mathcal{B}(\mathcal{H})$. It is well known that, for any proper two-sided ideal $\mathcal{I} \subset \mathcal{B}(\mathcal{H})$.

Denote $\mathcal{B}_h(\mathcal{H}) = \{x \in \mathcal{B}(\mathcal{H}) : x = x^*\}$, $\mathcal{B}_+(\mathcal{H}) = \{x \in \mathcal{B}_h(\mathcal{H}) : x \ge 0\}$, and let $\tau : \mathcal{B}_+(\mathcal{H}) \to [0, \infty]$ be the *canonical trace* on $\mathcal{B}(\mathcal{H})$, that is,

$$\tau(x) = \sum_{j\in J} (x\varphi_j, \varphi_j), \quad x \in \mathcal{B}_+(\mathcal{H}),$$

where $\{\varphi_j\}_{j\in J}$ is an orthonormal basis in $\mathcal{H}$ (see, for example, [5, Ch. 7, E. 7.5]).

Let $\mathcal{P}(\mathcal{H}) = \{e \in \mathcal{B}(\mathcal{H}) : e = e^2 = e^*\}$ be the lattice of projectors in $\mathcal{B}(\mathcal{H})$. If $\mathbf{1}$ is the identity of $\mathcal{B}(\mathcal{H})$ and $e \in \mathcal{P}(\mathcal{H})$, we will write $e^\perp = \mathbf{1} - e$.

Let $x \in \mathcal{B}(\mathcal{H})$, and let $\{e_\lambda(|x|)\}_{\lambda \ge 0}$ be the spectral family of projections for the absolute value $|x| = (x^*x)^{1/2}$ of x, that is, $e_\lambda(|x|) = \{|x| \le \lambda\}$. If $t > 0$, then the *t-th generalized singular number* of x, or the *non-increasing rearrangement* of x, is defined as

$$\mu_t(x) = \inf\{\lambda > 0 : \ \tau(e_\lambda(|x|)^\perp) \le t\}$$

(see [2]).

A non-zero linear subspace $X \subset \mathcal{B}(\mathcal{H})$ with a Banach norm $\|\cdot\|_X$ is called *symmetric* (*fully symmetric*) if the conditions

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \mu_t(y) \le \mu_t(x) \quad \text{for all} \ \ t > 0$$

(respectively,

$$x \in X, \ y \in \mathcal{B}(\mathcal{H}), \ \int_0^s \mu_t(y)dt \le \int_0^s \mu_t(x)dt \quad \text{for all} \ \ s > 0 \ \ (\text{writing } y \prec\prec x))$$

imply that $y \in X$ and $\|y\|_X \le \|x\|_X$.

The spaces $(\mathcal{B}(\mathcal{H}), \|\cdot\|_\infty)$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ as well as the classical Banach two-sided ideals

$$\mathcal{C}^p = \{x \in \mathcal{K}(\mathcal{H}) : \ \|x\|_p = \tau(|x|^p)^{1/p} < \infty\}, \ 1 \le p < \infty,$$

are examples of fully symmetric spaces.

It should be noted that for every symmetric space $(X, \|\cdot\|_X) \subset \mathcal{B}(\mathcal{H})$ and all $x \in X$, $a, b \in \mathcal{B}(\mathcal{H})$,

$$\|x\|_X = \|\,|x|\,\|_X = \|x^*\|_X, \quad axb \in X, \quad \text{and} \quad \|axb\|_X \leq \|a\|_\infty \|b\|_\infty \|x\|_X.$$

Remark 1. *If $X \subset \mathcal{B}(\mathcal{H})$ is a symmetric space and there exists a projection $e \in \mathcal{P}(\mathcal{H}) \cap X$ such that $\tau(e) = \infty$, that is, $\dim e(\mathcal{H}) = \infty$, then $\mu_t(e) = \mu_t(\mathbf{1}) = 1$ for every $t \in (0, \infty)$. Consequently, $\mathbf{1} \in X$ and $X = \mathcal{B}(\mathcal{H})$. If $X \neq \mathcal{B}(\mathcal{H})$ and $x \in X$, then $e_\lambda(|x|)^\perp = \{|x| > \lambda\}$ is a finite-dimensional projection, that is, $\dim e_\lambda(|x|)^\perp(\mathcal{H}) < \infty$ for all $\lambda > 0$. This means that $x \in \mathcal{K}(\mathcal{H})$, hence $X \subset \mathcal{K}(\mathcal{H})$. Therefore, either $X = \mathcal{B}(\mathcal{H})$ or $X \subset \mathcal{K}(\mathcal{H})$.*

