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Large entropy measures and their supports

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RESUME

In this article, the theorem of de Thélin and Dinh stating that the support of measures with large entropy for endomorphisms lies in the Julia set is proved by a new method, via the rate of convergence to the equidistributed measure.

Key words: Ergodic measure, entropy and endomorphism.

1. Introduction

The study of the dynamics of holomorphic endomorphisms of complex projective spaces $\mathbb{P}^k := \mathbb{P}^k(\mathbb{C})$ is a central topic in complex dynamics, see for instance [7, 12] for an overview of the subject. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of algebraic degree $d \geq 2$. There exists a canonical positive closed f^*-invariant $(1,1)$-current T, called the *Green current of* f, with the property that the sequence $d^{-n}(f^n)^*\omega_0$ converges to T for every smooth positive closed $(1,1)$-form ω_0 of mass 1. The current T has strong geometric properties, in particular, it has Hölder continuous potentials. As a consequence, the measure $\mu := T^{\wedge k}$ is well-defined, and it is the unique measure of maximal entropy $k \ln d$ of f [2, 10]. Its support is called the *Julia set* of f. By a result of de Thélin and Dinh [8, 9], every ergodic measure whose measure-theoretic entropy is strictly larger than $(k-1)\ln d$ is also supported on the Julia set of f.

Theorem 1. *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. If ν is a $f-$invariant measure with entropy $h_\nu(f) > (k-1)\ln d$ then the support of ν is supported on the Julia set J.*

The proof given in [8, 9] of the above property crucially relies on the existence of the Green current. In particular, it follows from a delicate induction which makes use of the successive self-intersections $T^{\wedge j}$ of the Green current T. In this paper, we provide an alternative proof based on the rate of convergence towards the measure of maximal entropy.

2. Preliminary notions and a proof of Theorem 1.

Let ω be the Fubini-Study metric on $\mathbb{P}^k$. The distance dist which we use below is with respect to the Fubini-Study metric. Let us recall some notions and definitions which we use in sequel. Let $f : \mathbb{P}^k \to \mathbb{P}^k$ be an endomorphism of degree $d \geq 2$ and ν be an ergodic $f-$invariant probability measure on $\mathbb{P}^k$. For $n \geq 0$, the Bowen metric defines as follows:

$$d_n(z,w) = \max_{0\leq j\leq n-1} \operatorname{dist}(f^j(z), f^j(w)).$$

For $z \in \mathbb{P}^k$, let

$$B_n(z,r) := \{w \in \mathbb{P}^k : d_n(z,w) \leq r\}$$

be the closed ball of radius r with respect to d_n. Then, the quantity

$$\sup_{r>0} \liminf_{n\to\infty} -\frac{1}{n} \ln \nu(B_n(z,r))$$

takes a constant value $c = c(f,\nu)$ ν-almost everywhere. We then define the metric entropy $h_\nu(f)$ of ν (with respect to f) as $h_\nu(f) := c$. It is easy to see that $h_\nu(f) \geq 0$. As we mentioned above, the Green measure μ is the measure of maximal entropy $k \ln d$ for f. An ergodic $f-$invariant probability measure ν is called the measure with large entropy if $h_\nu(f) > (k-1)\log d$. Before studying the measures of large entropy, we need to recall some result which we use in sequel.

We use the following theorem, which is proven by Dinh and Sibony (see [6, Theorem 1.1]).

Theorem 2. *Let f be a holomorphic endomorphism of algebraic degree $d \geq 2$ of $\mathbb{P}^k$. Let μ be the equilibrium measure of f and $1 < \lambda < d$ a constant. There is a invariant proper analytic subset E_λ, possibly empty, of $\mathbb{P}^k$ such that if a is a point out of E_λ and $\psi \in \mathcal{C}^2(\mathbb{P}^k)$ function, then*

$$|\langle d^{-nk}(f^n)^*\delta_a - \mu, \psi\rangle| \leq A\|\psi\|_2 \left(1 + \ln^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\right) \lambda^{-n} \tag{2}$$

where A is a constant independent of a and n.

We call the set E_λ which is given in Theorem 2 is λ-*exceptional set* of f. We have the following corollary of the above theorem.