If $x \in \mathcal{K}(\mathcal{H})$, then $|x| = \sum_{n=1}^{m(x)} s_n(x) p_n$ (if $m(x) = \infty$, the series converges uniformly), where $\{s_n(x)\}_{n=1}^{m(x)}$ is the set of singular values of x, that is, the set of eigenvalues of the compact operator $|x|$ in the decreasing order, and p_n is the projection onto the eigenspace corresponding to $s_n(x)$. Consequently, the non-increasing rearrangement $\mu_t(x)$ of $x \in \mathcal{K}(\mathcal{H})$ can be identified with the sequence $\{s_n(x)\}_{n=1}^\infty$, $s_n(x) \downarrow 0$ (if $m(x) < \infty$, we set $s_n(x) = 0$ for all $n > m(x)$).

Let $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ be a symmetric space. Fix an orthonormal basis $\{\varphi_n\}_{n \in \mathbb{N}}$ in $\mathcal{H}$. Let p_n be the one-dimensional projection on the subspace $\mathbb{C} \cdot \varphi_n \subset \mathcal{H}$. It is clear that the set

$$E(X) = \left\{ \xi = \{\xi_n\}_{n=1}^\infty \in c_0 : \; x_\xi = \sum_{n=1}^\infty \xi_n p_n \in X \right\}$$

(the series converges uniformly), is a symmetric sequence space with respect to the norm $\|\xi\|_{E(X)} = \|x_\xi\|_X$. Consequently, each symmetric subspace $(X, \|\cdot\|_X) \subset \mathcal{K}(\mathcal{H})$ uniquely generates a symmetric sequence space $(E(X), \|\cdot\|_{E(X)}) \subset c_0$. The converse is also true: every symmetric sequence space $(E, \|\cdot\|_E) \subset c_0$ uniquely generates a symmetric space $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) \subset \mathcal{K}(\mathcal{H})$ by the following rule (see, for example, [4, Ch. 3, Section 3.5]):

$$\mathcal{C}_E = \{x \in \mathcal{K}(\mathcal{H}) : \{s_n(x)\} \in E\}, \quad \|x\|_{\mathcal{C}_E} = \|\{s_n(x)\}\|_E.$$

In addition,

$$E(\mathcal{C}_E) = E, \; \|\cdot\|_{E(\mathcal{C}_E)} = \|\cdot\|_E, \; \mathcal{C}_{E(\mathcal{C}_E)} = \mathcal{C}_E, \; \|\cdot\|_{\mathcal{C}_{E(\mathcal{C}_E)}} = \|\cdot\|_{\mathcal{C}_E}.$$

The construction described above is known as Calkin's construction.

We will call the pair $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ a *Banach ideal of compact operators* (cf. [3, Ch. III]). It is known that $(\mathcal{C}^p, \|\cdot\|_p) = (\mathcal{C}_{l^p}, \|\cdot\|_{\mathcal{C}_{l^p}})$ for all $1 \leq p < \infty$ and $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty) = (\mathcal{C}_{c_0}, \|\cdot\|_{\mathcal{C}_{c_0}})$.

Hardy-Littlewood-Polya partial order in the Banach ideal $\mathcal{K}(\mathcal{H})$ is defined by

$$x \prec\prec y, \; x, y \in \mathcal{K}(\mathcal{H}) \iff \{s_n(x)\} \prec\prec \{s_n(y)\}.$$

We say that a Banach ideal $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is *fully symmetric* if conditions $y \in \mathcal{C}_E$, $x \in \mathcal{K}(\mathcal{H})$, $x \prec\prec y$ entail that $x \in \mathcal{C}_E$ and $\|x\|_{\mathcal{C}_E} \leq \|y\|_{\mathcal{C}_E}$. It is clear that $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ is a fully symmetric ideal if and only if $(E, \|\cdot\|_E)$ is a fully symmetric sequence space.

Examples of fully symmetric ideals include $(\mathcal{K}(\mathcal{H}), \|\cdot\|_\infty)$ as well as the Banach ideals $(\mathcal{C}^p, \|\cdot\|_p)$ for all $1 \leq p < \infty$. It is clear that $\mathcal{C}^1 \subset \mathcal{C}_E \subset \mathcal{K}(\mathcal{H})$ for every symmetric sequence space $E \subset c_0$ with $\|x\|_{\mathcal{C}_E} \leq \|x\|_1$ and $\|y\|_\infty \leq \|y\|_{\mathcal{C}_E}$ for all $x \in \mathcal{C}^1$ and $y \in \mathcal{C}_E$.

Abstract characterizations of spaces l^1 and $\mathcal{C}^1$

Let us consider

$$E = l^1 = \Big\{ x = \{x_n\}_{n=1}^\infty \subset \mathbb{C} : \sum_{n=1}^\infty |x_n| < \infty \Big\}.$$

In the linear space l^1 we define coordinate-wise multiplication

$$x \cdot y = \{x_n y_n\}_{n=1}^\infty, \quad x, y \in l^1,$$

and let $|x| = \{|x_n|\}_{n=1}^\infty$.

Proposition 2. *If $x, y \in l^1$ and $x \cdot y = \theta$, then*

$$\|x+y\|_1 = \|x\|_1 + \|y\|_1.$$

Proof. Since $x_n y_n = 0$ for each n, at least one of x_n, y_n vanishes for each coordinate. Thus

$$|x_n + y_n| = |x_n| + |y_n|.$$

Summing over n yields

$$\sum_{n=1}^{\infty} |x_n + y_n| = \sum_{n=1}^{\infty} |x_n| + \sum_{n=1}^{\infty} |y_n|,$$

that is, $\|x+y\|_1 = \|x\|_1 + \|y\|_1$. □

Motivated by Proposition 2, we would like to characterize all symmetric sequence spaces $(E, \|\cdot\|_E)$ for which this additivity property holds.

Theorem 3. *Let $(E, \|\cdot\|_E) \subset c_0$ be a fully symmetric space such that*

$$\|x+y\|_E = \|x\|_E + \|y\|_E, \quad \forall x, y \in E, \; x \cdot y = \theta.$$

Then $E = l^1$ and $\|x\|_E = \|x\|_1$ or $\|x\|_E = \alpha \cdot \|x\|_1$ for all $x \in E$, where α some positive number, depending on $\|\cdot\|_E$.

Proof. Pick any nonzero $x \in E$. Then $x^* \in E$, and $x_1^* > 0$. Consider $y = (x_1^*, 0, 0, \dots)$. Clearly $y \prec\prec x^*$, hence $y \in E$. Therefore $e_1 = \frac{1}{x_1^*} y = \{1, 0, 0, \dots\} \in E$. By permutation invariance, all unit vectors $e_n = \{0, 0, \dots, 0, 1, 0, \dots\} \in E$, where 1 in the n-th position, $n \in \mathbb{N}$.

Set $\alpha := \|e_n\|_E$ (independent of n). If $\alpha = 0$, then $E = \{0\}$, which we exclude. Rescale the norm if necessary so that $\alpha = 1$.

Now for any $x \in E$, with decreasing rearrangement $x^* = \{x_1^*, x_2^*, \dots\}$, define partial sums

$$x^{(k)} = \sum_{n=1}^{k} x_n^* e_n.$$

Since the vectors $x_n^* e_n$ are supported on disjoint coordinates, then from the condition of theorem we get

$$\|x^{(k)}\|_E = \sum_{n=1}^{k} \|x_n^* e_n\|_E = \sum_{n=1}^{k} x_n^* \|e_n\|_E = \sum_{n=1}^{k} x_n^*.$$

Thus $\|x^{(k)}\|_E = \sum_{n=1}^{k} x_n^*$.

Because $x^{(k)} \uparrow x^*$ and E has the Fatou property,

$$\|x^*\|_E = \lim_{k\to\infty} \|x^{(k)}\|_E = \sum_{n=1}^{\infty} x_n^*.$$

Hence $x^* \in l^1$ and $\|x^*\|_E = \|x\|_1$. Therefore $x \in l^1$ and $\|x\|_E = \|x\|_1$.