Lemma 3. *Let f, λ, ψ and E_λ be as in Theorem and ν be a finite measure on $\mathbb{P}^k$. Then for any ψ with* $\mathrm{supp}\psi \cap \mathrm{supp}\mu = \emptyset$ *there exists a constant $A > 0$ such that*

$$|\langle d^{-nk}(f^n)^*\nu, \psi\rangle| \leq A\|\psi\|_2\lambda^{-n} \int_{\mathbb{P}^k} \left(1 + \log^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\right) d\nu. \tag{3}$$

Proof. If $\nu = 0$ then (3) is clear. If ν is a probability measure on $\mathbb{P}^k$ then (3) directly follows from (2) by writing ν as a limit of a sum of Dirac measures. Assume now $\nu(\mathbb{P}^k) \neq 0$. Define $\tilde{\nu} = \frac{\nu}{\nu(\mathbb{P}^k)}$. Since $\tilde{\nu}$ is a probability measure we have

$$|\langle d^{-nk}(f^n)^*\tilde{\nu} - \mu, \psi\rangle| \leq A\|\psi\|_2\lambda^{-n} \int_{\mathbb{P}^k} \left(1 + \log^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\right) d\tilde{\nu}.$$

Since $\mathrm{supp}\psi \cap \mathrm{supp}\mu = \emptyset$ we have

$$\begin{aligned}
|\langle d^{-nk}(f^n)^*\nu, \psi\rangle| &= |\langle d^{-nk}(f^n)^*\nu - \nu(\mathbb{P}^k)\mu, \psi\rangle| \\
&= \nu(\mathbb{P}^k)|\langle d^{-nk}(f^n)^*\tilde{\nu} - \mu, \psi\rangle| \\
&\leq A\|\psi\|_2\lambda^{-n}\nu(\mathbb{P}^k) \int_{\mathbb{P}^k} \left(1 + \log^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\right) d\tilde{\nu} \\
&\leq A\|\psi\|_2\lambda^{-n} \int_{\mathbb{P}^k} \left(1 + \log^+ \frac{1}{\mathrm{dist}(a, E_\lambda)}\right) d\nu.
\end{aligned}$$

The proof is complete. □

We recall the definition of box-counting dimension. Let $E \subset \mathbb{P}^k$ be a set and $E_t := \{z \in \mathbb{P}^k : \mathrm{dist}(z, E) \leq t\}$ with $t > 0$. The upper box-counting dimension of E is defined as

$$\overline{\dim}_B E = 2k - \limsup_{t\to 0} \frac{\ln \mathrm{vol}(E_t)}{\ln t}.$$

From the definition we can see the following properties (for more details see [11]):

1. if upper box-counting dimension is less than $2k$ then there exists $\alpha > 0$ such that for small t we have $\mathrm{vol}(E_t) < t^\alpha$.
2. if E is a $p-$dimensional analytic set then $\overline{\dim}_B E = 2p$.

Let us prove the following lemma.

Lemma 4. *Let $E \subset \mathbb{P}^k$ be a set with upper box-counting dimension strictly less than $2k$. Then* $\ln^+ \frac{1}{\mathrm{dist}(z,E)} \in L^1(\mathbb{P}^k, \omega^k)$.

Proof. Since upper box-counting dimension of E is less than $2k$ there exists $\alpha > 0$ so that if E_t is a t-neighbourhood of E with $t > 0$ small enough we have $\mathrm{vol}(E_t) < t^\alpha$ (see [11]). We claim that

$$\int_{E_t} \ln^+ \frac{1}{\mathrm{dist}(z, E)}\omega^k = o(t^{\alpha/2}),\ t \to 0.$$

Indeed,

$$\begin{aligned}\int_{E_t}\ln^+\frac{1}{\operatorname{dist}(z,E)}\omega^k\leq&\sum_{l=0}^{\infty}\int_{E_{\frac{t}{2^l}}\setminus E_{\frac{t}{2^{l+2}}}}\ln^+\frac{1}{\operatorname{dist}(z,E)}\omega^k\\ \leq&\sum_{l=0}^{\infty}((l+2)\ln 2-\ln t)\int_{E_{\frac{t}{2^l}}\setminus E_{\frac{t}{2^{l+2}}}}\omega^k\\ \leq&\sum_{l=0}^{\infty}((l+2)\ln 2-\ln t)\frac{t^\alpha}{2^{l\alpha}}\\ \leq&t^{\alpha/2}\sum_{l=0}^{\infty}((l+2)\ln 2-\ln t)\frac{t^{\alpha/2}}{2^{l\alpha}}\\ =&o(t^{\alpha/2}),\ t\to 0.\end{aligned}$$

Since $\ln^+\frac{1}{\operatorname{dist}(z,E)}$ is continuous outside E_t, the above claim follows $\ln^+\frac{1}{\operatorname{dist}(z,E)}\in L^1(\mathbb{P}^k)$. The proof is complete. □