This shows $E = l^1$ as sets, and $\|\cdot\|_E = \|\cdot\|_1$ after normalization. Without normalization, $\|\cdot\|_E = \alpha\|\cdot\|_1$ with $\alpha > 0$. □

Now, using the Calkin's construction, we pass to operator ideals.

Theorem 4. *Let $(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E})$ be a fully symmetric ideal in $\mathcal{K}(\mathcal{H})$ such that*

$$\|A+B\|_{\mathcal{C}_E} = \|A\|_{\mathcal{C}_E} + \|B\|_{\mathcal{C}_E}$$

for all $A, B \in \mathcal{C}_E$ with $A = A^*$, $B = B^*$, and $AB = 0$. Then

$$(\mathcal{C}_E, \|\cdot\|_{\mathcal{C}_E}) = (\mathcal{C}^1, \alpha\|\cdot\|_1), \text{ where } \alpha > 0.$$

Proof. Let take $x, y \in E$ with $x \cdot y = \theta$. Choose an orthonormal basis $\{\varphi_n\}$ in $\mathcal{H}$ and define diagonal operators

$$A(\varphi_n) = |x_n|\varphi_n, \qquad B(\varphi_n) = |y_n|\varphi_n.$$

Then A, B are compact, self-adjoint, and $AB = 0$. Their singular values sequences are $\{s_n(A)\}_{n=1}^{\infty} = |x|^*$, $\{s_n(B)\}_{n=1}^{\infty} = |y|^*$, and $\{s_n(A+B)\}_{n=1}^{\infty} = |x+y|^* = (|x|+|y|)^*$.

The condition of theorem gives

$$\|A+B\|_{\mathcal{C}_E} = \|A\|_{\mathcal{C}_E} + \|B\|_{\mathcal{C}_E},$$

which translates to

$$\|x+y\|_E = \|x\|_E + \|y\|_E.$$

By Theorem 3, it follows that $E = l^1$ and $\|\cdot\|_E$ is (up to a constant) the l^1 norm. Hence $\mathcal{C}_E = \mathcal{C}^1$ and $\|\cdot\|_{\mathcal{C}_E}$ coincides with the norm $\alpha\|\cdot\|_1$ for some $\alpha > 0$. ☐

Conclusion. In the framework of Banach lattice theory, a norm is said to be *disjointly additive* if

$$\|x+y\| = \|x\| + \|y\| \qquad (x \perp y),$$

that is, whenever x and y are disjointly supported.

The results established above show that, within the class of fully symmetric sequence spaces, this property uniquely characterizes l^1. Thus, l^1 emerges as the only symmetric sequence space in which disjointness of elements is precisely encoded by additivity of the norm.

On the other hand, in the setting of operator ideals, the analogous additivity condition

$$\|A+B\| = \|A\| + \|B\| \qquad (A, B \text{ self-adjoint}, \; AB = 0)$$

uniquely characterizes the trace class $\mathcal{C}^1$ equipped with the trace norm (up to a constant factor). Hence, $\mathcal{C}^1$ appears as the only fully symmetric ideal in which orthogonality of operators is exactly reflected by additivity of the norm.

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REZYUME

Ushbu maqola l^1 ketma-ketlik fazosining va $\mathcal{C}^1$ Shatten idealining abstrakt xarakterizatsiyalarini berishga bag'ishlangan bo'lib, ularning har ikkisi ham o'z sinflarida tabiiy aksiomatik xossalari orqali yagona tarzda aniqlanishi ko'rsatiladi.

Kalit so'zlar: o'smaydigan o'rin almashtirish, simmetrik ketma-ketlik fazosi, kompakt operator, kompakt operatorlar Banax ideali, Kalkin munosabati.

РЕЗЮМЕ

Настоящая статья посвящена установлению абстрактных характеризаций пространства последовательностей l^1 и идеала Шаттена $\mathcal{C}^1$, показывая, что оба они уникально определяются в своих классах естественными аксиоматическими свойствами.

Ключевые слова: невозрастающая перестановка, симметричное пространство последовательностей, банаховы компактный оператор, Банаховы идеалы компактных операторов, соответствие Калкина.