Proof of Theorem 1. Let $F\subset\mathbb{P}^k\setminus J$ such that $\overline{F}\cap J=\emptyset$ and W be an open neighbourhood of F with $\overline{W}\cap J=\emptyset$. By using Gromov's contruction [10] we have

$$h_t(f,F)=h_t(f,F\cap K)\leq \operatorname{lov}(f,W):=\limsup_{n\to\infty}\frac{1}{n}\ln\operatorname{volume}(\Gamma_{\mathrm{n}}^{\mathrm{W}})$$

where

$$\Gamma_n^W:=\{(z,f(z),...,f^{n-1}(z)),z\in W\cap\mathbb{P}^k\}.$$

Note that

$$\operatorname{vol}(\Gamma_n^W)=\sum_{0\leq n_i\leq n-1}\int_{W\cap\mathbb{P}^k}(f^{n_1})^*\omega\wedge...\wedge(f^{n_k})^*\omega.$$

We shall estimate above each term of the sum in the right hand side of the last equation. Without lost of generality assume $n_1\leq n_i$ for $1\leq i\leq k$. Let $\Omega=\omega\wedge...\wedge(f^{n_k-n_1})^*\omega$. Take a smooth function $0\leq\psi\leq 1$ so that $\psi|_W=1$ and $\tilde{W}\cap J=\emptyset$ where $\tilde{W}=\operatorname{supp}\psi$. For λ with $1<\lambda<d$, let E_λ be the λ-exceptional set of f. Since $\mu|_{\tilde{W}}=0$ by Lemma 3 we have

$$\begin{aligned}\int_{\mathbb{P}^k}\psi\cdot(f^{n_1})^*\Omega=&|\langle(f^{n_1})^*\Omega-d^{kn_1}\mu,\psi\rangle|\\ \leq&Ad^{kn_1}\|\psi\|_2\lambda^{-n_1}\int_{\mathbb{P}^k}\left(1+\ln^+\frac{1}{\operatorname{dist}(z,E_\lambda)}\right)\Omega\\ \leq&A(d^{k-1})^{n-n_1}(d^k/\lambda)^{n_1}\|\psi\|_2\int_{\mathbb{P}^k}\left(1+\ln^+\frac{1}{\operatorname{dist}(z,E_\lambda)}\right)\omega^k.\end{aligned}$$

In the last estimate we used $\Omega\leq d^{n_k-n_1+n_{k-1}-n_1+...+n_2-n_1}\omega^k\leq(d^{k-1})^{n-n_1}\omega^k$. Since upper box-counting dimension of E_λ is less than $2k$ by using Lemma 4 we deduce that there exists A_0 independent of n so that

$$\int_{\mathbb{P}^k}\psi\cdot(f^{n_1})^*\Omega\leq A_0(d^{k-1})^{n-n_1}(d^k/\lambda)^{n_1}\leq A_0(d^k/\lambda)^n.\tag{4}$$

From (4) it follows that

$$\int_{W\cap\mathbb{P}^k}(f^{n_1})^*\omega\wedge...\wedge(f^{n_k})^*\omega\leq A_0(d^k/\lambda)^n.$$

Finally, we have

$$\operatorname{vol}(\Gamma_n^W)\leq A_0n^k(d^k/\lambda)^n$$

and

$$\limsup_{n\to\infty} \frac{1}{n} \ln \mathrm{vol}(\Gamma_n^W) \le \ln(d^k/\lambda).$$

Consequently, we have

$$(k-1)\ln d < h_\nu(f) \le \ln\left(\frac{d^k}{\lambda}\right).$$

Since λ is an arbitrary constant such that $1 < \lambda < d$, this leads to a contradiction. □

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REZYUME

Ushbu maqolada endomorfizmlar uchun de Thélin hamda Dinh tomonidan isbotlangan katta entropiyali o'lchovlarning havzasi Julia to'plamida yotishi haqidagi teorema tekis taqsimlangan o'lchovga yaqinlashish tezligi orqali yangicha usulda isbotlangan.

Kalit so'zlar: Ergodik o'lchov, entropiya, endomorfizm.

РЕЗЮМЕ

В данной статье приведено новое доказательство теоремы de Thélin и Dinh о том, что бассейн мер с большой энтропией для эндоморфизмов лежит в множестве Жюлиа. Доказательство основано на скорости сходимости к равномерно распределённой мере.

Ключевые слова: Эргодическая мера, энтропия, эндоморфизм.